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On the burning number of the generalized Heawood graphs

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Abstract: Graph burning is a discrete-time process that models the spread of contagion or information in a network. The burning number $b(G)$ of a graph G is defined as the minimum number of discrete time steps required to ensure that all vertices are "burned", assuming that in each step, a new vertex is ignited and the fire spreads to all adjacent vertices. Suppose n,k are two natural numbers where $k \geq 3$ is odd and $n \geq k$. The generalized Heawood graph, which is a cubic bipartite graph of girth 4 or 6, denoted as $H(n,k)$ is the graph consisting of a $2n$-cycle $r_0r_1r_2\dots r_{2n-1}r_0$ together with edges of the form $r_{2i}r_{2i+k}, i = 0,1,2,\dots,n-1$. Here the operations on the subscripts are reduced modulo $2n$. In this paper, we propose ways to burn the generalized Heawood graphs and hence determine the burning number of all $H(n,k)$ of girth 4 and that of $H(2k,k)$ which is of girth 6.

Keywords: Burning number; generalized Heawood graphs.

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1 Introduction

The burning number $b(G)$ of a graph G is a time-discrete process that is used to study the spread of contagion or viral outbreaks such as COVID-19 in a modern network. The spreading model includes vertices infecting the neighbours. The concept of graph burning is inspired by firefighters played on graphs in [3,4]. Burning number seeks to minimize the number of rounds to completely burn a graph. The smaller the burning number, the faster the spreading of the contagion in a network. Consider a graph G. Initially, all vertices in G are unburned. Then, the beginning of every time step $t \geq 1$, an unburned vertex is choose to burn (if such a vertex is available). After that, if a vertex is burned in time step $t-1$, then in time step t, any unburned vertex adjacent to the burned vertex will be burned. A burned vertex will remain burned throughout the process. The process is completed when all vertices of the graph have been burned and we said the graph is burned.

Suppose a graph G is burned in m time steps (or rounds) in a burning process. We denote the vertex chosen to burn at the beginning of time step i by x_i for $1 \leq i \leq m$. Each x_i is a *burning source* of G and the sequence $(x_1,x_2,\dots,x_m)$ is a *burning sequence* for G.

Here, we introduce some terminology. The distance between any two vertices u and v in a graph G, denoted as $d_G(u,v)$, is the length of a shortest path from u to v in G. Note that $d_G(u,u) = 0$. Furthermore, we will write $d(u,v)$ for $d_G(u,v)$ if the context is clear. The *girth* of a graph G is the length of a shortest cycle contained in G. If G contains no cycles (i.e., G is acyclic), its girth is defined to be infinity. Given a non-negative integer s, the *s-th closed neighbourhood* of a vertex u, is the set of vertices whose distance from u is at most s, and denoted as

$$N_s^G[u] = \{v \in V(G) : \, d_G(u,v) \leq s\}.$$

Again, we may write $N_s[u]$ for $N_s^G[u]$ if the context in clear. The *burning number* of a graph G, $b(G)$, is the length of a shortest burning sequence for G. It is clear that $b(K_n) = 2$.

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For paths and cycles, Bonato, Janssen and Roshanbin [4] determined their burning numbers exactly.
Theorem 1.[4, Theorem 9 and Corollary 10] *Let P_n be a path with n vertices and C_n be a cycle with n vertices. Then*

$$b(P_n) = \left\lceil n^{1/2} \right\rceil = b(C_n).$$

By referring to the proof of Theorem 1 in [4], we can completely burn P_n in this way: Assume $k = \lceil n^{1/2} \rceil$ and $P_n : v_1 v_2 \ldots v_n$. Then for $0 \le i \le k-2$, we choose $x_{k-i} = v_{n-i^2-i}$. Also, if $n \ge (k-1)^2 + k$, we take $x_1 = v_{n-(k-1)^2-(k-1)}$; otherwise we take $x_1 = v_1$. Therefore, we can burn P_n in exactly k steps by the burning sequence $(x_1, x_2, \ldots, x_k)$. This burning sequence is also similar to C_n. For example, to burn the path $P_5 = v_1v_2v_3v_4v_5$, we may choose $x_1 = v_1$, $x_2 = v_3$, and $x_3 = v_5$, hence $b(P_5) \le 3$. To see that $b(P_5) > 2$, note that with only two burning sources, at most four vertices can be burned within two time steps. Therefore, $b(P_5) = 3$.

Besides that, given the radius r and diameter d of a graph G, they showed that $\lceil (d+1)^{1/2} \rceil \le b(G) \le r+1$. In the same paper, they gave an upper bound on the burning number of any connected graph G of order n, showing that $b(n) \le 2\sqrt{n} - 1$. This upper bound was later improved to roughly $\frac{\sqrt{6}}{2}\sqrt{n}$ by Land and Lu [10]. Bonato et al. [4] conjectured that $b(G) \le \lceil \sqrt{n} \rceil$ in [4] for any connected graph G of order n. Following this, Bonato and Lidbetter [5] verified this conjecture for spider graphs, which are trees with exactly one vertex of degree at least 3. Recently, Li et al. [14] determined the exact value of the burning number of Q graphs and confirm this conjecture for Q graph.

