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On The Prime Graphs of Finite and Dihedral Groups

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Abstract: Let G be a finite group. The order of the group G, denoted $|G|$ is the number of elements in G. For any element $g \in G$, the order of g denoted $o(g)$ is the smallest positive integer $n \in \mathbb{N}$ such that $g^n = e$. A graph of a finite group is said to be a prime graph $\Gamma(G)$ if two vertices p, q are primes that divide $|G|$, with the vertices p and q connected by an edge if and only if the product pq divides $o(g)$, the order of a group element. In this research, we study the prime graph of finite groups by focusing on one of these finite groups, i.e. the dihedral group with n sides which has order $2n$. Next, we define and study the prime graph generated from 2 prime graphs of dihedral group $\Gamma(D_m) * \Gamma(D_n)$ and obtain the form of the prime graph of dihedral group which is forms the graph of K_n.

Keywords: Finite group, Dihedral group, Prime graph, Order of the group, Order of the group element.

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1 Introduction

In this research, the topic of group theory and graph theory will be studied. The group that will be used in this study is a finite group, i.e. the dihedral group with n sides which has order $2n$. Let G be a group, with G as a set and "$*$" as a binary operation on G that follows the axioms of being closed, associative, identities, and inverses of every element [3]. The number of elements in G is said to be its order [6]. The order of G is denoted by $|G|$, and the order of an elements g in G is denoted by $o(g)$. In between groups, graph theory will be studied as well. A graph G is a set of pairs (V, E) with V being a set and E being a set of unordered pairs of elements of V. The set of elements of V is called a vertex, and $V(G)$ is called the vertex set of the graph. The set of elements of E is called an edge, and $E(G)$ is called the set of edges of the graph [5]. In graph theory, we will discuss prime graphs in more detail, specifically prime graphs $\Gamma(G)$. Prime graphs were introduced for the first time by Gruenberg and Kegel in 1970 . Williams (1981) introduced the prime graph component of a finite group [8]. Kondratev (1990) discussed the components of the prime graph of a simple finite group [7]. Dolfi and friends (2019) studied prime graphs on some classes of finite groups having bipartite complements, and showed that prime graphs do not contain odd-length cycles, i.e. bipartite graphs [1]. Chris (2021) studied the prime graph of some classes of finite groups, by giving a complete characterization of some classes of groups [2].
From several previous studies on prime graphs, the author is interested in studying the Prime Graph of a finite group by taking the prime numbers of the order group as the basis for graph construction.
The paper will be divided into several sections. Section 1 is an introduction. Section 2, we will discuss the definitions used as guidelines in this research. Section 3, we will discuss the research that we have done, starting from the formation of prime graphs on the dihedral group with n sides which has $2n$ then developing the definition of prime graphs consisting of 2 dihedral groups , then constructing prime graphs of 2 dihedral groups.

2 Main Result

In this section, the author describes the definitions, lemmas and theorems obtained in the research on the Prime Graph of a Finite Group. The main results in this paper are presented in Theorem 1 and 2.

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2.1 Prime Graph of a Finite Group

In this section, we construct the prime graph of a finite group, i.e., the dihedral group with n sides which has order $2n$. Suppose we are given a Dihedral group $(D_n,*)$, with group orders $|G| = |D_n| = 2n$, with $n > 3$, is a group consisting of 2 elements s,r with $G = D_n = \{r,s|r^n = e,s^2 = e,srs^{-1} = r^{-1}\}$, then we will find a different form of prime graph between dihedral groups with n odd and n even. The two types of graphs generated represent the shape of the Prime Graph on the Dihedral Group.
Based on the research that the author conducted, 2 general forms of the dihedral group prime graph are obtained in the following theorem:

Theorem 1.*Suppose given a dihedral group D_n with n sides which has order $2n$, and the prime graph of dihedral group D_n is $\Gamma(D_n)$, then the following holds:*
1. For n odd, it will form a prime graph $\Gamma(D_n)$ which contains 2 component graphs ;
2. For n even, it will form a prime graph $\Gamma(D_n)$ which forms the complete graph K_n.

