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Projective Curvature Tensor of C_9-Manifolds

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Abstract: This article determines on C_9- manifold only fifteen non-zero components of the projective curvature tensor. It proves a C_9- manifold of dimension > 3 with flat projective curvature tensor is just a cosymplectic manifold with flat Ricci tensor. Whereas, the C_9- manifold of dimension 3 with flat projective curvature tensor is Einstein manifold. On the other hand, it discovers fifteen projective invariant classes that denoted by $P_1, P_2, \ldots, P_{15}$. It establishes some of these classes are equivalent on C_9- manifold and the C_9- manifold is Einstein manifold if it belongs to the class $P_1 \bigcap P_{11}$.

Keywords: C_9-manifold; projective curvature tensor; Einstein manifold.

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1 Introduction

In 1990, Chinea and Gonzalez [1] classified the almost contact metric manifolds into many classes. One of these classes is C_9-class whose geometry was studied by Rustanov et al. [2]. There is another important classes, for instance, manifold of Kenmotsu type that satisfying the identity $\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X$, for any vector fields X, Y and $C_{12}-$manifold that were introduced and examined by Abass, Abood [3], [4], [5] and Abass, Al-Zamil [6] respectively.

On the other hand, Abood and Mohammed focused on studying the geometric identities for projective curvature tensor of nearly cosymplectic manifold [7] and they discussed the geometry of the pseudo projective tensor on the same manifold that belongs to a class differ from the above mentioned classes [8].

Whereas, K. De and U. C. De [9] studied the $\phi-$projectively flat and $\xi-$ projectively flat for connected $3-$dimensional trans-Sasakian manifolds. They also studied the semi-symmetric projective property of aforementioned manifolds. While, Rustanov et al. [10] investigated the projective tensor and its invariants on almost $C(\lambda)-$manifolds.

Raghuwanshi et al. [11] generalized the projective curvature tensor and discussed it on the manifold of para-Kenmotsu structure. On the other side, there are many researches which are related to this article, such as [12], [13], [14].

Since the class of manifolds considered in this article is one of the most difficult to study and there few authors studying it and according to the fact that the projective curvature tensor plays a very significant role in studying different properties of Riemannian manifolds, then we shall discuss the projective curvature tensor on C_9- manifolds. So, this article is organized in four sections. After the introduction, section 2 is devoted to review the basic related definitions and facts. In section 3, the components of projective tensor are determined in C_9-manifolds and according to those components, the relationships among Einstein manifold, C_9-manifold and flat projective tensor are investigated. The last section is involved a new classification based on projective tensor.

2 Preliminaries

Let M be $(2n+1)-$dimensional manifold with $n \in \mathbb{Z}^+$, ∇ is Levi-Civita connection on M, and $\mathscr{X}(M)$ is $C^\infty(M)-$module of smooth vector fields on M.

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Definition 1.[1] *A quadruple (η,ξ,Φ,g) of tensor fields on M is called an almost contact metric $(AC-)$ structure on M, if η is a differential $1-$form, ξ is a vector field named a characteristic vector field, Φ is $(1,1)-$tensor field named a structure endomorphism of the module $X(M)$, and $g=\langle\cdot,\cdot\rangle$ is a Riemannian metric, such that the following satisfied:*

$$i)\ \eta(\xi)=1;\quad ii)\ \eta\circ\Phi=0;\quad iii)\ \Phi(\xi)=0;\quad iv)\ \Phi^2=-id+\eta\otimes\xi;$$
$$v)\ \langle\Phi X,\Phi Y\rangle=\langle X,Y\rangle-\eta(X)\eta(Y),\qquad\forall\, X,Y\in\mathscr{X}(M).$$

Additionally, a manifold M equipped with an $AC-$structure (η,ξ,Φ,g) is called an almost contact metric $(AC-)$ manifold.

Definition 2.[2] *An $AC-$manifold M is called a C_9-manifold if its $AC-$structure (η,ξ,Φ,g) satisfies the following identity:*

$$\nabla_X(\Phi)Y=\eta(Y)\nabla_{\Phi X}\xi-\langle\Phi X,\nabla_Y\xi\rangle\xi,\ for\ all\ X,Y\in\mathscr{X}(M),$$

where ∇ is Levi-Civita connection on M that defined by $\nabla(Y,X)=\nabla_Y X$.

Definition 3.[15] *A Riemannian curvature $(3,1)-$tensor R on $AC-$manifold M is known as:*

$$R(X,Y)Z=([\nabla_X,\nabla_Y]-\nabla_{[X,Y]})Z,$$

for all $X,Y,Z\in\mathscr{X}(M)$, where $[A,B]=A\circ B-B\circ A$.

The researchers can learn about the space of associated $G-$structure ($AG-$structure for short) from the references [10] and [16], where in this space, the tensors Φ and g of any $AC-$manifold are formulated by:

$$(\Phi^i_j)=\begin{pmatrix}0&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\sqrt{-1}\boldsymbol{I}_n&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&-\sqrt{-1}\boldsymbol{I}_n\end{pmatrix}\tag{1}$$

$$g_{ij}=\begin{cases}1 & ;\quad if\ i=j=0\\ \delta^a_b & ;\quad if\ i=b\ and\ j=\hat{a},\ or\ i=\hat{a}\ and\ j=b\\ 0 & ;\quad otherwise,\end{cases}\tag{2}$$

where $\boldsymbol{I}_n$ is the identity matrix of rank n, $i,j=0,1,2,\cdots,2n$, $a,b=1,2,\cdots,n$, $\hat{a}=a+n$, and δ^a_b is Krönecker delta.

Proposition 1.[2] *Let $S=(\eta,\xi,\Phi,g)$ be an $AC-$structure on a manifold M. Hence the next terms are equivalent:*
1) M is an $AC-$manifold of class C_9; 2) $B=C=E=G=0$;
3) $F_{ab}=-B_{ab}=-\sqrt{-1}\Phi^0_{a,b}$; $F^{ab}=-B^{ab}=\sqrt{-1}\Phi^0_{\hat{a},\hat{b}}$; $F^{ab}=\overline{F_{ab}}$; $F_{ab}=F_{ba}$; $F^{ab}=F^{ba}$, where B,C,D,E,F and G are structure tensors defined as follow [5], [17], [18]:

$$B(\mathscr{V},\mathscr{U})=-\frac{1}{8}\{\Phi\circ\nabla_{\Phi^2\mathscr{U}}(\Phi)(\Phi^2\mathscr{V})+\Phi\circ\nabla_{\Phi\mathscr{U}}(\Phi)(\Phi\mathscr{V})+\Phi^2\circ\nabla_{\Phi\mathscr{U}}(\Phi)(\Phi^2\mathscr{V})-\Phi^2\circ\nabla_{\Phi^2\mathscr{U}}(\Phi)(\Phi\mathscr{V})\};$$
$$C(\mathscr{V},\mathscr{U})=-\frac{1}{8}\{-\Phi\circ\nabla_{\Phi^2\mathscr{U}}(\Phi)(\Phi^2\mathscr{V})+\Phi\circ\nabla_{\Phi\mathscr{U}}(\Phi)(\Phi\mathscr{V})+\Phi^2\circ\nabla_{\Phi\mathscr{U}}(\Phi)(\Phi^2\mathscr{V})+\Phi^2\circ\nabla_{\Phi^2\mathscr{U}}(\Phi)(\Phi\mathscr{V})\};$$
$$D(\mathscr{V})=\frac{1}{4}\{2\Phi\circ\nabla_{\Phi^2\mathscr{V}}(\Phi)\xi-2\Phi^2\circ\nabla_{\Phi\mathscr{V}}(\Phi)\xi-\Phi\circ\nabla_{\xi}(\Phi)(\Phi^2\mathscr{V})+\Phi^2\circ\nabla_{\xi}(\Phi)(\Phi\mathscr{V})\};$$
$$E(\mathscr{V})=-\frac{1}{2}\{\Phi\circ\nabla_{\Phi^2\mathscr{V}}(\Phi)\xi+\Phi^2\circ\nabla_{\Phi\mathscr{V}}(\Phi)\xi\};$$
$$F(\mathscr{V})=\frac{1}{2}\{\Phi\circ\nabla_{\Phi^2\mathscr{V}}(\Phi)\xi-\Phi^2\circ\nabla_{\Phi\mathscr{V}}(\Phi)\xi\};$$
$$G=\Phi\circ\nabla_{\xi}(\Phi)\xi,$$

for all $\mathscr{V},\mathscr{U}\in\mathscr{X}(M)$. Moreover, F^{ab} and F_{ab} are the components of tensor F, B^{ab} and B_{ab} are the components of tensor D, and $\Phi^k_{i,j}$ are components of $\nabla\Phi$ on AG-structure space.