The burning problem is NP-complete [2]. In the past few years, the burning number has been widely studied for some families of graphs, for example, circulant graphs [7], generalised Petersen graphs [21], caterpillars [9, 15], grid and interval graphs[8], t-unicyclic graphs[24], hypercube [19], theta graphs [17], spiders and path forests [5, 6, 16]. Mitsche, Pralat and Roshanbin investigated the burning number of graph products in [19] and they also focused on the probabilistic aspects of the burning number in [18]. Bonato and Lidbetter [5] and Das et al. [6] proved that every spider of order m^2 is m-burnable. A graph G is said to be m-burnable if the burning number of G is at most m. A tight upper bound on the order of a spider to guarantee that it is m-burnable was then determined by Tan and Teh [22]. The authors proposed a stronger burning conjecture on tree that for $m > n \ge 2$, if T is a tree with n leaves and its order is at most $m^2 + n - 2$, then T is m-burnable. This strengthened form of the burning number conjecture has been verified for several important families of trees, including double spiders [23], balanced spider [11] and trees with sufficiently long arms [13]. Building on this line of research, Leong et al. [12] further demonstrated that for any disjoint union of an n-spider and a path, the same upper bound $m^2 + n - 2$ on the order ensures m-burnability, provided that $m \ge n + 3$, with only a few canonical exceptions. Collectively, these results reinforce the view that the number of leaves is a key structural parameter influencing the tightness of upper bounds on the burning number.

This paper is an initial step in exploring the case where G is cubic and of girth 4 and 6. The most natural example that comes to our mind is the generalized Heawood graph.

Definition 1.*Suppose n, k are two natural numbers where $k \ge 3$ is odd and $n \ge k$. The generalized Heawood graph, denoted as $H(n,k)$ is the graph consisting of a $2n$-cycle $r_0r_1r_2\ldots r_{2n-1}r_0$ together with edges of the form $r_{2i}r_{2i+k}, i = 0, 1, 2, \ldots, n-1$. Here the operations on the subscripts are reduced modulo $2n$. Edges of the $2n$-cycle are called the rim edges while the form $r_{2i}r_{2i+k}$ are called the diagonals.*

It is not difficult to see that $H(n,k)$ is a cubic bipartite graph. This class of cubic bipartite graphs was investigated in [1] for some other graph theoretical problems. The generalized Heawood graphs, $H(k,k)$ (which is isomorphic to Möbus ladder), $H(k+1,k)$ and $H(n,3)$ are of girth 4. Morever, it is of girth 6 if and only if $n \ge k+2$ and $k \ge 5$. In this paper, we determine the burning number of all $H(n,k)$ of girth 4 and that of $H(2k,k)$ which is of girth 6.

For a graph G, suppose $b(G) = m$. Then, we define the *maximum newly burned sequence by x_i* be the sequence of maximum number of vertices newly burned by x_i in $(i, i+1, \ldots, m)$-th round. It is denoted as $\{N_0[x_i], N_1[x_i] \setminus N_0[x_i], N_2[x_i] \setminus N_1[x_i], \ldots, N_{m-1}[x_i] \setminus N_{m-2}[x_i]\}$. In fact, $N_{j+1}[x_i] \setminus N_j[x_i]$ is the set of vertices of distance exactly $j+1$ from x_i.

2 Main results on burning number of $H(n,k)$ of girth 4

2.1 Burning number of $H(k+1,k)$

Theorem 2.*For every odd natural number $5 \le k \le 13$,*

$$b(H(k+1,k)) = \begin{cases} 3 & \text{if } k = 5; \\ 4 & \text{if } k = 7, 9, 11; \\ 5 & \text{if } k = 13. \end{cases}$$

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*Proof.*For $H(6,5)$, it can be burned by the burning sequence $(x_1,x_2,x_3) = (r_2,r_{10},r_5)$ and hence $b(H(6,5)) \leq 3$. For the lower bound, if $b(H(6,5)) \leq 2$, then two burning sources, x_1 and x_2 which maximally burn a total of 5 vertices, are not able to completely burn $H(6,5)$ and hence $b(H(6,5)) \geq 3$.

Note that $|H(8,7)| = 16, |H(10,9)| = 20, |H(12,11)| = 24$ and $|H(14,13)| = 28$. However, it can be easily observed that the number of vertices that can be maximally burned by (x_1,x_2,x_3) is $(1+3+5)+(1+3)+1 = 14$. Hence $b(H(k+1,k)) \geq 4$ for $7 \leq k \leq 13$.

To show the equality holds for $k = 7,9,11$, we only need to show that there exists a burning sequence (x_1,x_2,x_3,x_4) to completely burn the graph. For $H(8,7)$, we use the 4 burning sources provided by Theorem 1 to burn the path $r_0r_1\ldots r_{15}$ (consists of 16 vertices). For $H(10,9)$, let $(x_1,x_2,x_3,x_4) = (r_2,r_6,r_8,r_9)$. For $H(12,11)$, let $(x_1,x_2,x_3,x_4) = (r_3,r_9,r_{20},r_{11})$.