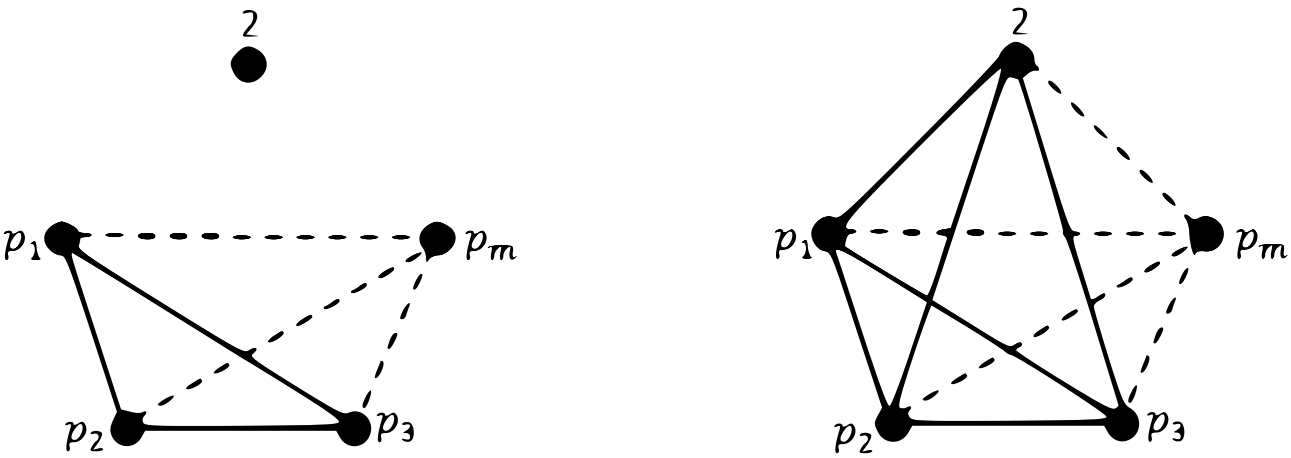
*Proof.*The proof will be in 2 cases. Case 1 for n odd and case 2 for n even. Suppose $|D_n| = 2, p_1^{k_1}, p_2^{k_2},, p_m^{k_m}$, with $p_1, p_2,, p_m$ being odd primes,and $k_1,k_2,\dots,k_m \in \mathbb{N}$. Case 1 for n odd. For n odd $(n \in 2k-1,\ k \in \mathbb{N})$, we will show the two points:
1. 2 not connected with p_i , where $p_i \in \{p_1, p_2,..., p_m\}$;
2. p_i connected with p_j , where $p_i, p_j \in \{p_1, p_2,..., p_m\}$.
Given $D_n = \{e,r,r^2,\dots,r^{n-1},s,sr,sr^2,\dots,sr^{n-1}\}$, with $|D_n| = 2n$, and

$$o(D_n) = \{o(r^i), o(r^i s) : i = 0,1,2,\dots\mathbb{N}\},\ \text{where} : o(r^i) = \frac{n}{\gcd(i,n)}\ \text{and}\ o(r^i s) = 2\ \text{for all}\ i.$$

