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Torse-forming vector field and Ricci–Bourguignon soliton on a generic submanifold of a Kenmotsu manifold

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Abstract: The aim of this paper is to derive significant results related to generic submanifolds and the generic product of Kenmotsu manifolds equipped with torse-forming vector fields. In particular, we investigate the existence and non-existence of tangential component u^T of a vector u, which is a torse-forming vector field on D^T, the invariant distribution of a generic submanifold N. Additionally, we establish necessary and sufficient condition for a submanifold of a Kenmotsu manifold to be umbilical. Furthermore, we present the necessary and sufficient conditions for D^T and $D^\perp$, representing the invariant and anti-invariant distributions of a generic submanifold admitting a Wicciourguignon soliton to be Einstein. Lastly, we illustrate a 5-dimensional generic submanifold within a 7-dimensional Kenmotsu manifold through an example.

Keywords: Generic submanifold; Torse-forming vector field; Ricci–Bourguignon soliton.

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1 Introduction

Submanifold theory is currently a vibrant area of research in differential geometry, playing a significant role in both the theoretical development of the field and its applications in areas such as theoretical physics and applied mathematics. Bejancu and Papaghiuc [3] first proposed the notion of semi-invariant submanifolds in a framework of Sasakian manifolds. This concept generalizes both invariant and anti-invariant submanifolds. In 2011, Alegre [1] examined the integrability of the distributions that define a semi-invariant submanifold within a Lorentzian Sasakian manifold. Subsequently, Atçeken and Uddin [2] derived necessary and sufficient criteria for a semi-invariant submanifold to be either an invariant, anti-invariant, or semi-invariant product.
The investigation of semi-invariant submanifolds has attracted considerable attention, with significant contributions from researchers such as Khan et al. [13], Kobayashi [14], and many others in the field. The results derived from generic semi-invariant submanifolds, a specific class of semi-invariant submanifolds, are even more remarkable and impactful. Yano and Kon [18] explored generic submanifolds of Sasakian manifolds, and more recently, Yoldas et al. investigated generic submanifolds in Sasakian manifolds admitting concurrent vector fields. Furthermore, generic submanifolds have been studied by Ghosh and Sarkar ([10], [15]), offering further insights into this interesting area of research.

Torse-forming is defined as a vector field u on a (pseudo) Riemannian manifold $(\tilde{M}, g)$ that satisfies the following condition [19]:

$$\tilde{\nabla}_{A_1} u = f A_1 + \alpha(A_1) u, \tag{1}$$

for every $A_1 \in \Gamma(T\tilde{M})$, where f is a function, α is a 1-form and $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$.
The vector field u in the equation above is called concircular if the 1-form $\alpha = 0$. If $f = 1$ and $\alpha = 0$, the vector field u is called concurrent. If equation (1) is satisfied by $f = 0$, then u is recurrent. In addition, u is called parallel if both $f = 0$ and $\alpha = 0$.

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Chen characterized and studied rectifying submanifolds in Riemannian manifolds equipped with a torqued vector field [7]. In [8], Chen classified all torqued vector fields on Riemannian manifolds and investigated Ricci solitons with torqued potential fields.
In 1988, Hamilton [11] introduced the concept of a Ricci soliton as a generalization of Einstein manifolds, and since then, various classes of Ricci solitons have been developed. A notable example is the Ricci–Bourguignon soliton. The Ricci–Bourguignon flow was a novel geometric flow that was first presented by Jean-Pierre Bourguignon [5] in 1981. The Ricci–Bourguignon flow is defined as [9]:

$$\frac{\partial g}{\partial t} = -2(S - \rho R g),$$

where S denotes the Ricci tensor, R represents the scalar curvature and $\rho \in \mathbb{R}$ is a constant.
This equation reduces to the Ricci flow [11] if $\rho = 0$. Dwivedi [9] introduced the Ricci–Bourguignon soliton in a manner similar to that of the Ricci soliton [11], which is a self-similar solution to the Ricci flow:

Definition 1.*A Riemannian manifold $(\tilde{M}, g)$ is called a Ricci–Bourguignon soliton (RB soliton for short) if there exists a vector field u on $\tilde{M}$, such that*

$$(L_u g)(A_1, A_2) + 2S(A_1, A_2) = 2\lambda g(A_1, A_2) + 2\rho R g(A_1, A_2), \tag{2}$$

for any vector fields $A_1, A_2 \in \tilde{M}$, where, R indicate the scalar curvature, the constants $\lambda, \rho \in \mathbb{R}$ and $L_u g$ denotes the Lie-derivative of the metric g with respect to u.

Depending on whether λ is greater than 0, λ is equal to 0, or λ is less than 0, the soliton is categorized as expanding, steady, or shrinking.
The vector field u is a conformal killing vector field if $L_u g = \gamma g$, where γ is a function and u is a Killing vector field if γ vanishes identically.
Moreover, Sarkar et al. [16] studied RB solitons and η-RB solitons on submanifolds that admit torse-forming vector fields. Many geometers have investigated the Ricci–Bourguignon soliton, including Sarkar et al. [16], Dwivedi [9], and Soylu [17].

Inspired by previous studies, this paper investigates generic submanifolds and the generic semi-invariant product of Kenmotsu manifolds admitting torse-forming vector fields. Specifically, we examine the existence and non-existence of the tangential component u^T of a vector u, which is a torse-forming vector field on D^T, the invariant distribution of a generic submanifold N. Additionally, we establish sufficient and necessary condition for a submanifold of a Kenmotsu manifold to be umbilical with regard to the normal component. The paper also investigates generic submanifolds that admit a RB soliton in Kenmotsu manifolds equipped with torse-forming vector fields. Furthermore, we give a sufficient and necessary condition that the anti-invariant distribution $D^\perp$ and the invariant distribution D^T of N to be Einstein. Finally, we illustrate that the semi-invariant submanifold is also a generic semi-invariant submanifold by presenting an example of a 5-dimensional generic submanifold of a 7-dimensional Kenmotsu manifold.