Now, we consider $H(14,13)$ and we will show that $b(H(14,13)) > 4$. Suppose $b(H(14,13)) = 4$ and let (x_1,x_2,x_3,x_4) be a burning sequence of $H(14,13)$. Since (x_1,x_2,x_3,x_4) can maximally burn $(1+3+5+5)+(1+3+5)+(1+3)+1 = 28$ vertices in $H(14,13)$, each of vertices in $H(14,13)$ is burned only once by any of the 4 burning sources. This means that $N_3[x_1]$, $N_2[x_2]$, $N_1[x_3]$ and $N_0[x_4]$ are disjoint sets, and $|N_3[x_1]| = 14$, $|N_2[x_2]| = 9$, $|N_1[x_3]| = 4$ and $|N_0[x_4]| = 1$. Since $H(14,13)$ is a vertex-transitive graph, without loss of generality, we place x_1 at r_0. Note that

$$N_3[x_1] = \{r_{25},r_{26},r_{27},r_0,r_1,r_2,r_3,r_{11},r_{12},r_{13},r_{14},r_{15},r_{16},r_{17}\}.$$

Let $J = H(14,13) \setminus N_3[x_1]$ be as in Figure 1. We will show that each vertex in J is impossible to be burned by the remaining 3 burning sources, namely x_2,x_3 and x_4, only once.

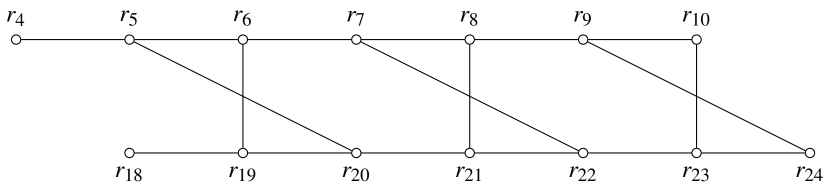


Fig. 1: $H(14,13) \setminus N_3[x_1]$

Consider the graph $J_1 = H(14,13) \setminus (N_3[x_1] \cup N_2[x_2])$. It has 5 vertices and at least one of the vertices must be of degree 3 because $x_3,x_4 \in J_1$, $|N_1[x_3]| = 4$ and $|N_0[x_4]| = 1$.

Now, $N_2[x_2] = 9$ and $N_2[x_2] \cap N_3[x_1] = \emptyset$ imply that $x_2 \notin \{r_4,r_5,r_9,r_{10},r_{18},r_{19},r_{23},r_{24}\}$. If $x_2 = r_6$, then J_1 is the graph spanned by the vertices $r_9,r_{10},r_{21},r_{23},r_{24}$. Clearly, none of the vertices is of degree 3. If $x_2 = r_7$, then J_1 is the graph spanned by the vertices $r_4,r_{10},r_{18},r_{20},r_{24}$. If $x_2 = r_8$, then J_1 is the graph spanned by the vertices $r_4,r_5,r_{18},r_{19},r_{23}$. Again, for $x_2 = r_7$ or r_8, none of the vertices in J_1 is of degree 3. Similarly, none of the vertices in J_1 is of degree 3 if $x_2 \in \{r_{20},r_{21},r_{22}\}$. Following these, $b(H(14,13)) \geq 5$. To show that the equality holds, $H(14,13)$ can be burned using burning sequence $(x_1,x_2,x_3,x_4,x_5) = (r_0,r_6,r_{10},r_{21},r_{22})$.

□

Figure 2 shows some generalized Heawood graph with burning source x_1,x_2,x_3,x_4,x_5 as shown in Theorem 2.

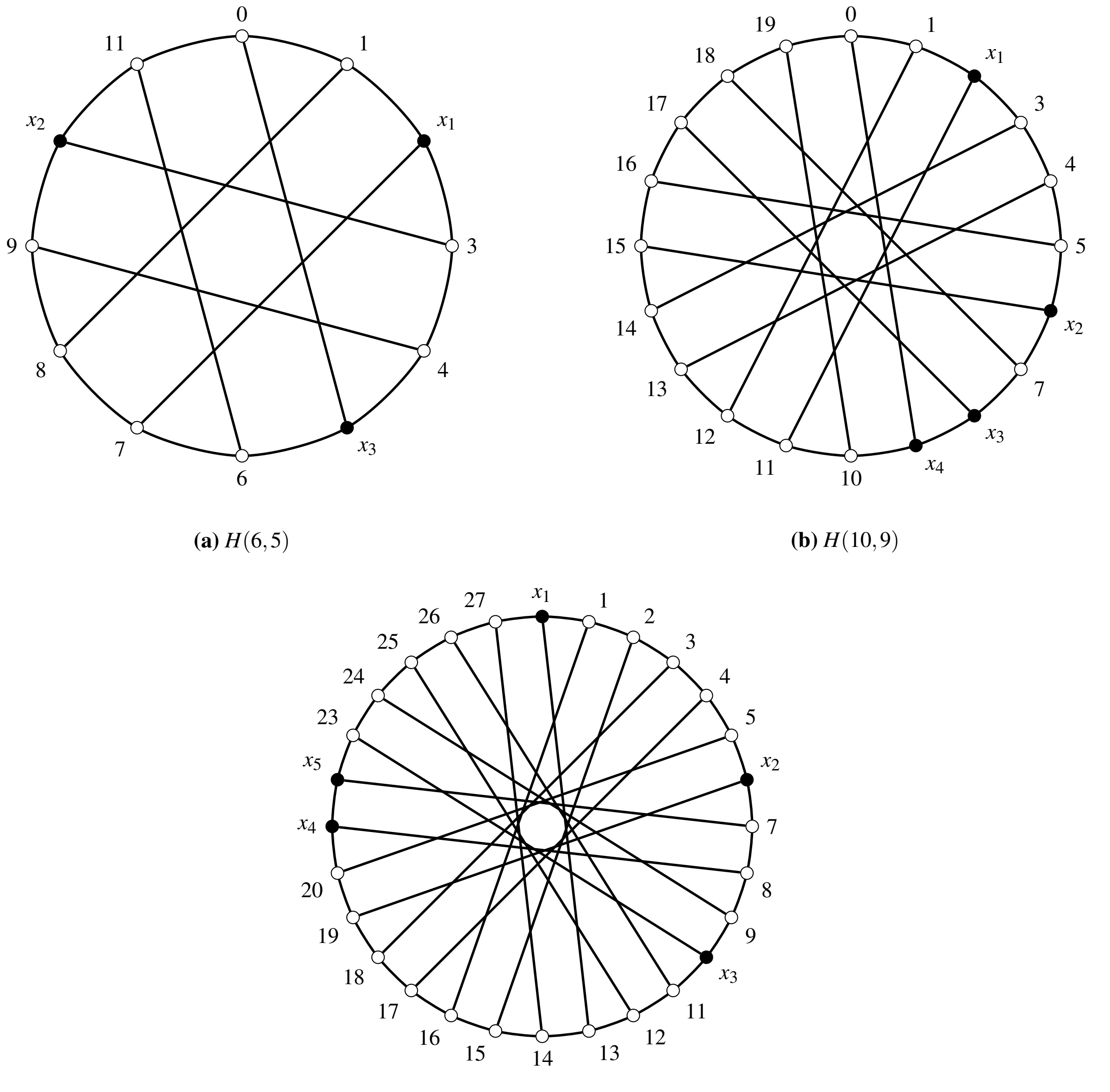
Lemma 1.*For every odd natural number $k \geq 15$,*

