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Finite Groups with Specific Subgroups Exhibiting Trivial Intersection Property

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Abstract: This paper studies finite groups in which every Sylow p-subgroup has a trivial intersection with its conjugates, forming the class of TIP-groups. While classical conditions, such as the normality of Sylow p-subgroups, are equivalent to the group being nilpotent, the trivial intersection property defines a broader and more flexible class. We provide structural characterizations of TIP-groups and examine their relationship with well-known classes of finite groups. In particular, we relate TIP-groups to the p-cores by introducing the β-condition and offer examples to illustrate their distinguishing features.

Keywords: TIP-subgroups, Sylow subgroups, Nilpotent group, Solvable group.

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1 Introduction

The study of Sylow p-subgroups has been central in group theory, and considerable attention has been given to their structural and intersection properties. One aspect of this involves groups for which certain subgroups have trivial intersection properties. This line of research originates from the foundational work conducted by Suzuki, who considered finite groups G such that for any p-group P, there exists $g \in G$ such that $P \cap P^g = \{e\}$, where $P^g = g^{-1}Pg$ is the conjugate of P by g. Suzuki in [6] showed that such groups G have the structure of semidirect products of P with sufficiently large faithful modules V over finite fields of characteristic different from p. For these groups, Suzuki studied the union of centralizers $C_V(x)$ and showed that if V is large enough, $P \cap P^v = \{e\}$ for some $v \in V$. Suzuki classified these groups as quaternion groups, elementary abelian groups, or cyclic groups for $p = 2$.

Building on these ideas, researchers have studied several generalizations of the described properties of groups, among which are those having certain subgroups with a trivial intersection with their conjugates. For example, Walls investigated groups in which every subgroup is a TI-subgroup [8]. Similarly, Guo, Li, and Flavell described groups in which all abelian subgroups are TI-subgroups [2]. In [4], Shi and Zhang showed that a group G has the property that all of its non-abelian subgroups are TI-subgroups if and only if those subgroups are normal in G. In a further generalization (see [5]), they also showed that all non-nilpotent subgroups being TI-subgroups is equivalent to all non-nilpotent subgroups being normal in G. Related studies on subgroup embedding properties and subgroup normality, such as weakly C-normal and Cs-normal subgroups, have been explored in [7], further enriching the classification framework for finite groups.

The structural behavior of TIP-groups, especially in relation to subgroup lattices and abelian group configurations, finds context in the analysis of characteristic subgroup lattices of finite abelian p-groups as examined in [3].

In this paper, we introduce a new class of groups called *TIP-groups*, which generalize the concept of trivial intersection for Sylow p-subgroups. A group G is defined to be a TIP-group if, for each prime p dividing the order of G, and for each P in the set of all Sylow p-subgroups of G, the intersection of P with any conjugate P^g is either trivial $\{e\}$ or the entire subgroup P, for all $g \in G$. The study of TIP-groups is closely related to the study of TI-subgroups, as explored in [4,5,8].

Our main goal is to study different algebraic properties of TIP-groups, with particular focus on their relationships to some well-known classes of finite groups like nilpotent and solvable groups. We examine their behavior under subgroup

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operations, homomorphic images, and direct products. Specifically, we explore whether the class of TIP-groups is closed under these operations and discuss the implications of our findings for broader group theory. For *TIP-groups*, the following result is established, with its proof provided in the subsequent section.

Relevance to JJMS. This work aligns with the scope of the *Jordan Journal of Mathematical and Statistical Sciences*, particularly its emphasis on algebraic structures and group theory. By introducing and analyzing TIP-groups, we contribute to the ongoing exploration of subgroup behavior and classification in finite group theory.

2 TIP-Groups and Their Structural Properties

In this section, we investigate the behavior of TIP-groups under several important algebraic operations. Our goal is to determine whether the class of TIP-groups is closed under certain group operations, such as taking subgroups, factor groups, and direct products. Studying these properties of TIP-groups provide an insights into their structure and how they relate to other well-known classes of finite groups, including nilpotent and solvable groups. For *TIP-groups*, the following result is established, with its proof provided in the subsequent section.

Theorem 1. *Every subgroup of a TIP-group is a TIP-group.*

This theorem demonstrates that the class of TIP groups is closed for subgroups. Another notable property of the class of TIP groups is given in the following result:

Theorem 2. *The class of TIP groups is closed under taking factor groups.*

The class of TIP groups is not always closed under direct products, even though that it is closed under homomorphic images and subgroups. However, by limiting the groups' orders, we can ensure that the direct product remains a TIP group. This result is formalized in the following theorem.