2 Preliminaries

Let $\tilde{M}$ be a m-dimensional almost contact metric manifold that has the almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a characteristic vector field, η is a 1-form, and g is a Riemannian metric that satisfies the following conditions [4]:

$$\phi^2 A_1 = -A_1 + \eta(A_1)\xi, \tag{3}$$

$$\phi\xi = 0, \quad \eta(\phi A_1) = 0, \quad \eta(\xi) = 1, \quad \eta(A_1) = g(A_1, \xi), \tag{4}$$

$$g(\phi A_1, \phi A_2) = g(A_1, A_2) - \eta(A_1)\eta(A_2), \quad g(\phi A_1, A_2) = -g(A_1, \phi A_2) \tag{5}$$

for all $A_1, A_2 \in \chi(T\tilde{M})$.
If an almost contact metric manifold holds the following conditions, it is referred to as a Kenmotsu manifold [12].

$$(\tilde{\nabla}_{A_1}\phi)A_2 = g(\phi A_1, A_2)\xi - \eta(A_2)\phi A_1, \tag{6}$$

$$\tilde{\nabla}_{A_1}\xi = A_1 - \eta(A_1)\xi, \tag{7}$$

where $\tilde{\nabla}$ stands for the Levi-Civita connection associated with the Riemannian metric g.

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Let N be a submanifold of a Kenmotsu manifold $\tilde{M}$ that is isometrically immersed and has an induced metric g. The tangent and normal subspaces of N in $\tilde{M}$ are represented by $\chi(TN)$ and $\chi(T^{\perp}N)$, respectively.
The Gauss and Weingarten formulas are expressed as follows [6]:

$$\tilde{\nabla}_{A_1}A_2 = \nabla_{A_1}A_2 + \sigma(A_1,A_2) \tag{8}$$

and

$$\tilde{\nabla}_{A_1}A_5 = -A_{A_5}A_1 + \nabla^{\perp}_{A_1}A_5 \tag{9}$$

for all $A_1, A_2 \in \chi(TN)$ and $A_5 \in \chi(T^{\perp}N)$, where ∇ and $\nabla^{\perp}$ represent the induced connections on TN, the tangent bundle and $T^{\perp}N$, the normal bundle of N, respectively.
The following equation relates the shape operator A with the second fundamental form σ:

$$g(A_{A_5}A_1,A_2) = g(\sigma(A_1,A_2),A_5), \tag{10}$$

for any $A_1, A_2 \in \chi(TN)$ and $A_5 \in \chi(T^{\perp}N)$.
The mean curvature vector H of N is expressed as:

$$H = \frac{1}{n}tr(\sigma) = \frac{1}{n}\sum_{f=1}^{n}\sigma(\hat{D}_f,\hat{D}_f), \tag{11}$$

where n represents the dimension of N and $(\hat{D}_1,\hat{D}_2,...\hat{D}_n)$ denotes a local orthonormal frame on N.
If a submanifold N is totally umbilical, then

$$\sigma(A_1,A_2) = g(A_1,A_2)H, \tag{12}$$

for any $A_1, A_2 \in \chi(T\tilde{M})$.
A submanifold N is considered umbilical with regard to a normal vector field U if the second fundamental form σ satisfies the following condition [10]:

$$g(\sigma(A_1,A_2),U) = \delta g(A_1,A_2), \tag{13}$$

where δ represents a function.
Let N be a n-dimensional submanifold of a m-dimensional Kenmotsu manifold $\tilde{M}$, such that ξ is tangent to N. If two differentiable distributions D^T and $D^{\perp}$ on N satisfy the following criteria, then N is referred to be a semi-invariant submanifold of $\tilde{M}$ [15]:

(i)The expression $\phi(D^T_v) = D^T_v$ implies that D^T is invariant under the map ϕ.
(ii)The expression $\phi(D^{\perp}_v) \subset T_vN^{\perp}$ implies that $D^{\perp}$ is anti-invariant under the map ϕ.

where $T_vN^{\perp}$ indicates the normal space to N at v, for all $v \in N$.
We denote $\dim\tilde{M} = m$, $\dim N = n$, $\dim D^T = s$, and $\dim D^{\perp} = t$. The submanifold N is referred to as a generic submanifold of $\tilde{M}$ if the condition $m - n = t$ is satisfied.
Let N be a semi-invariant submanifold of a Kenmotsu manifold $\tilde{M}$. According to the definition of a semi-invariant submanifold, the tangent and normal bundles of N admit the following orthogonal decompositions:

$$TN = D^T \oplus D^{\perp} \oplus \{\xi\}, \quad TN^{\perp} = \phi(D^{\perp}) \oplus \nu, \quad \phi(\nu) = \nu, \tag{14}$$

where ν is the orthogonal complement of $\phi(D^{\perp})$ in $\chi(TN^{\perp})$, and $\{\xi\}$ is the 1-dimensional distribution generated by ξ.
Additionally, if $D^T \oplus \xi$ and $D^{\perp}$ are totally geodesic in N. The submanifold N is referred to as a semi-invariant product.
Furthermore, the following lemma holds for a semi-invariant submanifold N of a Kenmotsu manifold $\tilde{M}$:

Lemma 1.*[18] The following properties are equivalent*

(i)N is a semi-invariant product,
(ii)$A_{\phi(A_3)}A_1 = 0$,
(iii)the second fundamental form, σ of N satisfies $\sigma(\phi A_1,A_2) = \phi\sigma(A_1,A_2)$,
for any $A_1 \in \chi(D^T)$, $A_2 \in \chi(TN)$ and $A_3 \in \chi(D^{\perp})$.

A semi-invariant product that satisfies the condition $m-n=t$ is also referred to as a generic semi-invariant product. In this case, we have $\nu=\{0\}$ in (14), and consequently, the following decomposition is valid:

$$TN=D^T\oplus D^{\perp}\oplus\{\xi\},\quad TN^{\perp}=\phi(D^{\perp}). \tag{15}$$

From (15), we can express the decomposition for a generic semi-invariant product as follows [15]:

$$u=u^T+u^{\perp}+\phi(u^{\perp})+\mu\xi, \tag{16}$$

where $u\in\chi(T\tilde{M})$, $u^T\in\chi(D^T)$ and $u^{\perp}\in\chi(D^{\perp})$. In this study, we assume that μ is a constant function.
Let N be a semi-invariant submanifold of a Kenmotau manifold $\tilde{M}$. If the following condition holds, the semi-invariant submanifold N is referred to as D^T-geodesic

$$\sigma(A_1,A_2)=0, \tag{17}$$

for any $A_1,A_2\in\chi(D^T)$. Likewise, N is said to be $D^{\perp}$-geodesic if the aforementioned relation is true for every $A_1,A_2\in\chi(D^{\perp})$ on N. Furthermore, if $A_1\in\chi(D^T)$ and $A_2\in\chi(D^{\perp})$ satisfy the condition, then N is referred to as a $(D^T,D^{\perp})$-geodesic or mixed geodesic.
If the distribution D^T satisfies $\tilde{\nabla}_{A_1}A_2=0\in\chi(D^T)$ for all $A_1\in\chi(T\tilde{M})$ and $A_2\in\chi(D^T)$, then it is considered parallel with respect to the Levi-Civita connection $\tilde{\nabla}$.