$$\sqrt{k+3} \leq b((H(k+1,k)) \leq \left\lceil \sqrt{k} \right\rceil + 1.$$

*Proof.*Let $G = H(k+1,k)$ and $(x_1,x_2,\ldots,x_i,\ldots,x_b)$ be the burning sequence of G. It is observed that for each burning source x_i, the maximum newly burned sequence by each x_i is

$$\{\overbrace{1,3,5,5,4,4,4,4\ldots}^{b-i+1 \text{ times}}\}.$$

If $b \leq 4$, then (x_1,x_2,x_3,x_4) can burn at most $(1+3+5+5)+(1+3+5)+(1+3)+1 = 28$ vertices. So, for $k \geq 15$, G contains at least 32 vertices and this requires at least 5 burning sources to burn the graph.



(a) $H(6,5)$ **(b)** $H(10,9)$

(c) $H(14,13)$

Fig. 2: Some generalized Heawood graph with minimum number of burning sources as in Theorem 2.

Hence the maximum number of vertices burned by $(x_1, x_2, \ldots x_b)$ where $b \geq 5$ is

$$
\begin{aligned}
&b+3(b-1)+5(b-2)+5(b-3)+4\sum_{r=1}^{b-4} r \\
=&14b-28+4\left(\frac{b-4}{2}\right)(b-3) \\
=&2b^2-4
\end{aligned}
$$

In order to completely burn G, then $2b^2-4 \geq 2(k+1)$ and this gives $b \geq \sqrt{k+3}$.

To show the upper bound, we burn the path $r_0r_1\dots r_{k-1}$ in the rim edges of G using $\left\lceil\sqrt{k}\right\rceil$ burning sources provided by Theorem 1, say $(x_1,x_2,\dots,x_{\lceil\sqrt{k}\rceil})$. For each $k\le i\le 2k-1$ (except possibly $i=k+1$), r_i is adjacent to a vertex in $\{r_0,r_1,\dots,r_{k-1}\}$. In fact, both r_{2k+1} and r_k are adjacent to r_0, r_{2k} is adjacent to r_{k-2}, r_{k+2} is adjacent to r_2, and for $k+3\le i\le 2k-1$, r_i is adjacent to r_{i-k} if i is odd, whereas r_i is adjacent to r_{i-k-2} if i is even. After that, by additionally burning r_{k+1} (or any unburned vertex if r_{k+1} is burned) as $x_{\lceil\sqrt{k}\rceil+1}$, we see that the burning sequence $(x_1,x_2,\dots,x_{\lceil\sqrt{k}\rceil},x_{\lceil\sqrt{k}\rceil+1})$ is able to burn all the vertices in G.

This completes the proof of the lemma.

□

Corollary 1. $b((H(16,15))=5$.

Proof. It follows from Lemma 1, by noting that

$$4<\sqrt{18}\le b((H(16,15))\le\left\lceil\sqrt{15}\right\rceil+1=5.$$

□

Corollary 2. *Let m,k be positive integers. If $m^2<k\le(m+1)^2$, then $\left\lceil\sqrt{k}\right\rceil=m+1$.*

Proof. Clearly, $m<\sqrt{k}\le m+1$. Therefore, $\left\lceil\sqrt{k}\right\rceil=m+1$.

□

Theorem 3. *Let $k\ge 17$ be odd. If m is the largest positive integer such that $m^2<k$, then*

$$b((H(k+1,k))=\begin{cases} m+1, & \text{if } m^2<k\le(m+1)^2-4;\\ m+2, & \text{if } (m+1)^2-2\le k\le(m+1)^2;\\ m+1 \text{ or } m+2, & \text{if } k=(m+1)^2-3.\end{cases}$$

Proof. By the choice of m, $m^2<k\le(m+1)^2$.