Proposition 2.[2] *C_9-manifold coincide with the cosymplectic manifold if and only if $F^{ab}=F_{ab}=0$ on the space of associated $G-$structure.*

Theorem 1.[2] *On C_9-manifold M, the components of Riemannian curvature $(3,1)-$ tensor R are given in the following formulas:*

$$1.\ R^0_{a\hat{b}0} = F_{ac}F^{cb}\ ; \quad 2.\ R^0_{ab0} = -F_{ab0}; \quad 3.\ R^0_{ab\hat{c}} = -F_{ab}{}^{c}\ ;$$
$$4.\ R^a_{bc\hat{d}} = A^{ad}_{bc} + F^{ad}F_{bc}\ ; \qquad 5.\ R^{\hat{a}}_{bcd} = -2F_{a[c}\,F_{|b|d]},$$

and the other components are matching to zero, where $[a|b|c] = \frac{1}{2}\{abc - cba\}$.

Remark.[2] Note that:

a)R has the following properties:
1. $R_{ijkl} = R^{\hat{i}}_{jkl}$; 2. $-R_{ijlk} = R_{ijkl} = -R_{jikl}$; 3. $R_{ijkl} = R_{klij}$;
4. $\overline{R_{ijkl}} = R_{\hat{i}\hat{j}\hat{k}\hat{l}}$; 5. $R_{ijkl} + R_{iklj} + R_{iljk} = 0 = R_{ijkl} + R_{kijl} + R_{jkil}$.

b)$A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$.

The above indexes have the ranges $i,j,k,l = 0,1,2,\dots,2n$, and $a,b,c,d = 1,2,\dots,n$. Moreover, $\hat{0} = 0$; $\hat{a} = a+n$, $\hat{\hat{a}} = a$, and $F_{ab0}, F_{ab}{}^{c}, A^{ad}_{bc}$ are suitable smooth functions.

Remark.[2] The Ricci tensor r of an $AC-$manifold satisfies the relation $r_{ij} = -R^k_{ijk} = r_{ji}$. So, the components of Ricci tensor on the space of $AG-$structure for C_9-manifold are matching the following:

$$r_{00} = -2F_{ab}F^{ba}; \quad r_{a0} = -F_{ba}{}^{b}; \quad r_{ab} = F_{ab0}; \quad r_{a\hat{b}} = A^{bc}_{ac}. \tag{3}$$

Definition 4.[10] *The projective curvature $(3,1)-$tensor P on $AC-$manifold M is defined as follows:*

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{r(Y,Z)X - r(X,Z)Y\}, \qquad \forall X,Y,Z \in \mathscr{X}(M),$$

where r is the Ricci tensor of type (2, 0).

Proposition 3.[10] *On $AC-$manifold M, the projective curvature $(4,0)-$tensor supported the following properties:*

$$1)\ P_{ijkl} = -P_{ijlk}; \quad 2)\ P_{ijkl} + P_{iklj} + P_{iljk} = 0\ ; \quad 3)\ P_{ijkl} = \overline{P_{\hat{i}\hat{j}\hat{k}\hat{l}}},$$

Definition 5.[2] *An $AC-$manifold is said to be Einstein manifold if it satisfies the identity $r_{ij} = \lambda\, g_{ij}$, where λ is a constant.*

3 Geometry of Projective Curvature Tensor

In this section, since the projective tensor P measures the deviation of Riemannian manifolds from being projectively flat $(P = 0)$, then we shall determine the components of P and the conditions that make $P = 0$. Furthermore, since Einstein manifolds have significant in Riemannian geometry, then we obtaining them from the discussion of P on C_9-manifolds because P shows how the geometry of manifolds differ from spaces whose geodesics are straight lines.

Theorem 2.*The components of projective curvature tensor on the space of $AG-$structure for C_9-manifold M are given by:*

1.$P_{000d} = -\frac{1}{2n}\, r_{0d}$;
2.$P_{0b0d} = \frac{2n-1}{2n} r_{bd}$;
3.$P_{0b0\hat{d}} = -F_{bc}\, F^{cd} - \frac{1}{2n} r_{b\hat{d}}$;
4.$P_{0bc\hat{d}} = -F_{bc}{}^{d}$;
5.$P_{a00d} = -r_{ad}$;
6.$P_{a00\hat{d}} = F_{ac}F^{cd} + \frac{1}{2n}\delta^d_a\, r_{00}$;
7.$P_{a0c\hat{d}} = F_{ac}{}^{d} + \frac{1}{2n}\delta^d_a\, r_{0c}$;
8.$P_{a0\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta^c_a\, r_{0\hat{d}} - \delta^d_a\, r_{0\hat{c}}\}$;
9.$P_{ab0\hat{d}} = \frac{1}{2n}\delta^d_a\, r_{b0}$;
10.$P_{abcd} = -2F_{a[c}\, F_{|b|d]}$;
11.$P_{abc\hat{d}} = \frac{1}{2n}\delta^d_a\, r_{bc}$;
12.$P_{ab\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta^c_a\, r_{b\hat{d}} - \delta^d_a\, r_{b\hat{c}}\}$;

13. $P_{a\hat{b}0\hat{d}} = F^{bd}_{a} + \frac{1}{2n}\delta^{d}_{a}\, r_{\hat{b}0}$;
14. $P_{a\hat{b}c\hat{d}} = -(A^{bd}_{ac} + F^{bd}F_{ac}) + \frac{1}{2n}\delta^{d}_{a}\, r_{\hat{b}c}$;
15. $P_{a\hat{b}\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta^{c}_{a}\, r_{\hat{b}\hat{d}} - \delta^{d}_{a}\, r_{\hat{b}\hat{c}}\}$,

and the remaining components are zero or can be determined by Proposition 3.