3 Torse-forming vector field and RB soliton on generic submanifold of a Kenmotsu manifold

This section examines the characteristics of a generic semi-invariant product and a generic submanifold within a Kenmotsu manifold $\tilde{M}$, which is equipped with a torse-forming vector field and a RB soliton.

Lemma 2. *Let N be a generic submanifold of a Kenmotsu manifold $\tilde{M}$ with torse-forming vector field u. Then we get the following*

$$\nabla_{A_1}u^T+\nabla_{A_1}u^{\perp}-A_{\phi u^{\perp}}A_1+\mu A_1-fA_1-\alpha(A_1)u^T-\mu\alpha(A_1)\xi=0, \tag{18}$$

and

$$\sigma(A_1,u^T)+\sigma(A_1,u^{\perp})+\nabla^{\perp}_{A_1}\phi(u^{\perp})-\alpha(A_1)u^{\perp}-\alpha(A_1)\phi(u^{\perp})=0. \tag{19}$$

Proof. Since u is a torse-forming vector field on $\tilde{M}$, equations (1) and (16) imply that

$$\tilde{\nabla}_{A_1}u^T+\tilde{\nabla}_{A_1}u^{\perp}+\tilde{\nabla}_{A_1}\phi(u^{\perp})+\mu\tilde{\nabla}_{A_1}\xi=fA_1+\alpha(A_1)u^T+\alpha(A_1)u^{\perp}+\alpha(A_1)\phi(u^{\perp})+\mu\alpha(A_1)\xi, \tag{20}$$

for any $A_1\in\chi(D^T)$.
Substituting equations (7), (8), and (9) in (20) yields:

$$\begin{aligned}&\nabla_{A_1}u^T+\sigma(A_1,u^T)+\nabla_{A_1}u^{\perp}+\sigma(A_1,u^{\perp})+\nabla^{\perp}_{A_1}\phi(u^{\perp})-A_{\phi u^{\perp}}A_1+\mu A_1\\&=fA_1+\alpha(A_1)u^T+\alpha(A_1)u^{\perp}+\alpha(A_1)\phi(u^{\perp})+\mu\alpha(A_1)\xi.\end{aligned}$$

By analyzing the tangential and normal components of the above equation, we obtain the desired results.

For the following theorem, we assume that the function μ in the equation (16) is nonzero.

Theorem 1. *Let N be a generic submanifold of a Kenmotsu manifold $\tilde{M}$ with torse-forming vector field u. If the submanifold N is D^T-geodesic, then vector field u^T on the invariant distribution D^T is not a torse-forming vector field.*

Proof. Assuming that u^T is a torse-forming vector field on D^T, equation (18) simplifies to:

$$\nabla_{A_1}u^{\perp}=-\mu A_1+\mu\alpha(A_1)\xi+A_{\phi u^{\perp}}A_1. \tag{21}$$

If we select $A_2=u$ in (6) for any $A_2\in\chi(T\tilde{M})$, we obtain

$$(\tilde{\nabla}_{A_1}\phi)u=g(\phi A_1,u)\xi-\eta(u)\phi A_1.$$

By applying the properties of the covariant derivative to the above equation, we get

$$\tilde{\nabla}_{A_1}\phi(u) - \phi(\tilde{\nabla}_{A_1}u) = g(\phi A_1, u)\xi - \eta(u)\phi A_1. \tag{22}$$

By substituting equation (16) in previous equality, we find

$$\begin{aligned}&\tilde{\nabla}_{A_1}\phi(u^T) + \tilde{\nabla}_{A_1}\phi(u^\perp) + \tilde{\nabla}_{A_1}\phi^2(u^\perp) - \phi\tilde{\nabla}_{A_1}u^T - \phi\tilde{\nabla}_{A_1}u^\perp - \phi\tilde{\nabla}_{A_1}\phi(u^\perp) - \mu\phi\tilde{\nabla}_{A_1}\xi\\ &= g(\phi A_1, u^T)\xi.\end{aligned} \tag{23}$$

Now, by using equations (3), (7), (8) and (9) in (23), we get

$$\begin{aligned}&\nabla_{A_1}\phi(u^T) + \sigma(A_1, \phi(u^T)) + \nabla^\perp_{A_1}\phi(u^\perp) - A_{\phi u^\perp}A_1 - \nabla_{A_1}u^\perp - \sigma(A_1, u^\perp) - \phi\nabla_{A_1}u^T\\ &-\phi\nabla_{A_1}u^\perp - \phi\sigma(A_1, u^\perp) - \phi\nabla^\perp_{A_1}\phi(u^\perp) + \phi A_{\phi u^\perp}A_1 = g(\phi A_1, u^T)\xi.\end{aligned}$$

Again, assuming that u^T is a torse-forming vector field on D^T, the equation above reduces to:

$$\begin{aligned}&\nabla_{A_1}\phi(u^T) + \sigma(A_1, \phi(u^T)) + \nabla^\perp_{A_1}\phi(u^\perp) - A_{\phi u^\perp}A_1 - \nabla_{A_1}u^\perp - \sigma(A_1, u^\perp) - f\phi A_1 - \alpha(A_1)\phi u^T\\ &-\phi\nabla_{A_1}u^\perp - \phi\sigma(A_1, u^\perp) - \phi\nabla^\perp_{A_1}\phi(u^\perp) + \phi A_{\phi u^\perp}A_1 = g(\phi A_1, u^T)\xi.\end{aligned}$$