Suppose $(m+1)^2-2\le k\le(m+1)^2$. Then, by Lemma 1 and Corollary 2,

$$m+1<\sqrt{(m+1)^2+1}\le\sqrt{k+3}\le b((H(k+1,k))\le\left\lceil\sqrt{k}\right\rceil+1=(m+1)+1=m+2.$$

Hence, $b((H(k+1,k))=m+2$. Similarly, by Lemma 1, if $k=(m+1)^2-3$, then $m+1\le b((H(k+1,k))\le m+2$.

Suppose $m^2<k\le(m+1)^2-4$. By Lemma 1, $b((H(k+1,k))\ge\sqrt{k+3}>\sqrt{m^2+3}>m$. Thus, $b\ge m+1$. We proceed to show that $b\le m+1$. Consider the path $r_0r_1r_2\dots r_{k-1}$ be a path in the rim edges of G. We will use the burning sources $(x_1,x_2,\dots,x_{m+1})$ where $x_{m+1-i}=r_{(k-1)-i^2-i}$ for $1\le i\le m-1$ and $x_{m+1}=r_{2k-5}$ to completely burn the graph. As for x_1, we will show our choice of x_1 in Case 1 later. Clearly, $(k-1)-i^2-i\le k-3$ and

$$(k-1)-i^2-i\ge(k-1)-(m-1)^2-(m-1)>(m^2-1)-(m^2-2m+1)-m+1=m-1>0.$$

So, the set of integers $\{(k-1)-i^2-i\ :\ 1\le i\le m-1\}$ is contained in $\{0,1,2,\dots,k-1\}$.

Now, since $k\ge 17$ then $m\ge 4$. In the rest of the proof, we show that, by these choices of burning sources $x_1,x_2,\dots x_{m+1}$ and analysing its associated neighbourhood, each vertex in $H(k+1,k))$ is burned by at least of one the burning sources in $m+1$ rounds.

Case 1: First, we consider the first burning source x_1.

Case 1.1: If $(k-1)-m^2-m<0$, then we let $x_1=r_0$. Then the set of vertices

$$S=\{r_{2k+1},r_{2k},r_{2k-1},r_{2k-2},r_k,r_{k-1}\},$$

is contained in $N_m[x_1]$, because $d(x,r_0)\le 4$ for all $x\in S$. Next, for $1\le j\le(k-1)-m^2$,

$$d(r_j,r_0)\le j\le(k-1)-m^2=((k-1)-m^2-m)+m<m,$$

and

$$d(r_{k+j},r_0)\le j+1\le(k-1)-m^2+1=((k-1)-m^2-m)+m+1\le m.$$

So, all these vertices r_j, r_{k+j} for $1 \le j \le (k-1)-m^2$ are contained in $N_m[x_1]$. We claim that $r_{(2k-1)-m^2+1} \in N_m[x_1]$. In fact, if $(2k-1)-m^2+1$ is even, then $r_{(2k-1)-m^2+1}$ is adjacent to $r_{(k-1)-m^2-1}$. So,

$$d(r_{(2k-1)-m^2+1}, r_0) \le 1 + d(r_{(k-1)-m^2-1}, r_0) \le (k-1)-m^2 < m,$$

and $r_{(2k-1)-m^2+1} \in N_m[x_1]$. Suppose $(2k-1)-m^2+1$ is odd. Assume that $(k-1)-m^2-2 \ge 0$. Then $r_{(2k-1)-m^2+1}$ is adjacent to $r_{(2k-1)-m^2}$. Since $(2k-1)-m^2$ is even, $r_{(2k-1)-m^2}$ is adjacent to $r_{(k-1)-m^2-2}$. So,

$$d(r_{(2k-1)-m^2+1}, r_0) \le 2 + d(r_{(k-1)-m^2-2}, r_0) \le (k-1)-m^2 < m,$$

and $r_{(2k-1)-m^2+1} \in N_m[x_1]$. Assume that $(k-1)-m^2-2 < 0$. Then $(2k-1)-m^2+1 = ((k-1)-m^2-2)+k+3 < k+3$. The path $r_0 r_k r_{k+1} r_{k+2} r_{k+3}$ is contained in G. Since $m \ge 4$, $r_{(2k-1)-m^2+1} \in N_m[x_1]$.

Case 1.2: If $(k-1)-m^2-m \ge 0$, then we choose $x_1 = r_{(k-1)-m^2-m}$. Since

$$(k-1)-m^2-m-(m-4) \le (m+1)^2-5-m^2-2m+4 \le 0,$$

we have $d(x_1, r_0) \le m-4$. Together with $d(x, r_0) \le 4$ for all $x \in S$, we conclude that $S \subseteq N_m[x_1]$. Furthermore, $N_m[x_1]$ contains the following two sets

$$S_1[1] = \{r_j \ : \ 0 \le j \le (k-1)-m^2\};$$
$$S_1[2] = \{r_{k+j} \ : \ 1 \le j \le (k-1)-m^2-1\}.$$