since $n = p_1^{k_1} \cdot p_2^{k_2} \cdot\ \dots\ \cdot p_m^{k_m}$, then

$$V(D_n) = \{2, p_1, p_2,\dots, p_m\}.$$

First, it will be shown that 2 and p_i are not connected because $2 \cdot p_i$ does not divide any of the orders elements of group. Known that $o(e) = 1$, we will show that $2p_i \nmid o(e)$ and known that there is no integer k such that $2p_i \times k = 1$. Since $2p_i$ is an even number greater than 1, it is clear that $2p_i \nmid o(e)$. Next, it will be shown that $2p_i \nmid o(r)$. It is known that $o(r) = n$, where n is odd, and $2p_i$ is even. It is clear that $2p_i \nmid o(r)$. Given $o(r^i) = \frac{n}{\gcd(i,n)}$. Since n is odd,$\gcd(i,n)$ is also odd. Therefore,$o(r^i)$ will be odd. It is known that $2p_i$ is even, so it cannot divide an odd number. Hence, $2p_i \nmid o(r^i)$. Given $o(s) = o(r^i s) = 2$, it will be shown that $2p_i \nmid o(s)$ and $2p_i \nmid o(r^i s)$. It is known that there is no integer k such that: $2p_i \times k = 2$. Since $2p_i$ is an even number greater than 2, it is clear that $2p_i \nmid o(s)$ or $2p_i \nmid o(r^i s)$. So that $2 \cdot p_i$ not divide any of the orders element of group for all $p_i \in \{p_1, p_2,\dots, p_m\}, m = 1,2,3,\dots,n$.
Next, it will be shown that p_i and p_j are connected because $p_i \cdot p_j$ divides one of the orders of the elements of the group. Take any $p_i, p_j \in \{p_1, p_2,..., p_m\}$, where $i \neq j$. Consider that $r \in D_n, n$ is odd, and $o(r) = n$. It is known that $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$, then $p_i \mid n$ and $p_j \mid n$. Hence, $p_i \cdot p_j \mid n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$. Meanwhile, $o(r) = n$, then $p_i \cdot p_j \mid o(r)$. Thus, for any pair of vertices $\{p_i, p_j\} \subseteq V(D_n)$ will be connected. So, the proof shows that the vertex $\{2, p_i\} \subseteq V(D_n)$ is not connected, and the vertices $\{p_i, p_j\} \subseteq V(D_n)$ are connected, so for n odd, we get a graph consisting of 2 graph components.
Case 2 for n even $(n \in 2k,\ k \in \mathbb{N})$.
Given $D_n = \{e,r,r^2,,r^{n-1},\ s,\ sr,\ sr^2,,\ sr^{n-1}\}$. $|D_n| = 2n, r \in D_n,\ o(r) = n$,
$n = 2^{k_0} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_m^{k_m}$, $k_0 \geq 1$ where $p_1, p_2,\dots, p_m$ are odd primes. and $V(D_n)$=$\{2, p_1, p_2,, p_m\}$, take any $u,v \in V(D_n)$, suppose $\{(u,v)\} \subseteq E(D_n)$, Next, it will be shown that $u \cdot v \mid o(r)$, for some $r \in D_n$, then $o(r) = n = 2^{k_0} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \cdots \cdot p_m^{k_m}$, where $k_0 \geq 1$. Then it is clear that $u \cdot v \mid o(r)$. Because $u \cdot v \mid o(r)$, any pair of vertices $\{u,v\} \subseteq V(D_n)$ will be connected.
So, since n is even, this shows that u,v are connected, and it forms K_n.



(a) The prime graph $\Gamma(D_n)$ which contains 2 component graphs for n odd

(b) The prime graph $\Gamma(D_n)$ which forms the complete group K_n for n even

Fig. 1: The prime graph of dihedral group D_n is $\Gamma(D_n)$.

2.2 The Combination of the Prime Graph of a finite group

In the previous section, it is known that the pattern of prime graphs from the dihedral group with n vertices is formed into 2, which are for prime graphs for n odd vertices and prime graphs for n even vertices.
In this section, we will show the prime graph of the dihedral group formed from two prime graphs $\Gamma(D_m)$ and $\Gamma(D_n)$, i.e. the prime graph $\Gamma(D_m) * \Gamma(D_n)$.
As a first step, based on the definition of prime graph in the previous section, the definition of prime graph for two prime graphs $\Gamma(D_m) * \Gamma(D_n)$ will be developed as follows:

Definition 1.*Let $\Gamma(D_m)$ and $\Gamma(D_n)$ be the prime graphs of dihedral groups D_m and D_n ,respectively. Define the graf $\Gamma(D_m) * \Gamma(D_n)$, as follows:*
(i) Vertex set of the graph $\Gamma(D_m) * \Gamma(D_n)$ are
$V[\Gamma(D_m) * \Gamma(D_n)] = \{_1x \mid x \in V[\Gamma(D_m)]\} \cup \{_2x \mid y \in V[\Gamma(D_n)]\}$
(ii)Edge set of the graph $\Gamma(D_m) * \Gamma(D_n)$ are $E[\Gamma(D_m) * \Gamma(D_n)]$ and defined:
- $_1x_1y \in E(\Gamma(D_m) * \Gamma(D_n))$ if and only if $xy \in E(D_m)$;
- $_2x_2y \in E(\Gamma(D_m) * \Gamma(D_n))$ if and only if $xy \in E(D_n)$;
- Let $x \in V(D_m)$ and $y \in V(D_n)$, then $_1x_2y \in E(\Gamma(D_m) * \Gamma(D_n))$
Let $u \in D_m$ and $v \in D_n$. Then, $(u,v) \in E(\Gamma(D_m) * \Gamma(D_n))$ if and only if $xy \mid \mathrm{lcm}(o(u), o(v))$,