Proof. The components of projective curvature $(4,0)-$tensor P on $AC-$manifold are provided in the following formula:

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}\{g_{ik}\, r_{jl} - g_{il}\, r_{jk}\}. \tag{4}$$

This formula yields 81 components of P which are separated into the following collections on C_9-manifold:
(i) There are 15 main non-zero components of P that are taken indexes as follow:

$$\{P_{000d}, P_{0b0d}, P_{0b0\hat{d}}, P_{0bc\hat{d}}, P_{a00d}, P_{a00\hat{d}}, P_{a0c\hat{d}}, P_{a0\hat{c}\hat{d}}, P_{ab0\hat{d}}, P_{abcd}, P_{abc\hat{d}}, P_{ab\hat{c}\hat{d}}, P_{a\hat{b}0\hat{d}}, P_{a\hat{b}c\hat{d}}, P_{a\hat{b}\hat{c}\hat{d}}\}.$$

(ii) There are also 15 non-zero components of P which are the conjugates of components in item (i) above and they written as:

$$\{P_{000\hat{d}}, P_{0\hat{b}0\hat{d}}, P_{0\hat{b}0d}, P_{0\hat{b}\hat{c}d}, P_{\hat{a}00\hat{d}}, P_{\hat{a}00d}, P_{\hat{a}0\hat{c}d}, P_{\hat{a}0cd}, P_{\hat{a}\hat{b}0d}, P_{\hat{a}\hat{b}\hat{c}\hat{d}}, P_{\hat{a}\hat{b}\hat{c}d}, P_{\hat{a}\hat{b}cd}, P_{\hat{a}b0d}, P_{\hat{a}b\hat{c}d}, P_{\hat{a}bcd}\}.$$

(iii) There are 13 zero components of P that are indicated by the following:

$$\{P_{00cd}, P_{00c\hat{d}}, P_{0b00}, P_{0bcd}, P_{0b\hat{c}\hat{d}}, P_{a000}, P_{a0cd}, P_{ab00}, P_{ab0d}, P_{abc0}, P_{a\hat{b}00}, P_{a\hat{b}cd}\} \cup \{P_{0000}\}.$$

(iv) There are also 12 zero components of P which are the conjugates of first group collection in item (iii) above and they described by:

$$\{P_{00\hat{c}\hat{d}}, P_{00\hat{c}d}, P_{0\hat{b}00}, P_{0\hat{b}\hat{c}\hat{d}}, P_{0\hat{b}cd}, P_{\hat{a}000}, P_{\hat{a}0\hat{c}\hat{d}}, P_{\hat{a}\hat{b}00}, P_{\hat{a}\hat{b}0\hat{d}}, P_{\hat{a}\hat{b}\hat{c}0}, P_{\hat{a}b00}, P_{\hat{a}b\hat{c}\hat{d}}\}.$$

(v) There is a collection that contains the component $P_{a\hat{b}0d}$ and its conjugate $P_{\hat{a}b0\hat{d}}$. The elements of this collection can be determined by using Proposition 3; items 2 and 3, respectively.

(vi) The following 12 components are determined by Proposition 3; item 1:

$$\{P_{00c0}, P_{0bc0}, P_{0b\hat{c}0}, P_{0b\hat{c}d}, P_{a0c0}, P_{a0\hat{c}0}, P_{a0\hat{c}d}, P_{ab\hat{c}0}, P_{ab\hat{c}d}, P_{a\hat{b}c0}, P_{a\hat{b}\hat{c}0}, P_{a\hat{b}\hat{c}d}\}.$$

(vii) The following 12 components are the conjugates of components in item (vi) above:

$$\{P_{00\hat{c}0}, P_{0\hat{b}\hat{c}0}, P_{0\hat{b}c0}, P_{0\hat{b}c\hat{d}}, P_{\hat{a}0\hat{c}0}, P_{\hat{a}0c0}, P_{\hat{a}0c\hat{d}}, P_{\hat{a}\hat{b}c0}, P_{\hat{a}\hat{b}c\hat{d}}, P_{\hat{a}b\hat{c}0}, P_{\hat{a}bc0}, P_{\hat{a}bc\hat{d}}\}.$$

Hence the above collections of components for P on C_9-manifold can determined by substituted the values of R_{ijkl} and g_{ij} from Theorem 1 and equation (2) respectively in equation (4). So, the desired results of the present theorem are obtained in the same way of the following chosen components:

$$\begin{aligned}
P_{0b0d} &= R_{0b0d} - \frac{1}{2n}\,\{g_{00}\, r_{bd} - g_{0d}\, r_{b0}\},\\
&= -R_{0bd0} - \frac{1}{2n} r_{bd},\\
&= -R^{0}_{bd0} - \frac{1}{2n} r_{bd},\\
&= F_{bd0} - \frac{1}{2n} r_{bd} = \frac{2n-1}{2n} r_{bd}. && \text{from equation (3).}
\end{aligned}$$

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$$
\begin{aligned}
P_{a00\hat{d}} &= R_{a00\hat{d}} - \frac{1}{2n}\,\{g_{a0}\,r_{0\hat{d}} - g_{a\hat{d}}\,r_{00}\},\\
&= R_{0a\hat{d}0} + \frac{1}{2n}\delta_a^d\,r_{00},\\
&= R^0_{a\hat{d}0} + \frac{1}{2n}\delta_a^d\,r_{00},\\
&= F_{ac}F^{cd} + \frac{1}{2n}\delta_a^d\,r_{00}.\\
P_{a\hat{b}c\hat{d}} &= R_{a\hat{b}c\hat{d}} - \frac{1}{2n}\,\{g_{ac}\,r_{\hat{b}\hat{d}} - g_{a\hat{d}}\,r_{\hat{b}c}\},\\
&= -R_{\hat{b}ac\hat{d}} + \frac{1}{2n}\delta_a^d\,r_{\hat{b}c},\\
&= -R^b_{ac\hat{d}} + \frac{1}{2n}\delta_a^d\,r_{\hat{b}c},\\
&= -(A^{bd}_{ac} + F^{bd}F_{ac}) + \frac{1}{2n}\delta_a^d\,r_{\hat{b}c}.\\
P_{0000} &= R_{0000} - \frac{1}{2n}\{g_{00}\,r_{00} - g_{00}\,r_{00}\},\\
&= -\frac{1}{2n}\{r_{00} - r_{00}\} = 0.
\end{aligned}
$$

The component $P_{a\hat{b}0d}$ will be proved by using Proposition 3; item (2) as follows:
$P_{a\hat{b}0d} + P_{a0d\hat{b}} + P_{ad\hat{b}0} = 0$. So, $P_{a\hat{b}0d} = -P_{a0d\hat{b}} - P_{ad\hat{b}0}$.
Since $P_{a0d\hat{b}}$ belongs to the collection of non-zero components for P and it has quantity $P_{a0d\hat{b}} = F_{ad}{}^{b} + \frac{1}{2n}\delta_a^b\,r_{0d}$ then from Proposition 3; item (1) we get:
$P_{ad\hat{b}0} = -P_{ad0\hat{b}} = -\frac{1}{2n}\delta_a^b\,r_{d0}$. Then

$$
\begin{aligned}
P_{a\hat{b}0d} &= -\,(F_{ad}{}^{b} + \frac{1}{2n}\delta_a^b\,r_{0d}) + \frac{1}{2n}\delta_a^b\,r_{d0}\\
&= -\,F_{ad}{}^{b} = -F_{da}{}^{b}.
\end{aligned}
$$

At the end, we shall calculate the component P_{00c0} by using Proposition 3; item (1) as follow:
$P_{00c0} = -P_{000c} = -(R_{000c} - \frac{1}{2n}\{g_{00}\,r_{0c} - g_{0c}\,r_{00}\}) = \frac{1}{2n}\,r_{0c}$.

Corollary 1.*If M in Theorem 2 is of dimension 3, then the non-zero components of P will be appeared in the following formulae:*

1)$P_{0112} = -2P_{0001} = -2P_{1012} = 2P_{1102} = r_{01}$*;*
2)$P_{1001} = -2P_{1112} = -2P_{0101} = -r_{11}$*;*
3)$P_{0102} = P_{1212} = \frac{1}{2}(r_{00} - r_{12})$*;*
4)$P_{1202} = -\frac{1}{2}r_{20}$.