We claim that $r_{(2k-1)-m^2+1}, r_{(2k-1)-m^2} \in N_m[x_1]$. In fact, if $(2k-1)-m^2$ is even, then $r_{(2k-1)-m^2}$ is adjacent to $r_{(k-1)-m^2-2}$. Since

$$d(r_{(k-1)-m^2-2}, x_1) \le m-2,$$

we have $d(r_{(2k-1)-m^2}, x_1) \le m-1$ and $d(r_{(2k-1)-m^2+1}, x_1) \le m$. So, $r_{(2k-1)-m^2+1}, r_{(2k-1)-m^2} \in N_m[x_1]$. If $(2k-1)-m^2$ is odd, then $(2k-1)-m^2+1$ is even. So, $r_{(2k-1)-m^2+1}$ is adjacent to $r_{(k-1)-m^2-1}$. Since

$$d(r_{(k-1)-m^2-1}, x_1) \le m-1,$$

we have $d(r_{(2k-1)-m^2+1}, x_1) \le m$ and $r_{(2k-1)-m^2+1} \in N_m[x_1]$. Now, $(2k-1)-m^2-1$ is even implies that $r_{(2k-1)-m^2-1}$ is adjacent to $r_{(k-1)-m^2-3}$. Since

$$d(r_{(k-1)-m^2-3}, x_1) \le m-3,$$

we have $d(r_{(2k-1)-m^2}, x_1) \le 2+(m-3) \le m-1$ and $r_{(2k-1)-m^2} \in N_m[x_1]$.

Hence, in either case for choices of x_1, $N_m[x_1]$ contains S and the following three sets

$$S_1[1] = \{r_j \ : \ 0 \le j \le (k-1)-m^2\};$$
$$S_1[2] = \{r_{k+j} \ : \ 1 \le j \le (k-1)-m^2-1\};$$
$$S_1[3] = \{r_j \ : \ (2k-1)-m^2 \le j \le (2k-1)-m^2+1\}.$$

Case 2: Now, we let $x_{m+1-i} = r_{(k-1)-i^2-i}$ where $1 \le i \le m-1$ be fixed.

Clearly, $(k-1)-i^2-i$ is even. So, $N_i[x_{m+1-i}]$ contains the following two sets

$$S_{m+1-i}[1] = \{r_{(k-1)-i^2-i+j} \ : \ -i \le j \le i\};$$
$$S_{m+1-i}[2] = \{r_{(2k-1)-i^2-i+j} \ : \ -i+1 \le j \le i-1\}.$$

Case 2.1: Suppose $3 \le i \le m-1$. We claim that $r_{(2k-1)-i^2}, r_{(2k-1)-i^2+1} \in N_i[x_{m+1-i}]$ for $3 \le i \le m-1$. If $(2k-1)-i^2$ is even, then $(2k-1)-i^2$ is adjacent to $r_{(k-1)-i^2-2}$. Since

$$d(r_{(k-1)-i^2-2}, x_{m+1-i}) \le i-2,$$

we have $d(r_{(2k-1)-i^2}, x_{m+1-i}) \leq i-1$ and $d(r_{(2k-1)-i^2+1}, x_{m+1-i}) \leq i$. So, $r_{(2k-1)-i^2+1}, r_{(2k-1)-i^2} \in N_i[x_{m+1-i}]$. If $(2k-1)-i^2$ is odd, then $(2k-1)-i^2+1$ is even. So, $r_{(2k-1)-i^2+1}$ is adjacent to $r_{(k-1)-i^2-1}$. Since

$$d(r_{(k-1)-i^2-1}, x_{m+1-i}) \leq i-1,$$

we have $d(r_{(2k-1)-i^2+1}, x_{m+1-i}) \leq i$ and $r_{(2k-1)-i^2+1} \in N_i[x_{m+1-i}]$. Now, $(2k-1)-i^2-1$ is even implies that $r_{(2k-1)-i^2-1}$ is adjacent to $r_{(k-1)-i^2-3}$. Note that $(k-1)-i^2-3 > (k-1)-i^2-i$, for $i \geq 4$. From

$$d(r_{(k-1)-i^2-3}, x_{m+1-i}) \leq i-3,$$

we have $d(r_{(2k-1)-i^2}, x_1) \leq 2+(i-3) \leq i-1$ and $r_{(2k-1)-i^2} \in N_i[x_{m+1-i}]$.

Case 2.2: Suppose $i = \{1,2\}$. $N_2[x_{m-1}]$ contains the following two sets

$$\begin{aligned} S_{m-1}[1] &= \{r_{(k-1)-6+j} \,:\, -2 \leq j \leq 2\}; \\ S_{m-1}[2] &= \{r_{(2k-1)-6+j} \,:\, -1 \leq j \leq 1\}. \end{aligned}$$

and $N_1[x_m]$ contains the following set two sets

$$\begin{aligned} S_m[1] &= \{r_{(k-1)-2+j} \,:\, -1 \leq j \leq 1\}; \\ S_m[2] &= \{r_{2k-3}\}. \end{aligned}$$

Case 3: Recall that $x_{m+1} = r_{2k-5}$. So, $r_{2k-5} \in N_0[x_{m+1}]$. Since r_{2k-4} is adjacent to $r_{k-6} \in S_{m-1}[1]$, we have $r_{2k-4} \in N_2[x_{m-1}]$. Collecting back all the vertices contained in $\bigcup_{0 \leq i \leq m} N_i[x_{m+1-i}]$ in the above argument, we see that all the vertices are burnt. Hence, $b \leq m+1$ and this completes the proof of the theorem.