Considering the tangential and normal components of the above equation yields:

$$\begin{aligned}&\nabla_{A_1}\phi(u^T) - A_{\phi u^\perp}A_1 - \nabla_{A_1}u^\perp - f\phi A_1 - \alpha(A_1)\phi u^T - \phi\nabla_{A_1}u^\perp + \phi A_{\phi u^\perp}A_1\\ &= g(\phi A_1, u^T)\xi\end{aligned} \tag{24}$$

and

$$\nabla^\perp_{A_1}\phi(u^\perp) + \sigma(A_1, \phi(u^T)) - \sigma(A_1, u^\perp) - \phi\sigma(A_1, u^\perp) - \phi\nabla^\perp_{A_1}\phi(u^\perp) = 0.$$

Similarly, by taking $A_2 = u^T$ in (6), we get

$$\tilde{\nabla}_{A_1}\phi(u^T) - \phi(\tilde{\nabla}_{A_1}u^T) = g(\phi A_1, u^T)\xi. \tag{25}$$

Substituting equations (8), (9) and utilizing the fact that N is D^T-geodesic and assuming u^T is a torse-forming vector field on D^T in (25), we observe

$$\nabla_{A_1}\phi(u^T) - f\phi A_1 - \alpha(A_1)\phi u^T = g(\phi A_1, u^T)\xi. \tag{26}$$

By using the equalities (21), (23) and (25), we arrive at

$$A_{\phi u^\perp}A_1 = \frac{1}{2}\{\mu(A_1 + \phi A_1) - \mu\alpha(A_1)\xi\}. \tag{27}$$

Taking the inner product of (27) with $A_2 \in \chi(D^T)$ and using equation (10), we get

$$g(\sigma(A_1, A_2), \phi u^\perp) = \frac{1}{2}\mu g(A_1, A_2) + \frac{1}{2}\mu g(\phi A_1, A_2). \tag{28}$$

In (28), by switching the roles of A_1 and A_2, we get

$$g(\sigma(A_2, A_1), \phi u^\perp) = \frac{1}{2}\mu g(A_2, A_1) + \frac{1}{2}\mu g(\phi A_2, A_1). \tag{29}$$

From equations (28) and (29), we derive

$$g(\sigma(A_1, A_2), \phi u^\perp) = \frac{1}{2}\mu g(A_1, A_2).$$

Since N is D^T-geodesic, this leads to a contradiction. Hence, the vector field u^T on D^T can never be a torse-forming vector field.

*Remark.*The above result extends the findings of ([15], [20]), where it was established that the tangential component of a concurrent vector field cannot be concurrent on the invariant distribution of a D^T-geodesic generic submanifold in Sasakian and trans-Sasakian manifolds. In this work, we generalize this result to torse-forming vector fields in the context of Kenmotsu manifolds. Since torse-forming vector fields include concurrent vector fields as a special case, our theorem presents a more general result within a different class of contact metric manifolds.

This generalization is significant as it provides a unified framework to study a wider class of vector fields, enriching the geometric structure of submanifolds and enabling further developments in both contact geometry and its potential applications in mathematical physics.

Theorem 2.*Let N be a generic semi-invariant product of a Kenmotsu manifold $\tilde{M}$ with a torse-forming vector field u. If the function μ vanishes identically and $\nabla_{A_1}u^T \in \Gamma(D^T)$ for any $A_1 \in \Gamma(D^T)$, then the vector field u^T is a torse-forming vector field on D^T.*

*Proof.*Given a Kenmotsu manifold $\tilde{M}$ and N as its generic semi-invariant product, utilizing part (ii) of Lemma (1) and equation (18), we obtain

$$\nabla_{A_1}u^T + \nabla_{A_1}u^{\perp} + \mu A_1 - fA_1 - \alpha(A_1)u^T - \mu\alpha(A_1)\xi = 0. \tag{30}$$

On the other hand, for any $A_2 \in \Gamma(T\tilde{M})$, by choosing $A_2 = u$ in (6), we get

$$(\tilde{\nabla}_{X_1}\phi)u = g(\phi X_1, u)\xi - \eta(u)\phi X_1.$$

By applying the properties of the covariant derivative to the above equation, we get

$$\tilde{\nabla}_{X_1}\phi(u) - \phi(\tilde{\nabla}_{X_1}u) = g(\phi X_1, u)\xi - \eta(u)\phi X_1.$$

Given that u is a torse-forming vector field on $\tilde{M}$, we employ (1) in previous equation, we find that

$$\tilde{\nabla}_{A_1}\phi u - f\phi A_1 - \alpha(A_1)\phi u = g(\phi A_1, u)\xi - \eta(u)\phi A_1.$$

By substituting (16) in above equation, we obtain

$$\begin{aligned}&\tilde{\nabla}_{A_1}\phi(u^T) + \tilde{\nabla}_{A_1}\phi(u^{\perp}) - \tilde{\nabla}_{A_1}u^{\perp} - f\phi A_1 - \alpha(A_1)\phi(u^T) - \alpha(A_1)\phi(u^{\perp}) + \alpha(A_1)u^{\perp}\\ &= g(\phi A_1, u^T)\xi - \mu\phi A_1.\end{aligned} \tag{31}$$

By applying equations (8), (9) and part (ii) of Lemma (1) in (31), we find

$$\begin{aligned}&\nabla_{A_1}\phi(u^T) + \sigma(A_1, \phi(u^T)) + \nabla^{\perp}_{A_1}\phi(u^{\perp}) - \nabla_{A_1}u^{\perp} - \sigma(A_1, u^{\perp}) - f\phi A_1 - \alpha(A_1)\phi u^T\\ &- \alpha(A_1)\phi u^{\perp} + \alpha(A_1)u^{\perp} = g(\phi A_1, u^T)\xi - \mu\phi A_1.\end{aligned}$$

By comparing the tangential and normal components of the above equation, we obtain

$$\nabla_{A_1}\phi(u^T) - \nabla_{A_1}u^{\perp} - f\phi A_1 - \alpha(A_1)\phi u^T = g(\phi A_1, u^T)\xi - \mu\phi A_1 \tag{32}$$

and

$$\nabla^{\perp}_{A_1}\phi(u^{\perp}) + \sigma(A_1, \phi(u^T)) - \sigma(A_1, u^{\perp}) - \alpha(A_1)\phi u^{\perp} + \alpha(A_1)u^{\perp} = 0.$$