*Proof.*Since M is a C_9-manifold of dimension $3 = 2n+1$, then $n = 1$. Hence $a = b = c = d = 1$ (because $a,b,c,d = 1,\dots,n$) and $\hat{a} = \hat{b} = \hat{c} = \hat{d} = 2$ (because $\hat{a} = a+n = 1+1 = 2$). Then the results of the present corollary are attained from Theorem 2 and equation (3) by changing the indexes into 1 and 2 according to the previous. So, equation (3) becomes:

$$F_{11}{}^{1} = -r_{01};\quad F^{11}{}_{1} = -r_{02};\quad A^{11}_{11} = r_{12};\quad F_{11}F^{11} = -\frac{1}{2}r_{00}.$$

Then for example, item 14 in Theorem 2 becomes:

$$P_{1212} = -(A^{11}_{11} + F_{11}F^{11}) + \frac{1}{2}\delta^1_1 r_{21} = -(r_{12} - \frac{1}{2}r_{00}) + \frac{1}{2}r_{12} = -\frac{1}{2}(r_{12} - r_{00}).$$

Theorem 3.*If a C_9-manifold M of dimension greater than 3 has flat projective curvature tensor, then M has flat Ricci tensor ($r = 0$. That is, $r_{ij} = 0$).*

Proof. From Theorem 2, we have the following components:

$$P_{000d} = -\frac{1}{2n}\, r_{0d}. \tag{5}$$

$$P_{0b0\hat{d}} = -F_{bc}\, F^{cd} - \frac{1}{2n}\{r_{b\hat{d}}\}. \tag{6}$$

$$P_{a00d} = -r_{ad}. \tag{7}$$

$$P_{a00\hat{d}} = F_{ac}F^{cd} + \frac{1}{2n}\{\delta_a^d\, r_{00}\}. \tag{8}$$

$$P_{ab\hat{c}\hat{d}} = -\frac{1}{2n}\{\delta_a^c\, r_{b\hat{d}} - \delta_a^d\, r_{b\hat{c}}\}. \tag{9}$$

Since P is flat ($P = 0$), then $P_{ijkl} = 0$ for all i, j, k, l. So, from equations (5) and (7), we get $r_{0d} = 0$ and $r_{ad} = 0$, respectively. Now, from equation (9), we obtain:

$$\delta_a^c\, r_{b\hat{d}} - \delta_a^d\, r_{b\hat{c}} = 0.$$

So, by contracting (a, c) in the above equation, we have:

$$\begin{aligned} \delta_a^a\, r_{b\hat{d}} - \delta_a^d\, r_{b\hat{a}} &= 0, \quad \textit{then we get}: \\ (n-1)\, r_{b\hat{d}} &= 0, \quad \textit{and this implies that}: \\ n = 1 \textit{ or } r_{b\hat{d}} &= 0. \end{aligned}$$

If $n = 1$, then $2n+1 = 3$. This contradicts the hypothesis $\dim(M) = 2n+1 > 3$. Then $r_{b\hat{d}} = 0$. Hence equation (6) gives:

$$F_{bc}\, F^{cd} = 0. \tag{10}$$

Thus equations (8) and (10) imply that $r_{00} = 0$. Then $r_{ij} = 0$ on M for all i, j and hence M has flat Ricci tensor.

Theorem 4. *If C_9-manifold M of dimension greater than 3 has flat projective curvature tensor, then M is cosymplectic manifold.*

Proof. Let $P_{ijkl} = 0$ on M. According to the proof of Theorem 3 and exactly equation (10), we get:

$$\begin{aligned} F_{bc}F^{cd} &= 0, \\ F_{bc}F^{cb} &= 0, \qquad \text{by contracting } (b,d) \\ \sum_{c,b} |F^{cb}|^2 &= 0. \end{aligned}$$

So, $F^{cb} = 0$ *and* $F_{bc} = 0$. Then the result is valid from Proposition 2.

Corollary 2. *If C_9-manifold M of dimension greater than or equal to 5 has flat projective curvature tensor, then $A_{ac}^{ba} = 0$.*

Proof. Let $P_{ijkl} = 0$ on M of dimension greater than or equal to 5. According to Theorem 3, M has flat Ricci tensor. Then $r_{ij} = 0$ and so $r_{a\hat{b}} = 0$. Thus, from equation (3), we have $A_{ca}^{ba} = A_{ac}^{ba} = 0$.

Theorem 5. *Let M be a C_9-manifold of dimension 3. M has flat projective curvature tensor if and only if M is Einstein manifold with $\lambda = -2|F_{11}|^2 = A_{11}^{11}$.*

Proof. Assume M has flat tensor P, then $P_{ijkl} = 0$ for all $i, j, k, l = 0, 1, 2$. Then from Corollary 1, we obtain $r_{00} = r_{12}$ and $r_{10} = r_{11} = 0$. According to the equation (3), we get:

$$-2|F_{11}|^2 = -2F_{11}\, F^{11} = r_{00} = r_{12} = A_{11}^{11}.$$

So, Definition 5 and equation (2) imply that M is Einstein manifold with $\lambda = -2|F_{11}|^2 = A_{11}^{11}$, because $r_{ij} = \lambda g_{ij}$ and if we put $i = j = 0$ in the previous identity, then we have $r_{00} = \lambda g_{00} = \lambda$. Since M is 3-dimensional C_9-manifold, then according to the proof of Corollary 1, we obtain the following:

$$\lambda = r_{00} = -2F_{11}F^{11} = -2F_{11}\overline{F_{11}} = |F_{11}|^2.$$

Also, if we put $i = 1$ and $j = \hat{1} = 2$, then $r_{12} = \lambda g_{12} = \lambda \delta_1^1 = \lambda$. So, $\lambda = r_{12} = A_{11}^{11}$. Hence, $\lambda = -2|F_{11}|^2 = A_{11}^{11}$.

Conversely, let M be Einstein manifold with $\lambda = -2|F_{11}|^2 = A_{11}^{11}$. Then we deduce from equation (3) that $r_{00} = r_{12}$ and $r_{01} = r_{11} = 0$. So, $r_{20} = r_{02} = \overline{r_{01}} = 0$. Then Corollary 1 gives $P_{ijkl} = 0$, for all $i, j, k, l = 0, 1, 2$. Hence P is flat.

Theorem 6. *If a Riemannian manifold M of dimension $2n+1$ satisfies the following property:*

$$P_{ijkl} + P_{kijl} + P_{jkil} = 0,$$

then M is an Einstein manifold.

Proof. Let $P_{ijkl} + P_{kijl} + P_{jkil} = 0$. Then from the fact $R_{ijkl} + R_{kijl} + R_{jkil} = 0$ and equation (4), we obtain:

$$-\frac{1}{2n}\{g_{ik}\, r_{jl} - g_{il}\, r_{jk} + g_{kj}\, r_{il} - g_{kl}\, r_{ij} + g_{ji}\, r_{kl} - g_{jl}\, r_{ki}\} = 0.$$

Thus,

$$\{g_{ik}\, r_{jl} - g_{il}\, r_{jk} + g_{kj}\, r_{il} - g_{kl}\, r_{ij} + g_{ji}\, r_{kl} - g_{jl}\, r_{ki}\} g^{ik} = 0.$$

According to the following identities [19]:

$$\begin{array}{lll} g_{ik}\, g^{ik}\, r_{jl} = (2n+1)\, r_{jl}; & g_{kj}\, g^{ik}\, r_{il} = \delta^i_j\, r_{il}; & g_{ji}\, g^{ik}\, r_{kl} = \delta^k_j\, r_{kl} \\ g_{il}\, g^{ik}\, r_{jk} = \delta^k_l\, r_{jk}; & g_{kl}\, g^{ik}\, r_{ij} = \delta^i_l\, r_{ij}; & g_{jl}\, g^{ik}\, r_{ki} = \gamma\, g_{jl}, \end{array}$$

where γ is a scalar curvature of the Riemannian manifold M, the above identity turns into the following:

$$(2n+1)\, r_{jl} - \gamma\, g_{jl} = 0 \Rightarrow r_{jl} = \frac{\gamma}{2n+1}\, g_{jl}.$$

Hence M is Einstein manifold with $\lambda = \dfrac{\gamma}{2n+1}$.