$\square$

2.2 *Burning number of $H(n,3)$ and $H(n,n)$*

Proposition 1. *$H(n,3)$ and $H(n,n)$ are isomorphic when n is odd.*

Proof. Let $V(H(n,3)) = V(H(n,n)) = \{0,1,2,\ldots,2n-1\}$ and $\varphi : V(H(n,3)) \to V(H(n,n))$ be defined by

$$\varphi(x) = \begin{cases} 2r, & \text{if } x = 4r, r = 0,1,2,\ldots,\frac{2n-2}{4} \\ n+2r, & \text{if } x = 4r+1, r = 0,1,2,\ldots,\frac{2n-2}{4}; \\ n+2r+1, & \text{if } x = 4r+2, r = 0,1,2,\ldots,\frac{2n-6}{4}; \\ 2r+1, & \text{if } x = 4r+3, r = 0,1,2,\ldots,\frac{2n-6}{4}. \end{cases}$$

It is not hard to see that φ is a bijection function. So, it is sufficient to show that for every vertex $x \in V(H(n,3))$, $\varphi(N(x)) = N(\varphi(x))$, i.e., the restriction of φ to $N(x)$ onto $N(\varphi(x))$ is bijective.

If $x = 0$, then $\varphi(x) = 0$ and

$$\begin{aligned} N(\varphi(x)) = N(0) &= \{2n-1, 1, n\} \\ &= \{\varphi(2n-1), \varphi(3), \varphi(1)\} \\ &= \varphi(\{2n-1, 3, 1\}) \\ &= \varphi(N(0)) = \varphi(N(x)). \end{aligned}$$

If $x = 4r$ and $r \geq 1$, then $\varphi(x) = 2r$ and

$$\begin{aligned} N(\varphi(x)) = N(2r) &= \{2r-1, 2r+1, 2r+n\} \\ &= \{2(r-1)+1, 2r+1, 2r+n\} \\ &= \{\varphi(4(r-1)+3), \varphi(4r+3), \varphi(4r+1)\} \\ &= \varphi(\{4r-1, 4r+3, 4r+1\}) \\ &= \varphi(N(4r)) = \varphi(N(x)). \end{aligned}$$

If $x=1$, then $\varphi(x)=n$ and

$$\begin{aligned}N(\varphi(x))=N(n)&=\{n-1,n+1,0\}\\&=\{\varphi(2n-2),\varphi(2),\varphi(0)\}\\&=\varphi(\{2n-2,2,0\})\\&=\varphi(N(1))=\varphi(N(x)).\end{aligned}$$

If $x=4r+1$ and $r\geq 1$, then $\varphi(x)=n+2r$ and

$$\begin{aligned}N(\varphi(x))=N(n+2r)&=\{n+2r-1,n+2r+1,2r\}\\&=\{n+2(r-1)+1,\varphi(4r+2),\varphi(4r)\}\\&=\{\varphi(4(r-1)+2),\varphi(4r+2),\varphi(4r)\}\\&=\varphi(N(4r+1))=\varphi(N(x)).\end{aligned}$$

If $x=4r+2$, then $\varphi(x)=n+2r+1$ and

$$\begin{aligned}N(\varphi(x))=N(n+2r+1)&=\{n+2r,n+2r+2,2n+2r+1\}\\&=\{\varphi(4r+1),n+2(r+1),2r+1\}\\&=\{\varphi(4r+1),\varphi(4(r+1)+1),\varphi(4r+3)\}\\&=\varphi(N(4r+2))=\varphi(N(x)).\end{aligned}$$

If $x=4r+3$, then $\varphi(x)=2r+1$ and

$$\begin{aligned}N(\varphi(x))=N(2r+1)&=\{2r,2r+2,2r+1-n\}\\&=\{\varphi(4r),\varphi(4(r+1)),2r+n+1\}\\&=\{\varphi(4r),\varphi(4(r+1)),\varphi(4r+2)\}\\&=\varphi(N(4r+3))=\varphi(N(x)).\end{aligned}$$

□

By Proposition 1, we may just consider $H(n,n)$ instead of $H(n,3)$ in the case when n is odd.

Lemma 2. *For every positive integer $n\geq 3$,*

$$b(H(n,3))\geq\sqrt{n-\frac{1}{4}}+\frac{1}{2};\qquad\text{and}\qquad b(H(n,n)\geq\sqrt{n-\frac{1}{4}}+\frac{1}{2}\quad\text{if } n \text{ is odd.}$$

Proof. Let G be the graph $H(n,3)$ or $H(n,n)$ and $(x_1,x_2,\ldots x_b)$ be the burning sequence of G. It is observed that for each burning source x_i, the maximum newly burned sequence by each x_i is

$$\overbrace{\{1,3,4,4,4,4,4,4\ldots\}}^{b-i+1\text{ times}}.$$

Clearly, for G, the burning number of G is at least 3. Otherwise, x_1 and x_2 can at most burn 5 vertices. Hence $b(G)\geq 3$.

Now, the maximum number of vertices burned by $(x_1,x_2,\ldots x_b)$ where $b\geq 3$ is

$$\begin{aligned}&b+3(b-1)+4\sum_{r=1}^{b-2}r\\=&4b-3+4\left(\frac{b-2}{2}\right)(b-1)\\=&2b^2-2b+1\end{aligned}$$

In order to completely burn G, we must have $2b^2-2b+1\geq 2n$. So, $\left(b-\frac{1}{2}\right)^2\geq n-\frac{1}{4}$. Since $b\geq 3$, we have $b\geq\sqrt{n-\frac{1}{4}}+\frac{1}{2}$.