Theorem 3. *Let A and B be two TIP groups with relatively prime orders $|G| = a$ and $|H| = b$, where $\gcd(a,b) = 1$. Then the direct product $A \times B$ is a TIP group.*

The restriction in Theorem 3 on orders to relatively prime is essential and cannot be omitted, as demonstrated by the following example .

Example 1. The connection between the class of the TIP groups and other classes generally studied in finite groups is given in this paper. Let $\mathbb{Z}_2 = \langle a : a^2 = e \rangle$ be the group of order 2 and S_3 be the non-commutative group of order 6. Both of them are TIP groups, as is shown below:

(i) $\mathbb{Z}_2$ is a TIP group: Since, $\mathbb{Z}_2$ is abelian group. Hence, every subgroup of $\mathbb{Z}_2$ is normal in $\mathbb{Z}_2$, so $\mathbb{Z}_2$ is a TIP group.

(ii) S_3 is a TIP group: The group S_3 has order 6, with Sylow subgroups of orders 2 and 3. Each Sylow subgroup intersects trivially with other Sylow subgroups, satisfying the TIP condition. Thus, S_3 is a TIP group. Now, consider the direct product $G = S_3 \times \mathbb{Z}_2$, a group of order $|G| = 2 \cdot 6 = 12$. To show that G is not a TIP group, we show that its Sylow 2-subgroups are not TI-subgroups. For that it is enough to consider the folowing two Sylow 2-subgroups of order 4 of in the group G namely: $P = \langle (1,2) \rangle \times \mathbb{Z}_2$, and $P^g = \langle (1,3) \rangle \times \mathbb{Z}_2$, where $g = ((2,3),0)$. The intersection $P \cap P^g = \langle e \rangle \times \mathbb{Z}_2$. So, not all Sylow subgroups in G are TI-subgroups. This example shows that the TIP groups are not closed under direct products in general.

The connection between the class of the TIP groups and others, such as the ones generally studied in finite groups, is given in this paper. It is noted that, within finite groups, every abelian is nilpotent, every nilpotent is supersolvable, and every supersolvable is solvable. Every nilpotent group is certainly a TIP group. Conversely, a supersolvable group is not necessarily a TIP group (see Example 3). On the other hand, A_4 is an example of a TIP group that is not supersolvable. The example of A_4 indeed demonstrates that even a solvable TIP group need not be supersolvable. Finally, A_5 is an example of a TIP group that is not solvable (see Example 4), showing that a TIP group need not be solvable.

In what follows, we introduce a new condition referred to as the *β-condition*—based on the concept of the r-core. Recall that for a finite group A, the largest normal subgroup of A whose order is a power of a prime r is called the r-core, denoted by $O_r(A)$. Equivalently, $O_r(A)$ is the intersection of all Sylow r-subgroups of A. Using this, we formulate the following condition:

Definition 1. *A group A satisfies the condition β_p for all p in the set of all prime divisors of the group A if and only if $O_p(A) = \{e\}$ or $O_p(A) = P$, for all P from the set of all Sylow p-subgroups of A. The group A satisfies the β-condition if it satisfies β_p for each prime p that divides the order of the group A.*

For example, consider the group $A = S_3$, which satisfies the β-condition since $O_2(S_3) = \{e\}$ and $O_3(S_3) = A_3$. So G satisfies the β-condition.

Another notable result about TIP groups is presented in the following theorem:

Theorem 4. *If a group G is a TIP-group, then G satisfies the β-condition.*

Remark. The converse of Theorem 4 does not always hold, as the following example shows.

Example 2. Let S_3 be the symmetric group on the set $\{1,2,3\}$, which is clearly a TIP group. Let $A = S_3 \times S_3$, a non-abelian group of order $36 = 2^2 \times 3^2$. The group A has 9 subgroups of order 4, which are the 2-Sylow subgroups, consider $P_1 = \langle(1,2)\rangle \times \langle(1,2)\rangle$ and $P_2 = \langle(2,3)\rangle \times \langle(1,2)\rangle$ be two of these 2-Sylow subgroups then $P_1 \cap P_2 = \{(e,e),(e,(12))\} \neq \{(e,e)\}$, this indicates that P_1 in $\mathrm{Syl}_2(S_3 \times S_3)$ is not a TI-subgroup, thus $G = S_3 \times S_3$ does not belong to the TIP class. However, $G = S_3 \times S_3$ satisfies the β-condition where $O_2(S_3 \times S_3) = \{(e,e)\}$ and $O_3(S_3 \times S_3) = \mathbb{Z}_3 \times \mathbb{Z}_3$.