On the other hand, by choosing $A_2 = u^T$ in (6), we have

$$\tilde{\nabla}_{A_1}\phi(u^T) - \phi(\tilde{\nabla}_{A_1}u^T) = g(\phi A_1, u^T)\xi.$$

Utilizing (8) and part (iii) of Lemma (1) in previous equation, we reach at

$$\nabla_{A_1}\phi(u^T) = g(\phi A_1, u^T)\xi + \phi(\nabla_{A_1}u^T). \tag{33}$$

From equations (30), (32) and (33), we derive

$$\nabla_{A_1}u^T + \phi\nabla_{A_1}u^T = (f-\mu)A_1 + (f-\mu)\phi A_1 + \alpha(A_1)u^T + \alpha(A_1)\phi u^T + \mu\alpha(A_1)\xi. \tag{34}$$

By applying ϕ on both sides of equation (34), we have

$$\phi\nabla_{A_1}u^T - \nabla_{A_1}u^T + \eta(\nabla_{A_1}u^T)\xi = (f-\mu)\phi A_1 - (f-\mu)A_1 + \alpha(A_1)\phi u^T - \alpha(A_1)u^T. \tag{35}$$

Subtracting equation (35) from equation (34), shows that

$$\nabla_{A_1} u^T = fA_1 + \alpha(A_1)u^T - \mu A_1 + \frac{1}{2}\mu\alpha(A_1)\xi + \frac{1}{2}\eta(\nabla_{A_1} u^T)\xi. \tag{36}$$

Setting $\mu = 0$ and noting that $\nabla_{A_1} u^T \in \chi(D^T)$ for any $A_1 \in \chi(D^T)$ in (36), we arrive at the desired result..
This concludes the proof..

*Remark.*The above result builds upon earlier findings on concurrent vector fields in generic semi-invariant product submanifolds of Sasakian and trans-Sasakian manifolds ([15], [20]). In those studies, the tangential component was shown to remain concurrent under specific conditions. Our present result extends this framework to torse-forming vector fields in Kenmotsu manifolds. As torse-forming vector fields generalize concurrent vector fields, the theorem provides a broader and more flexible setting under similar geometric assumptions.

Theorem 3.*Let N be a generic semi-invariant product of a Kenmotsu manifold $\tilde{M}$ with a torse-forming vector field u. If N is a mixed-geodesic submanifold and $\nabla_{A_1} u^\perp \in \Gamma(D^T)$ for any $A_1 \in \Gamma(D^T)$, then $u^\perp$ is a parallel vector field.*

*Proof.*Given that $\tilde{M}$ is a Kenmotsu manifold, by substituting $A_2 = u^\perp$ in (6), we obtain

$$\tilde{\nabla}_{A_1}\phi(u^\perp) - \phi(\tilde{\nabla}_{A_1} u^\perp) = 0.$$

By utilizing equations (8) and (9) in above equation, we have

$$\nabla^\perp_{A_1}\phi(u^\perp) - A_{\phi u^\perp}A_1 - \phi\nabla_{A_1} u^\perp - \phi\sigma(A_1, u^\perp) = 0.$$

Since N is a mixed-geodesic, we find that

$$\nabla^\perp_{A_1}\phi(u^\perp) - A_{\phi u^\perp}A_1 - \phi\nabla_{A_1} u^\perp = 0. \tag{37}$$

By considering the normal and tangential parts of (37), we arrive at

$$\phi\nabla_{A_1} u^\perp = -A_{\phi u^\perp}A_1 \tag{38}$$

and

$$\nabla^\perp_{A_1}\phi(u^\perp) = 0.$$

Using part (ii) of Lemma (1) in (38), we get

$$\phi\nabla_{A_1} u^\perp = 0.$$

Applying ϕ on both sides of the above equation, we get

$$-\nabla_{A_1} u^\perp + \eta(\nabla_{A_1} u^\perp)\xi = 0.$$

Based on the provided condition, we find that

$$\nabla_{A_1} u^\perp = 0.$$

$u^\perp$ is hence a parallel vector field.

Theorem 4.*Let N be a generic submanifold of a Kenmotsu manifold $\tilde{M}$ with a torse-forming vector field u. If $\nabla_{A_1} A_2 = -\nabla_{A_2} A_1$, then u^T is a conformal vector field if and only if N is umbilical with respect to $\phi u^\perp$.*

*Proof.*The Lie-derivative definition gives us

$$(L_u g)(A_1, A_2) = g(\nabla_{A_1} u, A_2) + g(A_1, \nabla_{A_2} u),$$

for any $A_1, A_2, u \in \chi(TN)$.
Substituting $u = u^T$ in above equation, we get

$$(L_{u^T} g)(A_1, A_2) = g(\nabla_{A_1} u^T, A_2) + g(A_1, \nabla_{A_2} u^T). \tag{39}$$

Utilizing (18) in (39), we obtain

$$\begin{aligned}(L_{u^T} g)(A_1,A_2) =& 2fg(A_1,A_2) - 2\mu g(A_1,A_2) + 2g(\sigma(A_1,A_2),\phi u^{\perp}) + \alpha(A_1)g(u^T,A_2)\\ &+ \alpha(A_2)g(u^T,A_1) - g(\nabla_{A_1}u^{\perp},A_2) - g(\nabla_{A_2}u^T,A_1).\end{aligned} \tag{40}$$

Now, we compute the final four terms on the right side of equation (40). Since u^T is a torse forming vector field on $\tilde{M}$ from theorem (2), using equations (1) and (8), we observe that

$$\nabla_{A_1}u^T + \sigma(A_1,u^T) = fA_1 + \alpha(A_1)u^T.$$

Comparing the previous equation's tangential and normal components yields the following:

$$\nabla_{A_1}u^T = fA_1 + \alpha(A_1)u^T \tag{41}$$

and

$$\sigma(A_1,u^T) = 0.$$

When we take the inner product of (41) with $A_2 \in \chi(D^T)$, we notice that

$$\alpha(A_1)g(u^T,A_2) = -fg(A_1,A_2) - g(u^T,\nabla_{A_1}A_2). \tag{42}$$

Similarly, we obtain:

$$\alpha(A_2)g(u^T,A_1) = -fg(A_1,A_2) - g(u^T,\nabla_{A_2}A_1). \tag{43}$$

We know that

$$(\nabla_{A_1}g)(u^{\perp},A_2) = \nabla_{A_2}g(u^{\perp},A_2) - g(\nabla_{A_1}u^{\perp},A_2) - g(u^{\perp},\nabla_{A_1}A_2).$$