Theorem 7. *If M is C_9-manifold with the property above in Theorem 6, then $A^{ac}_{ac} = -2nF_{ac}F^{ca}$.*

Proof. From Theorem 6, M is an Einstein manifold with $\lambda = \dfrac{\gamma}{2n+1}$.
Then $\gamma\, g_{ij} = (2n+1)\, r_{ij}$ for all $i,j,k,l = 0,1,...,2n$.
If we make $i = j = 0$ once and $i = a$ & $j = \hat{b}$ again in the last equation, then equation (3) implies that:

$$\gamma = -2(2n+1)F_{ab}F^{ba}, \tag{11}$$

$$\gamma\delta^b_a = (2n+1)A^{bc}_{ac}. \tag{12}$$

Thus, the contracting of (a,b) in equation (12) gives the following:

$$\gamma = \frac{2n+1}{n} A^{ac}_{ac}. \tag{13}$$

Making use of the equations (11) and (13) produced $A^{ac}_{ac} = -2nF_{ac}F^{ca}$.

4 Projective Invariant Classes

Consider the following collections of functions:

$$\begin{aligned} &P_1 = \{P_{000d}, P_{000\hat{d}}\}; P_2 = \{P_{0b0d}, P_{0\hat{b}0\hat{d}}\}; P_3 = \{P_{0b0\hat{d}}, P_{0\hat{b}0d}\}; P_4 = \{P_{0bc\hat{d}}, P_{0\hat{b}\hat{c}d}\}; \\ &P_5 = \{P_{a00d}, P_{\hat{a}00\hat{d}}\}; P_6 = \{P_{a00\hat{d}}, P_{\hat{a}00d}\}; P_7 = \{P_{a0c\hat{d}}, P_{\hat{a}0\hat{c}d}\}; P_8 = \{P_{a0\hat{c}\hat{d}}, P_{\hat{a}0cd}\}; \\ &P_9 = \{P_{ab0\hat{d}}, P_{\hat{a}\hat{b}0d}\}; P_{10} = \{P_{abcd}, P_{\hat{a}\hat{b}\hat{c}\hat{d}}\}; P_{11} = \{P_{abc\hat{d}}, P_{\hat{a}\hat{b}\hat{c}d}\}; P_{12} = \{P_{ab\hat{c}\hat{d}}, P_{\hat{a}\hat{b}cd}\}; \\ &P_{13} = \{P_{a\hat{b}0\hat{d}}, P_{\hat{a}b0d}\}; P_{14} = \{P_{a\hat{b}c\hat{d}}, P_{\hat{a}b\hat{c}d}\}; P_{15} = \{P_{a\hat{b}\hat{c}\hat{d}}, P_{\hat{a}bcd}\}, \end{aligned}$$

which are defined tensors on the $AC-$manifold M. Such tensors are called the *basic projective invariants* of M, where P_1, P_5, P_6 are of type $(1,1)$ and $P_2, P_3, P_7, P_8, P_9, P_{13}$ are of type $(2,1)$. Whereas, $P_4, P_{10}, P_{11}, P_{12}, P_{14}, P_{15}$ are of type $(3,1)$.

Definition 6. *An $AC-$manifold is named a manifold of class P_α, if $P_\alpha = 0$, where $\alpha = 1,2,\ldots,15$.*

Lemma 1. *If Φ is the $(1,1)-$tensor of an $AC-$manifold M, $\Pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$ and $\overline{\Pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$, then $\{\Pi(X)\}^i = X^a$ and $\{\overline{\Pi}(X)\}^i = X^{\hat{a}}$, $\forall\, X \in \mathscr{X}(M)$ and $i = 0, a, \hat{a}$.*

Proof. From equation (1), we have:

$$\begin{aligned}\{\Pi(X)\}^i &= -\frac{1}{2}\{(\Phi^2+\sqrt{-1}\Phi)X\}^i = -\frac{1}{2}\{\Phi^i_j\Phi^j_k+\sqrt{-1}\Phi^i_k\}X^k; \quad k=0,a,\hat{a},\\ &= 0-\frac{1}{2}\{(\sqrt{-1})^2+(\sqrt{-1})^2\}X^a-\frac{1}{2}\{(-\sqrt{-1})^2-(\sqrt{-1})^2\}X^{\hat{a}}\\ &= X^a.\end{aligned}$$

In the same way, we can prove that $\{\overline{\Pi}(X)\}^i = X^{\hat{a}}$.

Theorem 8. *The basic invariant tensors P_α on $AC-$manifold M with $AC-$structure $(\eta,\xi,\ \Phi,g)$ are determined for $\alpha=1,2,\ldots,15$ by the following formulas:*

$$\begin{aligned}
P_1(X) &= P(\xi,X)\xi+\Phi^2\circ P(\xi,X)\xi;\\
P_2(X,Y) &= P(\xi,\Pi(X))\Pi(Y)+\Phi^2\circ P(\xi,\Pi(X))\Pi(Y)+P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y)+\Phi^2\circ P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y);\\
P_3(X,Y) &= P(\xi,\overline{\Pi}(X))\Pi(Y)+\Phi^2\circ P(\xi,\overline{\Pi}(X))\Pi(Y)+P(\xi,\Pi(X))\overline{\Pi}(Y)+\Phi^2\circ P(\xi,\Pi(X))\overline{\Pi}(Y);\\
P_4(X,Y)Z &= P(\Pi(X),\overline{\Pi}(Y))\Pi(Z)+\Phi^2\circ P(\Pi(X),\overline{\Pi}(Y))\Pi(Z)+P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z)+\Phi^2\circ P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z);\\
P_5(X) &= \frac{1}{2}\{\Phi\circ P(\xi,\Phi X)\xi-\Phi^2\circ P(\xi,X)\xi\};\\
P_6(X) &= \frac{1}{2}\{\Phi^2\circ P(X,\xi)\xi+\Phi\circ P(\Phi X,\xi)\xi\};\\
P_7(X,Y) &= \overline{\Pi}\circ P(\Pi(X),\overline{\Pi}(Y))\xi+\Pi\circ P(\overline{\Pi}(X),\Pi(Y))\xi;\\
P_8(X,Y) &= \overline{\Pi}\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\xi+\Pi\circ P(\Pi(X),\Pi(Y))\xi;\\
P_9(X,Y) &= \overline{\Pi}\circ P(\xi,\overline{\Pi}(Y))\Pi(X)+\Pi\circ P(\xi,\Pi(Y))\overline{\Pi}(X);\\
P_{10}(X,Y)Z &= \overline{\Pi}\circ P(\Pi(X),\Pi(Y))\Pi(Z)+\Pi\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z);\\
P_{11}(X,Y)Z &= \overline{\Pi}\circ P(\Pi(X),\overline{\Pi}(Y))\Pi(Z)+\Pi\circ P(\overline{\Pi}(X),\Pi(Y))\overline{\Pi}(Z);\\
P_{12}(X,Y)Z &= \overline{\Pi}\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\Pi(Z)+\Pi\circ P(\Pi(X),\Pi(Y))\overline{\Pi}(Z);\\
P_{13}(X,Y) &= \overline{\Pi}\circ P(\xi,\overline{\Pi}(X))\overline{\Pi}(Y)+\Pi\circ P(\xi,\Pi(X))\Pi(Y);\\
P_{14}(X,Y)Z &= \overline{\Pi}\circ P(\Pi(X),\overline{\Pi}(Y))\overline{\Pi}(Z)+\Pi\circ P(\overline{\Pi}(X),\Pi(Y))\Pi(Z);\\
P_{15}(X,Y)Z &= \overline{\Pi}\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)+\Pi\circ P(\Pi(X),\Pi(Y))\Pi(Z),
\end{aligned}$$

for all $X,Y,Z\in\mathscr{X}(M)$.