To construct a prime graph using definition 1, it is important to know the element order of two finite groups , where the element order of the direct product of two finite groups is the least common multiple of the element order of the finite group. The Concept of the least common multiple (lcm) becomes an essential tool in determining the order of elements group. As Gallian [4] notes,in the context of direct product groups, the order of an element is equal to the least common multiple of the orders of its components. Specifically, if an element $g = (g_1, g_2, ..., g_k)$ belongs to a direct product of finite groups $G_1 * G_2 * ... * G_k$, then $o(g) = \mathrm{lcm}(o(g_1), (g_2), ..., (g_k))$.
Based on Definition 1, we can find some graph patterns formed by the graph $\Gamma(D_m) * \Gamma(D_n)$ which are presented in the following lemma:
Let $D_n = \{e, r, r^2, , r^{n-1}, s, sr, sr^2, , sr^{n-1}\}$, with $|D_n| = 2n$,
where $o(e) = 1, o(r^i) = \frac{n}{\gcd(i,n)}, o(s) = o(sr^i) = 2$.
then $D_{p_1} = \{e_1, r_1, r_1^2, , r_1^{p_1-1}, s_1, s_1r_1, s_1r_1^2, s_1r_1^{p_1-1}\}$.
$D_{p_2} = \{e_2, r_2, r_2^2, , r_2^{p_2-1}, s_2, s_2r_2, s_2r_2^2, s_2r_2^{p_2-1}\}$.

Based on the definition of prime graph in Definition 1, we know :
$V(\Gamma(D_{p_1})) = \{2, p_1\}$ and $V(\Gamma(D_{p_2})) = \{2, p_2\}$, therefore based on the definition of $\Gamma(D_{p_1}) * \Gamma(D_{p_2})$, we know $V(\Gamma(D_{p_1}) * \Gamma(D_{p_2})) = \{_12, _1p_1, _22, _2p_2\}$

Lemma 1 and Lemma 2 represents the prime graph formed when given two dihedral groups with n odd, with p odd and p is equal or p is not equal.

Lemma 1. *Given a prime graph of dihedral group $\Gamma(D_p)$, with p odd primes, then $\Gamma(D_p)*\Gamma(D_p)$ forms the graph $2P_2$.*

Proof. It is known that $o(D_p)=\{o(e),o(r^i),o(s)\}=\{1,p,2\}$,
By $\Gamma(D_p)*\Gamma(D_p)$, known that $V(\Gamma(D_p)*\Gamma(D_p)) = \{_12,_1p,_22,_2p\}$.
By definition 1, we know that:
$2\cdot 2 \nmid \operatorname{lcm}(o(u),o(v))$, for all $u\in D_p$ and $v\in D_p$,
then vertices $_12$ and $_22$ not connected or $_12_22 \notin E(\Gamma(D_p)*\Gamma(D_p))$;
$2p \mid \operatorname{lcm}(o(s),o(r^i))$, then vertices $_12$ and $_2p$ connected or $_12_2p\in E(\Gamma(D_p)*\Gamma(D_p))$,
and vertices $_1p$ and $_22$ connected or $_1p_22\in E(\Gamma(D_p)*\Gamma(D_p))$;
$p^2 \nmid \operatorname{lcm}(o(u),o(v))$, for all $u\in D_p$ and $v\in D_p$,
then vertices $_1p$ and $_2p$ not connected or $_1p_2p\notin E(\Gamma(D_p)*\Gamma(D_p))$.
So, Γ (D_p) * Γ (D_p) forms the graph $2P_2$.

Lemma 2. *Given 2 prime graphs of dihedral group $\Gamma(D_{p_1^{k_1}})$ and $\Gamma(D_{p_2^{k_2}})$, with p_1,p_2 odd primes and $k_1,k_2\in N$, then $\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2^{k_2}})$ forms the graph P_4 , if it has one of the following 3 conditions:*
1. $p_1=p_2$, $(k_1\geq 2$ and $k_2=1)$ or $(k_1=1$ and $k_2\geq 2)$;
2. $p_1\neq p_2$, $(k_1\geq 2$ and $k_2=1)$ or $(k_1=1$ and $k_2\geq 2)$;
3. $p_1\neq p_2$, $k_1=k_2=1$.