3 Proofs and Fundamental Properties

The basic findings about TIP groups, including their structural characteristics and behavior under operations such as direct products and homomorphic images, are established in this section. These results are demonstrated by important arguments and examples, emphasizing the unique features of TIP groups.

Lemma 1. *If every Sylow p-subgroup of a group G is normal, then the group is a TIP-group.*

Proof. Let G be a group where every Sylow p-subgroup is normal. This means that for each Sylow p-subgroup N of G, we have $N^g = N$ for all $g \in G$, where N^g denotes the conjugate of N by g.

Since normal subgroups are invariant under conjugation, each Sylow p-subgroup is a TI-subgroup, as conjugation does not affect it. Therefore, if every Sylow p-subgroup is normal, all these subgroups are TI-subgroups.

Consequently, the group G will have all of its Sylow p-subgroups as TI-subgroups, and thus, by definition, G is a TIP-group.

Proposition 1. *All groups that are nilpotent are TIP-groups.*

Proof. It is well known that in a nilpotent group, every Sylow p-subgroup is normal. By Proposition 1, this implies that each Sylow p-subgroup is a TI-subgroup. Therefore, nilpotent groups satisfy the definition of TIP-groups.

Proposition 2. *Suppose that P_1 and P_2 are two Sylow p-subgroups of the group A. Then P_1 and P_2 are both TI-subgroups if one of them is.*

Proof. Let P_1 be a TI-subgroup and P_2 be conjugate to P_1, but assume P_2 is not a TI-subgroup. Then $P_2 = P_1^g$ and there exists $x \in A - P_2$ such that $P_2^x \cap P_2 \neq \{e\}$, which means $(P_1^g)^x \cap P_1^g \neq \{e\}$. This leads to a contradiction with the assumption that P_1 is a TI-subgroup.

Proposition 3. *For each prime p that divides the order of A, if A is a group with the condition that P the Sylow p-subgroup of A is isomorphic to $\mathbb{Z}_p$, then P is a TI-subgroup. Consequently, A is a TIP-group.*

Proposition 4. *Every group of square-free order is a TIP-group.*

Lemma 2. *Let G be a finite group. If every Sylow p-subgroup of G is either normal or isomorphic to $\mathbb{Z}_p$, then G is a TIP-group.*

Proof. Let R be one of A's Sylow r-subgroups. Two cases will be investigated.:

- **Case 1:** R is normal in A. By Proposition 1, R is a TI-subgroup.
- **Case 2:** R is isomorphic to $\mathbb{Z}_r$, where r is a prime number, then R has prime order. By Proposition 3, R is a TI-subgroup.

In both cases, R is a TI-subgroup. Therefore, A itself is a TIP-group.

Corollary 1. *If A is a group of order $\alpha^n\beta$, where α and β are distinct primes such that $\alpha \nmid (\beta - 1)$, then A is a TIP-group..*

*Proof.*If $|A| = \alpha^n\beta$, where α and β are prime divisors of $|A|$, and α does not divide $(\beta - 1)$. Then from Sylow's Theorem, P the Sylow α-subgroup must be normal in A, since the number of Sylow α-subgroups must be 1. So, P is TI-subgroup of A. Let Q be a Sylow β-subgroup of A. Then $Q \simeq \mathbb{Z}_\beta$. From Lemma2 we have A is TIP-group.

*Remark.*A group of order $\alpha^n\beta$, where α and β are distinct primes and $n \geq 2$, may not be a TIP-group. To illustrate this, consider the dihedral group $G = D_{12}$ of order $12 = 2^2 \times 3$, which is not a TIP-group (See Example 3). The proof for Theorem 1 is presented next.

Proof.(Theorem 1) Let G be a TIP-group and H be a subgroup of G. If H is a p-group, then H is a TIP-group. Let p be the smallest prime dividing the order of H, and let P_H be a p-Sylow subgroup of H such that P_H is not a TI-subgroup. Suppose h is an element of H not contained in P_H such that $P_H \cap P_H^h \neq \{e\}$. Now, there exist a Sylow subgroups P_1 and P_2 from $\mathrm{Syl}_p(G)$, such that $P_1 \cap H = P_H$ and $P_2 \cap H = P_H^h$. That implies $P_1 \cap P_2 \geq P_H \cap P_H^h \neq \{e\}$, a contradiction with G being a TIP-group.