Since N is a generic submanifold, the above equation gives:

$$g(\nabla_{A_1}u^{\perp},A_2) = -g(u^{\perp},\nabla_{A_1}A_2). \tag{44}$$

Similarly, we get the following:

$$g(\nabla_{A_2}u^{\perp},A_1) = -g(u^{\perp},\nabla_{A_2}A_1). \tag{45}$$

By substituting equations (42), (43), (44) and (45) in (40), we obtain

$$\begin{aligned}(L_{u^T} g)(A_1,A_2) =& -2\mu g(A_1,A_2) - g(u^T,\nabla_{A_1}A_2 + \nabla_{A_2}A_1) + g(u^{\perp},\nabla_{A_1}A_2 + \nabla_{A_2}A_1)\\ &+ 2g(\sigma(A_1,A_2),\phi u^{\perp}).\end{aligned}$$

Using the given conditions in the previous equation, we observe:

$$(L_{u^T} g)(A_1,A_2) = -2\mu g(A_1,A_2) + 2g(\sigma(A_1,A_2),\phi u^{\perp}). \tag{46}$$

Let us now assume that u^T is a conformal vector field. A conformal vector field is defined as follows:

$$(L_{u^T} g)(A_1,A_2) = \gamma g(A_1,A_2), \tag{47}$$

where γ is a function.
Considering equations (46) and (47), we arrive at

$$g(\sigma(A_1,A_2),\phi u^{\perp}) = (\frac{\gamma}{2} - \mu)g(A_1,A_2). \tag{48}$$

Equation (48) clearly shows that N is umbilical with respect to $\phi u^{\perp}$.

By contrast, if N is umbilical with respect to $\phi u^{\perp}$, then we get

$$g(\sigma(A_1,A_2),\phi u^{\perp}) = \delta g(A_1,A_2), \tag{49}$$

where δ is a function. Taking into account equations (46) and (49), we obtain

$$(L_{u^T} g)(A_1,A_2) = 2(\delta - \mu)g(A_1,A_2). \tag{50}$$

Therefore, u^T is a conformal vector field.
Thus, the theorem is proved.

*Remark.*The above result generalizes a known theorem in [10] LP-Sasakian manifolds involving concurrent vector fields. Here, we consider torse-forming vector fields in the setting of Kenmotsu manifolds, which provides a more general framework. The condition of umbilicity with respect to $\phi u^{\perp}$ reflects the influence of the contact structure.

Now we examine the generic submanifolds that admit the RB soliton in Kenmotsu manifold with a torse-forming vector field.

Lemma 3.*Let N be a generic submanifold admitting RB soliton of a Kenmotsu manifold $\tilde{M}$ with torse-forming vector field u. Then the Ricci tensor S_{D^T} of an invariant distribution D^T is given by*

$$\begin{aligned} S_{D^T}(A_1,A_2) =&(\lambda+\mu+\rho R)g(A_1,A_2) - g(\sigma(A_1,A_2),\phi u^{\perp}) + \frac{1}{2}g(u^T,\nabla_{A_1}A_2+\nabla_{A_2}A_1) \\ &- \frac{1}{2}g(u^{\perp},\nabla_{A_1}A_2+\nabla_{A_2}A_1), \end{aligned}$$

for any $A_1,A_2 \in \chi(D^T)$, where ∇ representing the Levi-Civita connection on N.

*Proof.*According to the Lie-derivative definition, we have

$$(L_{u^T}g)(A_1,A_2) = g(\nabla_{A_1}u^T,A_2) + g(A_1,\nabla_{A_2}u^T),$$

for any $A_1,A_2,u^T \in \chi(D^T)$.
Using (18) in above equation, we obtain

$$\begin{aligned} (L_{u^T}g)(A_1,A_2) =&2fg(A_1,A_2) - 2\mu g(A_1,A_2) + 2g(\sigma(A_1,A_2),\phi u^{\perp}) + \alpha(A_1)g(u^T,A_2) \\ &+ \alpha(A_2)g(u^T,A_1) - g(\nabla_{A_1}u^{\perp},A_2) - g(\nabla_{A_2}u^{\perp},A_1). \end{aligned} \tag{51}$$

By employing (42), (43), (44) and (45) in (51), we obtain

$$\begin{aligned} (L_{u^T}g)(A_1,A_2) =& -2\mu g(A_1,A_2) + 2g(\sigma(A_1,A_2),\phi u^{\perp}) - g(u^T,\nabla_{A_1}A_2+\nabla_{A_2}A_1) \\ &+ g(u^{\perp},\nabla_{A_1}A_2+\nabla_{A_2}A_1). \end{aligned} \tag{52}$$

Given that a RB soliton is admitted by the generic submanifold, the following can be obtained using the equation (2).

$$(L_{u^T}g)(A_1,A_2) + 2S_{D^T}(A_1,A_2) = 2\lambda g(A_1,A_2) + 2\rho R g(A_1,A_2). \tag{53}$$

Combining the equations (52) and (53), we obtain

$$\begin{aligned} S_{D^T}(A_1,A_2) =&(\lambda+\mu+\rho R)g(A_1,A_2) - g(\sigma(A_1,A_2),\phi u^{\perp}) + \frac{1}{2}g(u^T,\nabla_{A_1}A_2+\nabla_{A_2}A_1) \\ &- \frac{1}{2}g(u^{\perp},\nabla_{A_1}A_2+\nabla_{A_2}A_1). \end{aligned}$$

Which is the desired result.

*Remark.*The above expression for the Ricci tensor of the invariant distribution D^T generalizes similar results obtained for Ricci solitons on generic submanifolds of LP-Sasakian and trans-Sasakian manifolds with concurrent vector fields ([10], [15]). In those works, the Ricci tensor is expressed in terms of the concurrent vector field and the second fundamental form. Our result extends this framework to Ricci–Bourguignon solitons and torse-forming vector fields in Kenmotsu manifolds. Unlike previous cases, both the tangential and normal components of the torse-forming vector field contribute to the curvature, and particularly, the presence of $u^{\perp}$ in the expression indicates that the normal component has a direct influence on the Ricci tensor of the invariant distribution.