Proof. 1.Choose $k_1\geq 2$ and $k_2=1,$ and $p_1=p_2=p$.
It is known that $o(D_{p_1^{k_1}})=\{o(e_1),o(r_1^i),o(s_1)\}=\{1,p,p^2,p^3,\dots,p^{k_1},2\}$,
and $o(D_{p_2})=\{o(e_2),o(r_2^i),o(s_2)\}=\{1,p,2\}$. By $\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2})$,
it is known that $V(\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2}))=\{_12,_1p,_22,_2p\}$.
And by definition 1, we know that :
$2\cdot 2\nmid \operatorname{lcm}(o(u),o(v))$, for all $u\in D_{p_1^{k_1}}$ and $v\in D_{p_2}$,
then vertices $_12$ and $_22$ not connected or $_12_22\notin E(\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2}))$;
$2p\mid \operatorname{lcm}(o(s_1),o(r_2^i))$, then vertices $_12$ and $_2p$ connected or $_12_2p\in E(\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2}))$,
and vertices $_1p$ and $_22$ connected or $_1p_22\in E(\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2}))$;
$p^2\mid \operatorname{lcm}(o(r_1^i),o(e_2))$, then vertices $_1p$ and $_2p$ connected or $_1p_2p\in E(\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2}))$.
So, $\Gamma(D_{p_1^{k_1}})*\Gamma(D_{p_2})$ with $p_1=p_2, k_1\geq 2$ and $k_2=1$ forms the graph P_4.
The same thing also passed when applied to conditions 2 and 3.

Lemma 3 and Lemma 4 represents the prime graph formed when given two dihedral groups with n even, with p odd and p is equal or p is not equal.

Lemma 3. *Given a prime graph of dihedral group $\Gamma(D_{2p})$, with p odd primes,
then $\Gamma(D_{2p})*\Gamma(D_{2p})$ forms the graph C_4.*

Proof. It is known that $|D_{2p}|=2\cdot 2p$ and $o(D_{2p})=\{o(e),o(r^i),o(s)\}=\{1,p,2p,2\}$, by Γ (D_{2p}) * Γ (D_{2p}) known that $V(\Gamma(D_{2p})*\Gamma(D_{2p}))=\{_12,_1p,_22,_2p\}$.
And by definition 1, we know that :
$2\cdot 2\nmid \operatorname{lcm}(o(u),o(v))$, for all $u\in D_{2p}$ and $v\in D_{2p}$,
then vertices $_12$ and $_22$ not connected or $_12_22\notin E(\Gamma(D_{2p})*\Gamma(D_{2p}))$;
$2p\mid \operatorname{lcm}(o(s),o(r))$, then vertices $_12$ and $_2p$ connected or $_12_2p\in E(\Gamma(D_{2p})*\Gamma(D_{2p}))$,
and vertices $_1p$ and $_22$ connected or $_1p_22\in E(\Gamma(D_{2p})*\Gamma(D_{2p}))$;
$p^2\nmid \operatorname{lcm}(o(u),o(v))$, for all $u\in D_{2p}$ and $v\in D_{2p}$,
then vertices $_1p$ and $_2p$ not connected or $_1p_2p\notin E(\Gamma(D_{2p})*\Gamma(D_{2p}))$.
So, $\Gamma(D_{2p})*\Gamma(D_{2p})$ forms the graph C_4.

Lemma 4. *Given 2 prime graphs of dihedral group $\Gamma(D_{2^{t_1}p_1^{k_1}})$ and $\Gamma(D_{2^{t_2}p_2^{k_2}})$ with p odd primes, and $t_1,t_2,k_1,k_2 \in N$ then $\Gamma(D_{2^{t_1}p_1^{k_1}}) * \Gamma(D_{2^{t_2}p_2^{k_2}})$ forms the graph $P_2 + \overline{P_2}$, if it has one of the following 4 conditions:*
1. $p_1 = p_2, \quad t_1 = t_2 = 1, \quad (k_1 \geq 2$ and $k_2 = 1)$ or $(k_1 = 1$ and $k_2 \geq 2)$;
2. $p_1 \neq p_2, \quad t_1 = t_2 = 1, \quad (k_1 \geq 2$ and $k_2 = 1)$ or $(k_1 = 1$ and $k_2 \geq 2)$;
3. $p_1 \neq p_2, \quad t_1 = t_2 = 1, \quad k_1 = k_2 = 1$;
4. $p_1 = p_2, \quad (t_1 \geq 2$ and $t_2 = 1)$ or $(t_1 = 1$ and $t_2 \geq 2), \quad k_1 = k_2 = 1$.