The proof of Theorem 2 is provided below:

Proof.(Theorem 2) **Step 1: Assume G is a TIP group.** By definition, in G, every Sylow p-subgroup P satisfies $P \cap P^g = \{e\}$ for all $g \notin N_G(P)$, where $N_G(P)$ is the normalizer of P in G. We aim to show that G/N is a TIP-group for any normal subgroup N.

Step 2: Structure of Sylow subgroups oin G/N. The Sylow p-subgroups of G/N are of the form P/N, where P is a Sylow p-subgroup of G. For $gN \notin N_{G/N}(P/N)$, we have $P/N \neq (P/N)^{gN} = P^g/N$.

Step 3: Trivial intersection in G/N. Since G is a TIP group, $P \cap P^g = \{e\}$ for all $g \notin N_G(P)$. Applying the correspondence theorem, $(P \cap P^g)N/N = \{e\}/N = \{e_{G/N}\}$. Hence, $(P/N) \cap (P^g/N) = \{e_{G/N}\}$, which implies that Sylow p-subgroups in G/N intersect trivially.

Step 4: Conclusion. Therefore, G/N satisfies the TIP property, as every Sylow p-subgroup of G/N intersects trivially with other Sylow p-subgroups of G/N. This shows that TIP groups are closed under taking quotient groups.

To proof Theorem 3 we will need the following result.

Lemma 3.*Let A and B be groups with relatively prime orders $|A| = a$ and $|B| = b$, where a and b are relatively prime integers. For any prime α, the number of Sylow α-subgroups in the direct product $A \times B$ is equal to the number of Sylow α-subgroups in the component A (if $\alpha \mid a$) or B (if $\alpha \mid b$).*

*Proof.*Let $|A| = a$ and $|B| = b$, were a and b are relatively prime integers, and let α be a prime. Then $|A \times B| = ab$. The following cases are taken into consideration:

Case 1: $\alpha \mid a$ and $\alpha \nmid b$

Since $\alpha \nmid a$, the group B contains only the trivial α-Sylow subgroup $\{e_B\}$. Any Sylow α-subgroup of $A \times B$ is of the form $P_A \times \{e_B\}$, where P_A is a Sylow α-subgroup of A.

The number of Sylow α-subgroups in $A \times B$ is thus the same as the number of Sylow α-subgroups in A.

Case 2: $\alpha \mid b$ and $\alpha \nmid a$, The proof of this case is identical to that of case 1.

Case 3: $\alpha \nmid a$ and $\alpha \nmid b$

If α divides neither a nor b, then A and B both have only the trivial Sylow α-subgroup. In this case, $A \times B$ also has only the trivial Sylow α-subgroup. Conclusion In all cases, the number of Sylow α-subgroups in $A \times B$ is determined by the component whose order is divisible by α. Therefore, the number of α-Sylow subgroups in $A \times B$ equals the number of Sylow α-subgroups in A (if $\alpha \mid a$) or B (if $\alpha \mid b$). This complete the proof.

The proof of Theorem 3 is presented next.

Proof(Proof of Theorem 3). Denote $|A| = a$ and $|B| = b$, where $\gcd(a,b) = 1$. By Lemma 3, let p be any prime that divides ab. Then the Sylow p-subgroups of $A \times B$ are given whichever subgroup contains A or B and contains the order that is equal to p. Consequently, we can present three distinct subcases based on this categorization for primes p that belongs to $A \times B| = ab$:

Case 1: $p \mid a$ and $p \nmid b$. In this case, the Sylow p-subgroups of $A \times B$ are of the type $P_A \times \{e\}$, where P_A is the Sylow p-subgroup of A. By definition, two different p-Sylow subgroups from A cannot intersect, which means their relevant in $A \times B$ do not intersect either.

Case 2: $p \mid b$ and $p \nmid a$. In this case, the Sylow p-subgroups of $A \times B$ are of the type $e \times P_B$, where P_B is the Sylow p-subgroup of B. This argument applies by symmetric exchange of the components with that in case 1. As B is a TIP group, any two distinct p-Sylow subgroups of B intersect trivially, so their counterparts in $A \times B$.

Case 3: $p \nmid a$ and $p \nmid b$. In this case, both A and B contain only the trivial Sylow p-subgroup. Therefore, due to the fact that $A \times B$ contains only the trivial Sylow p-subgroup, the TIP property is satisfied in this case as well.