Lemma 4.*Let N be a generic submanifold admitting RB soliton of a Kenmotsu manifold $\tilde{M}$ with torse-forming vector field u. Then the Ricci tensor $S_{D^{\perp}}$ of an anti-invariant distribution $D^{\perp}$ is given by*

$$S_{D^{\perp}}(A_1,A_2) = (\lambda+\mu+\rho R - f)g(A_1,A_2) - g(\sigma(A_1,A_2),\phi u^{\perp}) - \frac{1}{2}g(u^T,\nabla_{A_1}A_2+\nabla_{A_2}A_1),$$

for any $A_1,A_2 \in \chi(D^{\perp})$, where ∇ representing the Levi-Civita connection on N.

*Proof.*The Lie-derivative definition gives us

$$(L_{u^\perp}g)(A_1,A_2)=g(\nabla_{A_1}u^\perp,A_2)+g(A_1,\nabla_{A_2}u^\perp),$$

for every $A_1,A_2,u^\perp \in \chi(D^\perp)$.

Using (18) in above equation, we obtain

$$\begin{aligned}(L_{u^\perp}g)(A_1,A_2)=&2fg(A_1,A_2)-2\mu g(A_1,A_2)+2g(\sigma(A_1,A_2),\phi u^\perp)\\&-g(\nabla_{A_1}u^T,A_2)-g(\nabla_{A_2}u^T,A_1).\end{aligned} \tag{54}$$

We know that

$$(\nabla_{A_1}g)(u^T,A_2)=\nabla_{A_2}g(u^T,A_2)-g(\nabla_{A_1}u^T,A_2)-g(u^T,\nabla_{A_1}A_2).$$

Since N is a generic submanifold, above equation gives

$$g(\nabla_{A_1}u^T,A_2)=-g(u^T,\nabla_{A_1}A_2). \tag{55}$$

Similarly, we get

$$g(\nabla_{A_2}u^T,A_1)=-g(u^T,\nabla_{A_2}A_1). \tag{56}$$

Employing (55) and (56) in (54), we observe

$$(L_{u^\perp}g)(A_1,A_2)=2(f-\mu)g(A_1,A_2)+2g(\sigma(A_1,A_2),\phi u^\perp)+g(u^T,\nabla_{A_1}A_2+\nabla_{A_2}A_1). \tag{57}$$

Given that the generic submanifold admits a RB soliton, the following can be derived from the equation (2).

$$(L_{u^\perp}g)(A_1,A_2)+2S_D(A_1,A_2)=2\lambda g(A_1,A_2)+2\rho Rg(A_1,A_2). \tag{58}$$

Equations (57) and (58) are combined to yield

$$S_{D^\perp}(A_1,A_2)=(\lambda+\mu+\rho R-f)g(A_1,A_2)-g(\sigma(A_1,A_2),\phi u^\perp)-\frac{1}{2}g(u^T,\nabla_{A_1}A_2+\nabla_{A_2}A_1).$$

which is the required result.

As a result of Lemma (3) and Lemma (4), we have the following two theorems.

Theorem 5.*Let N be a generic submanifold admitting RB soliton of a Kenmotsu manifold $\tilde{M}$ with torse-forming vector field u. If the invariant distribution D^T is D^T-parallel, then the invariant distribution D^T is Einstein.*

Theorem 6.*Let N be a generic submanifold admitting RB soliton of a Kenmotsu manifold $\tilde{M}$ with torse-forming vector field u. If the anti-invariant distribution $D^\perp$ is $D^\perp$-parallel, then the anti-invariant distribution $D^\perp$ is Einstein.*

4 Example: 5-dimensional submanifold of an 7-dimensional Kenmotsu manifold

In this section, we provide an example of a 7-dimensional Kenmotsu manifold. Using this example, we then construct a five-dimensional submanifold and demonstrate that N is a generic semi-invariant submanifold of $\tilde{M}$.

Consider a 7-dimensional manifold $\tilde{M}$, defined as $\tilde{M}=\{\hat{a}_1,\hat{a}_2,\hat{a}_3,\hat{a}_4,\hat{a}_5,\hat{a}_6,\hat{a}_7 \in \mathbb{R}^7\}$, where $(\hat{a}_1,\hat{a}_2,\hat{a}_3,\hat{a}_4,\hat{a}_5,\hat{a}_6,\hat{a}_7)$ denote the standard coordinates of $\mathbb{R}^7$. The linearly independent vector fields $\{\hat{D}_1,\hat{D}_2,\hat{D}_3,\hat{D}_4,\hat{D}_5,\hat{D}_6,\hat{D}_7\}$ on $\tilde{M}$ are defined as

$$\hat{D}_1=2\hat{a}_3\frac{\partial}{\partial\hat{a}_1},\quad \hat{D}_2=2\hat{a}_3\frac{\partial}{\partial\hat{a}_2},\quad \hat{D}_3=-\hat{a}_3\frac{\partial}{\partial\hat{a}_3},\quad \hat{D}_4=2\hat{a}_3\frac{\partial}{\partial\hat{a}_4},\quad \hat{D}_5=2\hat{a}_3\frac{\partial}{\partial\hat{a}_5},$$
$$\hat{D}_6=2\hat{a}_3\frac{\partial}{\partial\hat{a}_6},\quad \hat{D}_7=2\hat{a}_3\frac{\partial}{\partial\hat{a}_7}.$$

Let us now define g, the Riemannian metric on $\tilde{M}$, as follows:

$$g(\hat{D}_f,\hat{D}_g)=\begin{cases}1, & \text{if} \quad f=g,\\ 0, & \text{if} \quad f\neq g, \quad for \quad 1\leq f,g\leq 7.\end{cases}$$