Proof. 1. Choose $k_1 \geq 2$ and $k_2 = 1, \quad t_1 = t_2 = 1$, and $p_1 = p_2 = p$.
It is known that $|D_{2p_1^{k_1}}| = 2 \cdot 2p_1^{k_1}$ and $o(D_{2p_1^{k_1}}) = \{o(e_1),\ o(r_1^i),\ o(s_1)\} =$
$\{1, p, p^2, \ldots, p^{k_1}, 2p, 2p^2, \ldots, 2p^{k_1}, 2\}$
and $|D_{2p_2}| = 2 \cdot 2p_2$, $o(D_{2p_2}) = \{o(e_2), \quad o(r_2^i), \quad o(s_2)\} = \{1, 2p, 2\}$.
By $\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2})$, known that $V(\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2})) = \{_1 2, {_1 p}, {_2 2}, {_2 p}\}$.
And by Definition 1, we know that :
$2 \cdot 2 \nmid \operatorname{lcm}(o(u), o(v))$, for all $u \in D_{2p_1^{k_1}}$ and $v \in D_{2p_2}$,
then vertices $_1 2$ and $_2 2$ not connected or $_1 2 _2 2 \notin E(\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2}))$;
$2p \mid \operatorname{lcm}(o(s_1), o(r_2^i))$, then vertices $_1 2$ and $_2 p$ connected or $_1 2 _2 p \in E(\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2}))$,
then vertices $_1 p$ and $_2 2$ connected or $_1 p _2 2 \in E(\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2}))$;
$p^2 \mid \operatorname{lcm}(o(r_1^i), o(e_2))$, then vertices $_1 p$ and $_2 p$ connected or $_1 p _2 p \in E(\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2}))$.
So, $\Gamma(D_{2p_1^{k_1}}) * \Gamma(D_{2p_2})$ with $p_1 = p_2, t_1 = t_2 = 1, k_1 \geq 2$ and $k_2 = 1$ forms the graph
$P_2 + \overline{P_2}$. The same thing also passed when applied to conditions $2, 3$ and 4.

From the preceding lemmas, it follows that there exists a structural similarity among the graphs derived from the dihedral groups. Therefore, we define an equivalnece relation $\sim$ on the set of all dihedral groups D_n as follows :

Definition 2. *Let $D = \{D_n | n \in \mathbb{N}, n \geq 3\}$ be the family of all dihedral groups of order $2n$. Define a relation $\sim$ on D by:*
$D_n \sim D_m \iff \pi(n) = \pi(m)$,
where denotes the set of all primes divisors of n.

This relation partitions the family of dihedral groups into equivalence classes according to the prime factors of their orders. Consequently the study of the resulting graph structures can be focused on these equivalence classes, each of which exhibits similar structural properties.
Based on the equivalence relation devined above, we obtain a fundamental result formulated in the following theorem :

Theorem 2. *Every dihedral group belonging to the same equivalence class under $\sim$ generates the same graph.*

Proof. Let D_n and D_m be two dihedral groups such that $D_n \sim D_m$. By Definition 2, the equivalence relation $\sim$ is defined as follows:
$D_n \sim D_m \iff \pi(n) = \pi(m)$, where $\pi(n)$ denotes the set of all prime divisors of n. It is known that the dihedral group D_n has order $2n$. Therefore, the element orders in D_n are entirely determined by the prime factorization of n.

Next, suppose that for each group D_n, a prime graph $\Gamma(D_n)$ is constructed, where the construction of the graph depends on the orders of the elements in the group. For instance, the vertex set of the graph consists of all prime divisors of the group order, and two distinct vertices are connected by an edge if there exists an element in D_n whose order is divisible by the product of those two primes.

Since $\pi(n) = \pi(m)$, it follows that:

The groups D_n and D_m share the same set of prime divisors. Hence, the types of element orders that appear in D_n and D_m are equivalent, as a result, the rule for constructing the prime graph yields an identical outcome for both $\Gamma(D_n)$ and $\Gamma(D_m)$.