Proof. By using the facts $\xi^a=0=\xi^{\hat{a}},\xi^0=1$ and equation (1), we obtain the following:

$$\begin{aligned}
\{P(\xi,X)\xi\}^i &= P^i_{jkl}\xi^j\xi^kX^l = P_{\hat{i}jkl}\xi^j\xi^kX^l,\\
&= P_{000d}X^d+P_{000\hat{d}}X^{\hat{d}}+P_{a00d}X^d+P_{\hat{a}00\hat{d}}X^{\hat{d}}+P_{a00\hat{d}}X^{\hat{d}}+P_{\hat{a}00d}X^d.\\
\{\Phi^2\circ P(\xi,X)\xi\}^i &= \Phi^i_t\Phi^t_sP^s_{jkl}\xi^j\xi^kX^l = \Phi^i_t\Phi^t_sP_{\hat{s}00l}\,X^l, \qquad s,t=0,1,2,\ldots,2n\\
&= -\{P_{a00d}X^d+P_{\hat{a}00\hat{d}}X^{\hat{d}}+P_{a00\hat{d}}X^{\hat{d}}+P_{\hat{a}00d}X^d\}.
\end{aligned}$$

Then $\{P(\xi,X)\xi+\Phi^2\circ P(\xi,X)\xi\}^i = P_{000d}X^d+P_{000\hat{d}}X^{\hat{d}}$. Hence, the functions $\{P_{000d},P_{000\hat{d}}\}$ are components of the tensor $\{P(\xi,X)\xi+\Phi^2\circ P(\xi,X)\xi\}$, and therefore this tensor coincides with the tensor $P_1(X)$. So, the first equality is satisfied.

Now, by using Lemma 1 and equation (1), we have the following:

$$\begin{aligned}
\{P(\xi,\overline{\Pi}(X))\Pi(Y)\}^i &= P^i_{jkl}\{\Pi(Y)\}^j\xi^k\{\overline{\Pi}(X)\}^l,\\
&= P_{0b0\hat{d}}Y^bX^{\hat{d}} + P_{ab0\hat{d}}Y^bX^{\hat{d}} + P_{\hat{a}b0\hat{d}}Y^bX^{\hat{d}}.\\
\{\Phi^2\circ P(\xi,\overline{\Pi}(X))\Pi(Y)\}^i &= \Phi^i_t\Phi^t_sP^s_{jkl}\{\Pi(Y)\}^j\xi^k\{\overline{\Pi}(X)\}^l,\\
&= -\{P_{ab0\hat{d}}Y^bX^{\hat{d}} + P_{\hat{a}b0\hat{d}}Y^bX^{\hat{d}}\}.\\
\{P(\xi,\Pi(X))\overline{\Pi}(Y)\}^i &= P^i_{jkl}\{\overline{\Pi}(Y)\}^j\xi^k\{\Pi(X)\}^l,\\
&= P_{0\hat{b}0d}Y^{\hat{b}}X^d + P_{a\hat{b}0d}Y^{\hat{b}}X^d + P_{\hat{a}\hat{b}0d}Y^{\hat{b}}X^d.\\
\{\Phi^2\circ P(\xi,\Pi(X))\overline{\Pi}(Y)\}^i &= \Phi^i_t\Phi^t_sP^s_{jkl}\{\overline{\Pi}(Y)\}^j\xi^k\{\Pi(X)\}^l,\\
&= -\{P_{a\hat{b}0d}Y^{\hat{b}}X^d + P_{\hat{a}\hat{b}0d}Y^{\hat{b}}X^d\}.
\end{aligned}$$

Then

$$\begin{aligned}
P_{0b0\hat{d}}Y^bX^{\hat{d}} + P_{0\hat{b}0d}Y^{\hat{b}}X^d = \{&P(\xi,\overline{\Pi}(X))\Pi(Y) + \Phi^2\circ P(\xi,\overline{\Pi}(X))\Pi(Y)\\
&+ P(\xi,\Pi(X))\overline{\Pi}(Y) + \Phi^2\circ P(\xi,\Pi(X))\overline{\Pi}(Y)\}^i.
\end{aligned}$$

Hence, the functions $\{P_{0b0\hat{d}}, P_{0\hat{b}0d}\}$ are the components of the following tensor:

$$P(\xi,\overline{\Pi}(X))\Pi(Y) + \Phi^2\circ P(\xi,\overline{\Pi}(X))\Pi(Y) + P(\xi,\Pi(X))\overline{\Pi}(Y) + \Phi^2\circ P(\xi,\Pi(X))\overline{\Pi}(Y).$$

Therefore, this tensor coincides with the tensor $P_3(X,Y)$. Thus, the third equality is satisfied.
At this moment, when we prove the fifth equality for P_5, we need to do the following:

$$\begin{aligned}
\{\Phi\circ P(\xi,\Phi X)\xi\}^i &= \Phi^i_tP^t_{jkl}\xi^j\xi^k\Phi^l_sX^s,\\
&= P_{a00d}X^d + P_{\hat{a}00\hat{d}}X^{\hat{d}} - P_{a00\hat{d}}X^{\hat{d}} - P_{\hat{a}00d}X^d.\\
\{\Phi^2\circ P(\xi,X)\xi\}^i &= \Phi^i_t\Phi^t_sP^s_{jkl}\xi^j\xi^kX^l,\\
&= -P_{a00d}X^d - P_{\hat{a}00\hat{d}}X^{\hat{d}} - P_{a00\hat{d}}X^{\hat{d}} - P_{\hat{a}00d}X^d.
\end{aligned}$$

Then $\frac{1}{2}\{\Phi\circ P(\xi,\Phi X)\xi - \Phi^2\circ P(\xi,X)\xi\}^i = P_{a00d}X^d + P_{\hat{a}00\hat{d}}X^{\hat{d}}$. Hence, the functions $\{P_{a00d}, P_{\hat{a}00\hat{d}}\}$ are the components of the tensor $\frac{1}{2}\{\Phi\circ P(\xi,\Phi X)\xi - \Phi^2\circ P(\xi,X)\xi\}$, and this tensor coincides with the tensor $P_5(X)$. Therefore, the fifth equality is satisfied.
The proof of P_{10} based on the following:

$$\begin{aligned}
\{\overline{\Pi}\circ P(\Pi(X),\Pi(Y))\Pi(Z)\}^i &= \{\overline{\Pi}(P(\Pi(X),\Pi(Y))\Pi(Z))\}^i,\\
&= \{P(\Pi(X),\Pi(Y))\Pi(Z)\}^{\hat{a}},\\
&= P^{\hat{a}}_{jkl}\{\Pi(X)\}^k\{\Pi(Y)\}^l\{\Pi(Z)\}^j,\\
&= P_{abcd}X^cY^dZ^b.\\
\{\Pi\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^i &= \{\Pi(P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z))\}^i,\\
&= \{P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^a,\\
&= P^a_{jkl}\{\overline{\Pi}(X)\}^k\{\overline{\Pi}(Y)\}^l\{\overline{\Pi}(Z)\}^j,\\
&= P_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{c}}Y^{\hat{d}}Z^{\hat{b}}.
\end{aligned}$$

Then

$$\{\overline{\Pi}\circ P(\Pi(X),\Pi(Y))\Pi(Z) + \Pi\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\overline{\Pi}(Z)\}^i = P_{abcd}X^cY^dZ^b + P_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{c}}Y^{\hat{d}}Z^{\hat{b}}.$$