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The 1-form η for the metric g is written as $\eta(A_1) = g(A_1, \xi)$ and setting $\hat{D}_3 = \xi$ we observe that $\eta(\hat{D}_3) = 1$, and $\eta(\hat{D}_g) = 0$ for f=1, 2, 4, 5, 6, 7.
The (1, 1)-tensor field ϕ is also defined as follows:

$$\phi(\hat{D}_1) = -\hat{D}_7, \quad \phi(\hat{D}_2) = -\hat{D}_5, \quad \phi(\hat{D}_3) = 0, \quad \phi(\hat{D}_4) = -\hat{D}_6, \quad \phi(\hat{D}_5) = \hat{D}_2, \quad \phi(\hat{D}_6) = \hat{D}_1, \quad \phi(\hat{D}_7) = \hat{D}_4.$$

Considering the equations above, it is straightforward to verify that $\phi^2 A_1 = -A_1 + \eta(A_1)\xi$ and $g(\phi A_1, \phi A_2) = g(A_1, A_2) - \eta(A_1)\eta(A_2)$, for any vector fields $A_1, A_2 \in T\tilde{M}$. This shows that the structure $\tilde{M}(\phi, \xi, \eta, g)$ represents an almost contact metric manifold.
Suppose $\tilde{\nabla}$ represents the Levi-Civita connection associated with the metric g. This results in the following:

$$[\hat{D}_f, \hat{D}_g] = \begin{cases} \hat{D}_f, & \text{if} \quad f = 1,2,4,5,6,7; g = 3 \\ 0, & \text{otherwise}, \end{cases} \tag{59}$$

here [., .] represents the Lie bracket.
By applying Koszul's formula and (59), we derive the following:

$$\tilde{\nabla}_{\hat{D}_1}\hat{D}_1 = -\hat{D}_3, \tilde{\nabla}_{\hat{D}_1}\hat{D}_3 = \hat{D}_1, \tilde{\nabla}_{\hat{D}_2}\hat{D}_2 = -\hat{D}_3, \tilde{\nabla}_{\hat{D}_2}\hat{D}_3 = \hat{D}_2,$$
$$\tilde{\nabla}_{\hat{D}_4}\hat{D}_3 = \hat{D}_4, \tilde{\nabla}_{\hat{D}_4}\hat{D}_4 = \hat{D}_3, \tilde{\nabla}_{\hat{D}_5}\hat{D}_3 = \hat{D}_5, \tilde{\nabla}_{\hat{D}_5}\hat{D}_5 = -\hat{D}_3,$$
$$\tilde{\nabla}_{\hat{D}_6}\hat{D}_3 = \hat{D}_6, \tilde{\nabla}_{\hat{D}_6}\hat{D}_6 = -\hat{D}_3, \tilde{\nabla}_{\hat{D}_7}\hat{D}_3 = \hat{D}_7, \tilde{\nabla}_{\hat{D}_7}\hat{D}_7 = -\hat{D}_3.$$

remaining values $\tilde{\nabla}_{\hat{D}_f}\hat{D}_g = 0$.
As a consequence of above equations, the manifold $\tilde{M}$ satisfies

$$\tilde{\nabla}_{A_1}\xi = A_1 - \eta(A_1)\xi,$$

and

$$(\tilde{\nabla}_{A_1}\phi)A_2 = g(\phi A_1, A_2)\xi - \eta(A_2)\phi A_1,$$

Hence, from the above, we can conclude that $\tilde{M}$ is a Kenmotsu manifold.

Now, let us consider the five-dimensional submanifold N of $\tilde{M}(\phi, \xi, \eta, g)$, which is given by the isometric immersion $\psi : N \to \tilde{M}$ defined by $\psi(\hat{a}_1, \hat{a}_3, \hat{a}_5, \hat{a}_6, \hat{a}_7) = (\hat{a}_1, 0, \hat{a}_3, 0, \hat{a}_5, \hat{a}_6, \hat{a}_7)$.
It is clear that $N = \{(\hat{a}_1, \hat{a}_3, \hat{a}_5, \hat{a}_6, \hat{a}_7) \in \mathbb{R}^5\}$, where $(\hat{a}_1, \hat{a}_3, \hat{a}_5, \hat{a}_6, \hat{a}_7)$ represent standard coordinates in $\mathbb{R}^5$, and it forms a 5-dimensional submanifold of $\tilde{M}$. The vector fields $\{\hat{D}_1, \hat{D}_3, \hat{D}_5, \hat{D}_6, \hat{D}_7\}$ are given by

$$\hat{D}_1 = 2\hat{a}_3\frac{\partial}{\partial \hat{a}_1}, \quad \hat{D}_3 = -\hat{a}_3\frac{\partial}{\partial \hat{a}_3}, \qquad \hat{D}_5 = 2\hat{a}_3\frac{\partial}{\partial \hat{a}_5}, \quad \hat{D}_6 = 2\hat{a}_3\frac{\partial}{\partial \hat{a}_6}, \quad \hat{D}_7 = 2\hat{a}_3\frac{\partial}{\partial \hat{a}_7}.$$

ϕ, which represents the (1, 1)-tensor field, is defined as follows:

$$\phi(\hat{D}_1) = -\hat{D}_7, \quad \phi(\hat{D}_3) = 0, \quad \phi(\hat{D}_5) = \hat{D}_2, \quad \phi(\hat{D}_6) = \hat{D}_1, \quad \phi(\hat{D}_7) = \hat{D}_4.$$

Since $\phi(\hat{D}_1)$ and $\phi(\hat{D}_6)$ are tangent to TN, and $\phi(\hat{D}_5)$ and $\phi(\hat{D}_7)$ are orthogonal to TN, we conclude that D^T=span$\{\hat{D}_1, \hat{D}_6\}$ and $D^\perp$=span$\{\hat{D}_5, \hat{D}_7\}$ are the invariant and anti-invariant distributions of N.
Thus, we obtain the decomposition $TN = D^T \oplus D^\perp \oplus \{\hat{D}_3\}$. Therefore, N is a semi-invariant submanifold of $\tilde{M}$ of dimension five.

Here, dim($\tilde{M}$)=7, dim(N)=5, dim(D^T)=2, and dim($D^\perp$)=2 are present.
Additionally, we observe that dim($\tilde{M}$)-dim(N)=7-5=2=dim($D^\perp$).
Hence, the tangent space decomposition is given by $TN = D^T \oplus D^\perp \oplus \{\xi\}$, and the normal space is $TN^\perp = \phi(D^\perp)$. Thus, this semi-invariant submanifold N is also a generic semi-invariant submanifold of $\tilde{M}$.

Declarations

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