In other words, the structure of the graph is the same as that derived from D_n and D_m .
$\rightarrow \Gamma(D_n) = \Gamma(D_m)$

Thus, every dihedral group that belongs to the same equivalence class under $\sim$ generates the same prime graph.

Theorem 3 represents the general form of the prime graph formed when given two dihedral groups with even n, with p odd primes where p is equal, p is not equal.

Theorem 3. *Given 2 prime graphs of the dihedral group $\Gamma\left(D_{2^{m_0}\prod_{i=1}^{n_1} p_{1i}^{m_i}}\right)$ and $\Gamma\left(D_{2^{l_0}\prod_{j=1}^{n_2} p_{2j}^{l_j}}\right)$, where $n_1, n_2 \in \mathbb{N}$, with p_{1i}, p_{2j} odd primes, then:*

$$\Gamma\left(D_{2^{m_0}\prod_{i=1}^{n_1} p_{1i}^{m_i}}\right) * \Gamma\left(D_{2^{l_0}\prod_{j=1}^{n_2} p_{2j}^{l_j}}\right)$$

with $m_0, m_1, \ldots, m_i, l_0, l_1, \ldots, l_j \in \mathbb{N}$ and $m_0, m_1, \ldots, m_i, l_0, l_1, \ldots, l_j \geq 2$ forms the graph K_n .

Proof. Let $i = 1, 2, 3, \ldots, n_1, j = 1, 2, 3, \ldots, n_2$
and $m_0, m_1, m_2, m_3, \ldots, m_{n_1}, l_0, l_1, l_2, l_3, \ldots, l_{n_2} \geq 1$,
it is known that $|D_{2^{m_0} p_{11}^{m_1} p_{12}^{m_2} \ldots p_{1n_1}^{m_{n_1}}}| = 2 \cdot 2^{m_0} p_{11}^{m_1} p_{12}^{m_2} \ldots p_{1n_1}^{m_{n_1}}$
and $|D_{2^{l_0} p_{21}^{l_1} p_{22}^{l_2} \ldots p_{2n_2}^{l_{n_2}}}| = 2 \cdot 2^{l_0} p_{21}^{l_1} p_{22}^{l_2} \ldots p_{2n_2}^{l_{n_2}}$ then
$o\left(D_{2^{m_0} p_{11}^{m_1} p_{12}^{m_2} \ldots p_{1n_1}^{m_{n_1}}}\right) = \{o(e_1), o(r_1^i), o(s_1)\}$
$= \{2^{i_0} p_{11}^{i_1} p_{12}^{i_2} \cdot \ldots \cdot p_{1n_1}^{i_{n_1}} \mid i_0 = 0, 1, \ldots, m_0;\ i_1 = 0, 1, \ldots, m_1;\ \ldots; i_{n_1} = 0, 1, \ldots, m_{n_1}\}$
and
$o\left(D_{2^{l_0} p_{21}^{l_1} p_{22}^{l_2} \ldots p_{2n_2}^{l_{n_2}}}\right) = \{o(e_2), o(r_2^i), o(s_2)\}$
$= \{2^{j_0} p_{21}^{j_1} p_{22}^{j_2} \cdot \ldots \cdot p_{2n_2}^{j_{n_2}} \mid j_0 = 0, 1, \ldots, l_0;\ j_1 = 0, 1, \ldots, l_1;\ \ldots; j_{n_2} = 0, 1, \ldots, l_{n_2}\}$
By
$\Gamma\left(D_{2^{m_0} p_{11}^{m_1} p_{12}^{m_2} \ldots p_{1n_1}^{m_{n_1}}}\right) \times \Gamma\left(D_{2^{l_0} p_{21}^{l_1} p_{22}^{l_2} \ldots p_{2n_2}^{l_{n_2}}}\right)$,
it is known that:

$$V\left(\Gamma\left(D_{2^{m_0} p_{11}^{m_1} \ldots p_{1n_1}^{m_{n_1}}}\right) \times \Gamma\left(D_{2^{l_0} p_{21}^{l_1} \ldots p_{2n_2}^{l_{n_2}}}\right)\right) = \{{}_1 2, {}_1 p_{11}, \ldots, {}_1 p_{1n_1}, {}_2 2, {}_2 p_{21}, \ldots, {}_2 p_{2n_2}\}.$$