Hence, the functions $\{P_{abcd}, P_{\hat{a}\hat{b}\hat{c}\hat{d}}\}$ are the components of the following tensor:

$$\overline{\Pi} \circ P(\Pi(X), \Pi(Y))\Pi(Z) + \Pi \circ P(\overline{\Pi}(X), \overline{\Pi}(Y))\overline{\Pi}(Z),$$

and this coincides with the tensor $P_{10}(X,Y)Z$. To prove P_{14}, we do the following:
$\{\overline{\Pi} \circ P(\Pi(X), \overline{\Pi}(Y))\overline{\Pi}(Z)\}^i = P_{a\hat{b}c\hat{d}} X^c Y^{\hat{d}} Z^{\hat{b}}$, and $\{\Pi \circ P(\overline{\Pi}(X), \Pi(Y))\Pi(Z)\}^i = P_{\hat{a}b\hat{c}d} X^{\hat{c}} Y^d Z^b$. Then

$$\{\overline{\Pi} \circ P(\Pi(X), \overline{\Pi}(Y))\overline{\Pi}(Z) + \Pi \circ P(\overline{\Pi}(X), \Pi(Y))\Pi(Z)\}^i = P_{a\hat{b}c\hat{d}} X^c Y^{\hat{d}} Z^{\hat{b}} + P_{\hat{a}b\hat{c}d} X^{\hat{c}} Y^d Z^b.$$

Hence, the functions $\{P_{a\hat{b}c\hat{d}}, P_{\hat{a}b\hat{c}d}\}$ are the components of the following tensor:

$$\overline{\Pi} \circ P(\Pi(X), \overline{\Pi}(Y))\overline{\Pi}(Z) + \Pi \circ P(\overline{\Pi}(X), \Pi(Y))\Pi(Z),$$

and this coincides with the tensor $P_{14}(X,Y)Z$. The rest equalities in the present theorem can prove in the same manner that presented above.

Corollary 3.*Some of the basic invariant tensors in Theorem 8 can be written as follow:*

$$\begin{aligned}
P_2(\mathscr{V}_1,\mathscr{V}_2) &= \frac{1}{2}\Phi^2 \circ \{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2 - P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\} + \frac{1}{2}\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2 - P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\};\\
P_3(\mathscr{V}_1,\mathscr{V}_2) &= \frac{1}{2}\Phi^2 \circ \{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2 + P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\} + \frac{1}{2}\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2 + P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\};\\
P_4(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \frac{-1}{4}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&\quad - \frac{1}{4}\{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\};\\
P_7(\mathscr{V}_1,\mathscr{V}_2) &= \frac{-1}{4}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\xi + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\xi\} + \frac{1}{4}\Phi \circ \{P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\xi - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\xi\};\\
P_8(\mathscr{V}_1,\mathscr{V}_2) &= \frac{-1}{4}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\xi - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\xi\} + \frac{1}{4}\Phi \circ \{P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\xi + P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\xi\};\\
P_9(\mathscr{V}_1,\mathscr{V}_2) &= \frac{-1}{4}\Phi^2 \circ \{P(\xi,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_1 + P(\xi,\Phi\mathscr{V}_2)\Phi\mathscr{V}_1\} + \frac{1}{4}\Phi \circ \{P(\xi,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_1 - P(\xi,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_1\};\\
P_{10}(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \frac{1}{8}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&\quad + \frac{1}{8}\Phi \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\};\\
P_{11}(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \frac{1}{8}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&\quad + \frac{1}{8}\Phi \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\};\\
P_{12}(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \frac{1}{8}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&\quad + \frac{1}{8}\Phi \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\};\\
P_{13}(\mathscr{V}_1,\mathscr{V}_2) &= \frac{-1}{4}\Phi^2 \circ \{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2 - P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\} + \frac{1}{4}\Phi \circ \{P(\xi,\Phi^2\mathscr{V}_1)\Phi\mathscr{V}_2 + P(\xi,\Phi\mathscr{V}_1)\Phi^2\mathscr{V}_2\};\\
P_{14}(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \frac{1}{8}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&\quad + \frac{1}{8}\Phi \circ \{-P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\};\\
P_{15}(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \frac{1}{8}\Phi^2 \circ \{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&\quad + \frac{1}{8}\Phi \circ \{-P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3 - P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3 - P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3 + P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\},
\end{aligned}$$

where $\mathscr{V}_1,\mathscr{V}_2,\mathscr{V}_3 \in \mathscr{X}(M)$.

*Proof.*We shall prove only P_2 and P_{15}, while the others can be proven in the same way. So, according to the definitions of Π and $\overline{\Pi}$ in Lemma 1, P_2 appears as:

$$\begin{aligned}
P(\xi,\Pi(\mathscr{V}_1))\Pi(\mathscr{V}_2) &= P(\xi,\frac{-1}{2}\Phi^2\mathscr{V}_1+\frac{-\sqrt{-1}}{2}\Phi\mathscr{V}_1)(\frac{-1}{2}\Phi^2\mathscr{V}_2+\frac{-\sqrt{-1}}{2}\Phi\mathscr{V}_2),\\
&= \frac{1}{4}\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2+\sqrt{-1}P(\xi,\Phi^2\mathscr{V}_1)\Phi\mathscr{V}_2+\sqrt{-1}P(\xi,\Phi\mathscr{V}_1)\Phi^2\mathscr{V}_2-P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\}.\\
\Phi^2\circ P(\xi,\Pi(\mathscr{V}_1))\Pi(\mathscr{V}_2) &= \frac{1}{4}\Phi^2\circ\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2+\sqrt{-1}P(\xi,\Phi^2\mathscr{V}_1)\Phi\mathscr{V}_2+\sqrt{-1}P(\xi,\Phi\mathscr{V}_1)\Phi^2\mathscr{V}_2\\
&\quad -P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\}.\\
P(\xi,\overline{\Pi}(\mathscr{V}_1))\overline{\Pi}(\mathscr{V}_2) &= P(\xi,\frac{-1}{2}\Phi^2\mathscr{V}_1+\frac{\sqrt{-1}}{2}\Phi\mathscr{V}_1)(\frac{-1}{2}\Phi^2\mathscr{V}_2+\frac{\sqrt{-1}}{2}\Phi\mathscr{V}_2),\\
&= \frac{1}{4}\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2-\sqrt{-1}P(\xi,\Phi^2\mathscr{V}_1)\Phi\mathscr{V}_2-\sqrt{-1}P(\xi,\Phi\mathscr{V}_1)\Phi^2\mathscr{V}_2-P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\}.\\
\Phi^2\circ P(\xi,\overline{\Pi}(\mathscr{V}_1))\overline{\Pi}(\mathscr{V}_2) &= \Phi^2\circ P(\xi,\frac{-1}{2}\Phi^2\mathscr{V}_1+\frac{1}{2}\Phi\mathscr{V}_1)(\frac{-1}{2}\Phi^2\mathscr{V}_2+\frac{1}{2}\Phi\mathscr{V}_2),\\
&= \frac{1}{4}\Phi^2\circ\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2-\sqrt{-1}P(\xi,\Phi^2\mathscr{V}_1)\Phi\mathscr{V}_2-\sqrt{-1}P(\xi,\Phi\mathscr{V}_1)\Phi^2\mathscr{V}_2\\
&\quad -P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\}.
\end{aligned}$$

Then

$$\begin{aligned}
P_2(\mathscr{V}_1,\mathscr{V}_2) &= P(\xi,\Pi(\mathscr{V}_1))\Pi(\mathscr{V}_2)+\Phi^2\circ P(\xi,\Pi(\mathscr{V}_1))\Pi(\mathscr{V}_2)+P(\xi,\overline{\Pi}(\mathscr{V}_1))\overline{\Pi}(\mathscr{V}_2)+\Phi^2\circ P(\xi,\overline{\Pi}(\mathscr{V}_1))\overline{\Pi}(\mathscr{V}_2),\\
&= \frac{1}{2}\Phi^2\circ\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2-P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\}+\frac{1}{2}\{P(\xi,\Phi^2\mathscr{V}_1)\Phi^2\mathscr{V}_2-P(\xi,\Phi\mathscr{V}_1)\Phi\mathscr{V}_2\}.
\end{aligned}$$