In all cases, the Sylow p-subgroups of $A \times B$ are all TI-subgroups. This completes the proof of Theorem 4.

Lastly, we close this section with the following proof of Theorem 4

Proof.(Theorem 4) Assume G is a TIP group. By definition, for every prime $p \in \pi(G)$ and every Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$, we have: $P \cap P^g = \{e\}$ or P. This implies $O_p(G) \leq P \cap P^g = \{e\}$ or P, ensuring that $O_p(G) = \{e\}$ or P itself for all $p \in \pi(G)$. Thus, G satisfies the condition β_p for all $p \in \pi(G)$. Therefore, G satisfies the β-condition.

4 Illustrative Examples

In this section, we provide two illustrative examples that highlight key aspects of TIP groups. The first example demonstrates that a supersolvable group need not be a TIP group (see Example 3), while the second example shows that a TIP group is not necessarily solvable, as illustrated by the alternating group A_5 . These examples help to clarify the distinctions between TIP groups and other well-known classes of groups.

*Example 3.*Consider the dihedral group D_{12}, which is a non-abelian supersolvable group of order $12 = 2^2 \times 3$. It is defined by the presentation: $D_{12} = \langle r, s \mid r^6 = s^2 = \{e\}, rs = sr^{-1}\rangle$. The group D_{12} has a unique (hence normal) Sylow 3-subgroup, so $P \in Syl_3(D_{12})$ is a **TI-subgroup**. However, there exists $P \in Syl_2(D_{12})$ such that P is not a TI-subgroup. That is: $Syl_2(D_{12}) = \{P_2, P_2, P_3\}$, with $P_1 = \langle r^3, s\rangle = \{e, r^3, s, r^3 s\}$, $P_2 = \langle r^3, rs\rangle = \{e, r^3, rs, r^4 s\}$, and $P_3 = \langle r^3, r^2 s\rangle = \{e, r^3, r^2 s, r^5 s\}$. We observe that $P_1 \cap P_2 = P_1 \cap P_3 = \{e, r^3\} = \langle r^3\rangle \neq \{e\}$, which means P_1 is not a **TI-subgroup**. Thus, the group D_{12} is not a **TIP-group**.

*Example 4.*The group A_5 is recognized as the smallest simple and non-solvable group of order $60 = 2^2 \times 3 \times 5$. It can be presented as follows:

$$G = \langle A, B \mid A^5 = B^2 = (AB)^3 = e\rangle,$$

where A and B are the generators of the group. The group A_5 contains five Sylow 2-subgroups, each of which is isomorphic to V_4:

$$\begin{aligned}
P_1 &= \{e, (12)(34), (14)(23), (13)(24)\},\\
P_2 &= \{e, (12)(35), (13)(25), (15)(23)\},\\
P_3 &= \{e, (12)(45), (14)(25), (15)(24)\},\\
P_4 &= \{e, (13)(45), (14)(35), (15)(34)\},\\
P_5 &= \{e, (23)(45), (24)(35), (25)(34)\}.
\end{aligned}$$

This shows that every Sylow 2-subgroup is indeed a TI-subgroup. Additionally, A_5 has ten Sylow 3-subgroups and six Sylow 5-subgroups, both of which are cyclic of prime order and hence by Proposition 3 also TIP-subgroups. Furthermore, A_5 is not solvable, So, A_5 is an example of non-solvable TIP group.

Conclusions and Final Remarks

In this work, we introduced the concept of TIP-groups—groups in which every Sylow p-subgroup is a TI-subgroup—and examined their key properties and relationships with other familiar classes of finite groups. Our results demonstrate where TIP-groups fit within the hierarchy of group classes such as nilpotent, supersolvable, and solvable groups.

A possible direction for future research involves exploring what happens when the TI-property is applied not to Sylow subgroups but to *all maximal subgroups* of a group. In particular, we may ask:

What kind of group-theoretic behavior arises when all maximal subgroups are required to intersect trivially with their distinct conjugates?

This naturally leads to questions such as:

– How does this new class of groups relate to TIP-groups?
– Is there any overlap, or are they fundamentally distinct?
– What kind of structure or constraints does this condition impose on a group?

We emphasize that this proposed class is not necessarily a generalization or specialization of TIP-groups, but rather a parallel concept worth investigating on its own. Understanding the impact of the TI-condition on broader families of subgroups, such as maximal ones, may open new avenues in the structural analysis of finite groups.

We leave these questions for future exploration.

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