By Definition 1, we know that:
$2 \cdot 2 \mid \mathrm{lcm}(o(e_1), o(r_2^i))$, then vertices

$${}_1 2 \text{ and } {}_2 2 \text{ connected or } {}_1 2 {}_2 2 \in E\left(\Gamma\left(D_{2^{m_0} p_{11}^{m_1} \ldots p_{1n_1}^{m_{n_1}}}\right) \times \Gamma\left(D_{2^{l_0} p_{21}^{l_1} \ldots p_{2n_2}^{l_{n_2}}}\right)\right);$$

Similarly, for all other terms such as $2p_{21}$, $2p_{22}$, etc., we follow the same logic to show that they are connected in the graph. Thus, the general form of the prime graph on the dihedral group is obtained as follows:

$p_{1n_1} p_{2n_2} \mid \mathrm{lcm}(o(r_1^i), o(r_2^i))$, then vertices,

$${}_1 p_{1n_1} \text{ and } {}_2 p_{2n_2} \text{ connected or } {}_1 p_{1n_1} {}_2 p_{2n_2} \in E\left(\Gamma\left(D_{2^{m_0} p_{11}^{m_1} \ldots p_{1n_1}^{m_{n_1}}}\right) \times \Gamma\left(D_{2^{l_0} p_{21}^{l_1} \ldots p_{2n_2}^{l_{n_2}}}\right)\right).$$

So, $\Gamma\left(D_{2^{m_0} p_{11}^{m_1} p_{12}^{m_2} p_{13}^{m_3} \ldots p_{1n_1}^{m_{n_1}}}\right) \times \Gamma\left(D_{2^{l_0} p_{21}^{l_1} p_{22}^{l_2} p_{23}^{l_3} \ldots p_{2n_2}^{l_{n_2}}}\right)$

with $i = 1, 2, 3, \ldots, n_1,\ j = 1, 2, 3, \ldots, n_2$,
and $m_0, m_1, m_2, m_3, \ldots, m_{n_1}, l_0, l_1, l_2, l_3, \ldots, l_{n_2} \geq 1$ forms the graph K_n .

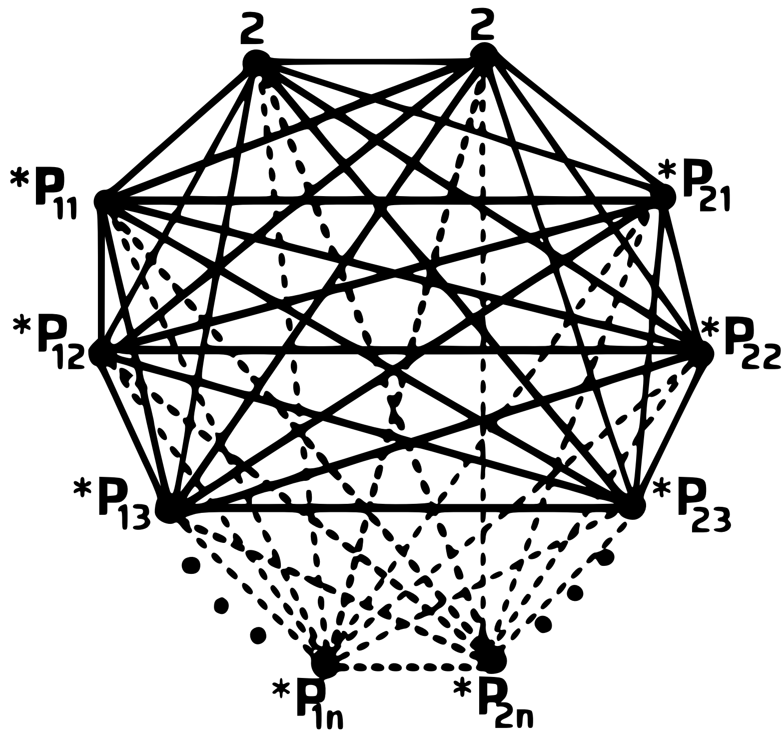


Fig. 2: $\Gamma\left(D_{2^{m_0} p_{11}^{m_1} p_{12}^{m_2} p_{13}^{m_3} \cdots p_{1n_1}^{m_{n_1}}}\right) * \Gamma\left(D_{2^{l_0} p_{21}^{l_1} p_{22}^{l_2} p_{23}^{l_3} \cdots p_{2n_2}^{l_{n_2}}}\right)$

Declarations

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