Now, P_{15} appears as:

$$\begin{aligned}
\overline{\Pi}\circ P(\overline{\Pi}(\mathscr{V}_1),\overline{\Pi}(\mathscr{V}_2))\overline{\Pi}(\mathscr{V}_3) &= \{\frac{-1}{2}\Phi^2+\frac{\sqrt{-1}}{2}\Phi\}\circ P(\frac{-1}{2}\Phi^2\mathscr{V}_1+\frac{\sqrt{-1}}{2}\Phi\mathscr{V}_1,\frac{-1}{2}\Phi^2\mathscr{V}_2\\
&\quad +\frac{\sqrt{-1}}{2}\Phi\mathscr{V}_2)(\frac{-1}{2}\Phi^2\mathscr{V}_3+\frac{\sqrt{-1}}{2}\Phi\mathscr{V}_3),\\
&= \frac{1}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3-\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&\quad -\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3-\frac{1}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3\\
&\quad -\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3-\frac{1}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&\quad -\frac{1}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3+\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\\
&\quad -\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3-\frac{1}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&\quad -\frac{1}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3+\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3\\
&\quad -\frac{1}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3+\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&\quad +\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3+\frac{1}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3.
\end{aligned}$$

$$\begin{aligned}
\Pi\circ P(\Pi(\mathscr{V}_1),\Pi(\mathscr{V}_2))\Pi(\mathscr{V}_3) &= \{\frac{-1}{2}\Phi^2+\frac{-\sqrt{-1}}{2}\Phi\}\circ P(\frac{-1}{2}\Phi^2\mathscr{V}_1+\frac{-\sqrt{-1}}{2}\Phi\mathscr{V}_1,\frac{-1}{2}\Phi^2\mathscr{V}_2\\
&+\frac{-\sqrt{-1}}{2}\Phi\mathscr{V}_2)(\frac{-1}{2}\Phi^2\mathscr{V}_3+\frac{-\sqrt{-1}}{2}\Phi\mathscr{V}_3),\\
&=\frac{1}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3+\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&+\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3-\frac{1}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3\\
&+\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3-\frac{1}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&-\frac{1}{16}\Phi^2\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3-\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\\
&+\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3-\frac{1}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&-\frac{1}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3-\frac{-\sqrt{1}}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3\\
&-\frac{1}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3-\frac{\sqrt{-1}}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&-\frac{\sqrt{-1}}{16}\Phi^2\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3+\frac{1}{16}\Phi\circ P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3.
\end{aligned}$$

So,

$$\begin{aligned}
P_{15}(\mathscr{V}_1,\mathscr{V}_2)\mathscr{V}_3 &= \overline{\Pi}\circ P(\overline{\Pi}(\mathscr{V}_1),\overline{\Pi}(\mathscr{V}_2))\overline{\Pi}(\mathscr{V}_3)+\Pi\circ P(\Pi(\mathscr{V}_1),\Pi(\mathscr{V}_2))\Pi(\mathscr{V}_3),\\
&=\frac{1}{8}\Phi^2\circ\{P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3-P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\\
&-P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3-P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\}\\
&+\frac{1}{8}\Phi\circ\{-P(\Phi^2\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi\mathscr{V}_3-P(\Phi^2\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi^2\mathscr{V}_3\\
&-P(\Phi\mathscr{V}_1,\Phi^2\mathscr{V}_2)\Phi^2\mathscr{V}_3+P(\Phi\mathscr{V}_1,\Phi\mathscr{V}_2)\Phi\mathscr{V}_3\}.
\end{aligned}$$

Corollary 4.*The identity $P_\alpha(X,Y)Z=-P_\alpha(Y,X)Z$ satisfies for $\alpha=4,10,11,12,14,15$. Moreover, the identity $P_\alpha(X,Y)=-P_\alpha(Y,X)$ satisfies for $\alpha=7,8$, where $X,Y,Z\in\mathscr{X}(M)$.*

Proof.

$$\begin{aligned}
P_8(X,Y) &= \overline{\Pi}\circ P(\overline{\Pi}(X),\overline{\Pi}(Y))\xi+\Pi\circ P(\Pi(X),\Pi(Y))\xi,\\
&=-\overline{\Pi}\circ P(\overline{\Pi}(Y),\overline{\Pi}(X))\xi-\Pi\circ P(\Pi(Y),\Pi(X))\xi,\\
&=-\{\overline{\Pi}\circ P(\overline{\Pi}(Y),\overline{\Pi}(X))\xi+\Pi\circ P(\Pi(Y),\Pi(X))\xi\},\\
&=-P_8(Y,X).
\end{aligned}$$

In the same way, we can prove the rest identities of the invariant tensors P_α and $\alpha=4,7,10,11,12,14,15$.

Theorem 9.*Let M be a C_9-manifold. If M is a manifold of classes P_1 and P_{11}, then M is an Einstein manifold with $\lambda=n^{-1}A^{ac}_{ac}=-2F_{ac}F^{ac}$.*

*Proof.*Suppose that M is fall in both classes P_1 and P_{11}. Then Definition 6 and Theorem 2 imply that $r_{0a}=0$ & $r_{ab}=0$. Moreover, Definition 5 and equation (3) produced $\lambda=n^{-1}A^{ac}_{ac}=-2F_{ac}F^{ac}$. Then the result attains.

Theorem 10.*The C_9-manifold belongs to class P_1 iff it belongs to class P_9.*

*Proof.*Suppose that M is a C_9-manifold. If M of class P_1, then Theorem 2 gives $-\frac{1}{2n}r_{0d}=0$. Hence $r_{0d}=0$. Now, since $r_{0d}=r_{d0}$ and $b,d\in\{1,2,...,n\}$ then $r_{b0}=0$. But Theorem 2 also shown that $P_{ab0\hat{d}}=\frac{1}{2n}\{\delta^d_a r_{b0}\}$. Hence $P_{ab0\hat{d}}=0=P_{\hat{a}\hat{b}0d}$. So, $P_9=0$, and this means that M is a C_9-manifold of class P_9.

Conversely, we can show that if M is a C_9-manifold of class P_9, then M of class P_1 by the same way that follower above.

Theorem 11. *Let M be a C_9-manifold of dimension equal or greater than 5. Then*

1. the following statements are equivalent:
(a) M of class P_1;
(b) M of class P_8;
(c) M of class P_9.
2. the following statements are equivalent:
(a) M of class P_2;
(b) M of class P_5;
(c) M of class P_{11};
(d) M of class P_{15}.

Proof. Let M be a C_9-manifold of dimension $2n+1$ with $n \geq 2$. We shall prove $1.(b) \Rightarrow (c)$ and $2.(a) \Rightarrow (d)$. The rest cases can be proved in similar way.
For $1.(b) \Rightarrow (c)$, assume M of class P_8, then from Definition 6 and Theorem 2, we get $\delta_a^c r_{0\hat{d}} - \delta_a^d r_{0\hat{c}} = 0$. So, by contracting (a, c), we obtain either $n = 1$, or $r_{0\hat{d}} = 0$. The case $n = 1$ contradicts our assumption $n \geq 2$. Then $r_{0\hat{d}} = 0$ and this gives $r_{b0} = 0$. Thus, $P_{ab0\hat{d}} = 0$, and according to Definition 6, we have M of class P_9.
Now, for $2.(a) \Rightarrow (d)$, suppose M of class P_2. Then Definition 6 and Theorem 2 imply that $r_{bd} = 0$. So, from Theorem 2, we get $P_{a\hat{b}\hat{c}\hat{d}} = 0$. Hence, Definition 6 gives M of class P_{15}.

Declarations

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