□

Lemma 3. *For every odd natural number $n \geq 3$,*

$$b(H(n,n)) \leq \lceil \sqrt{n} \rceil + 1.$$

Proof. First, we burn the path $P = r_0 r_1 r_2 \dots r_{n-1}$ in $H(n,n)$ by using the $\lceil \sqrt{n} \rceil$ burning sources provided by Theorem 1. Then with additional one burning sources added in any unburned vertices, all vertices in $\{r_n, r_{n+1}, \dots, r_{2n-1}\}$ are burned. This completes the proof.

□

Theorem 4. *For an odd integer $n \geq 3$, let G be $H(n,3)$ or $H(n,n)$. If m is the largest positive integer such that $m^2 < n$, then*

$$b(G) = \begin{cases} m+1, & \text{if } m^2 < n \leq (m+1)^2 - m - 1; \\ m+2, & \text{if } (m+1)^2 - m \leq n \leq (m+1)^2. \end{cases}$$

Proof. Note that $m^2 < n \leq (m+1)^2$. By Corollary 2, $\lceil \sqrt{n} \rceil = m+1$. Suppose $n \geq (m+1)^2 - m$. Then by Lemmas 2 and 3,

$$m+1 < \sqrt{(m+1)^2 - m - \frac{1}{4}} + \frac{1}{2} \leq \sqrt{n - \frac{1}{4}} + \frac{1}{2} \leq b(G) \leq \lceil \sqrt{n} \rceil + 1 = (m+1) + 1 = m+2.$$

Hence, $b(G) = m+2$.

Suppose $n \leq (m+1)^2 - m - 1 = m^2 + m$. Since $n \geq 3$, then $m \geq 2$. By Lemma 2, $b(G) \geq \sqrt{n - \frac{1}{4}} + \frac{1}{2} > \sqrt{m^2 - \frac{1}{4}} + \frac{1}{2} > m$. Thus, $b(G) \geq m+1$. We proceed to show that $b(G) \leq m+1$. We may assume that $G = H(n,n)$ as $H(n,n)$ and $H(n,3)$ are isomorphic (see Proposition 1). In the rest of the proof, we will assign the burning sources $\{x_1, x_2, \dots, x_{m+1}\}$ in G to completely burn the graph. First, we assign x_m and x_{m-1}. Then, the set of burning sources $\{x_2, x_3, \dots, x_{m-2}\}$ is assigned based on Conditions (I) or (II). Following this, we assign x_1 based on (III) or (IV), which illustrate the unburned vertices in the graph after the associated burned neighbood of the set of vertices $\{x_2, x_3, \dots, x_m\}$ in $m+1$ rounds. Lastly, we assigned x_{m+1} to any unburned vertex in the last round.

First, we set $x_m = r_{n-3}$. Then $N_1[x_m] = \{r_{n-4}, r_{n-3}, r_{n-2}, r_{2n-3}\}$ (see Figure 3).

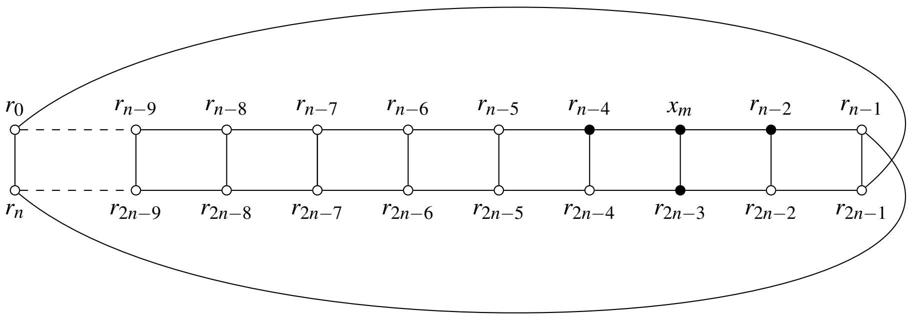


Fig. 3: $H(n,n) : N_1[x_m]$

Set $x_{m-1} = r_{2n-6}$. Then $N_2[x_{m-1}] = \{r_{2n-8}, r_{2n-7}, r_{2n-6}, r_{2n-5}, r_{2n-4}, r_{n-7}, r_{n-6}, r_{n-5}\}$ (see Figure 4).

So, $\bigcup_{1 \leq i \leq 2} N_i[x_{m+1-i}]$ contains all the vertices except r_{n-1}, r_{2n-2}, r_{2n-1}, and the vertices below,

$$\begin{array}{c} r_0,\ r_1,\ \cdots,\ r_{n-9},\ r_{n-8}, \\ r_n,\ r_{n+1},\ \cdots,\ r_{2n-9}. \end{array}$$

Now, we demonstrate how to assign $x_{m-2}, x_{m-3}, \dots, x_2$ and show the associated neighbourhood of these vertices in $m+1$ rounds. Following the fixed x_m and x_{m-1}, either Condition (I) or (II) (see below in this proof) holds after each x_{m+1-j} has been chosen for some $2 \leq j \leq m-2$. Then, each x_{m-j} is chosen based on Condition (I) or (II). That is, for instance, let $j = 2$. After selecting x_{m-1}, one of the Conditions (I) or (II) holds. Subsequently, x_{m-2} is chosen based on whether Condition (I) or Condition (II) applies. Once x_{m-2} is determined, again either Condition (I) or (II) holds, and x_{m-3} is then chosen again accordingly. This process continues until x_3 (i.e. $j = m-2$) is chosen, and finally, x_2 is determined based on whether Condition (I) or (II) holds.

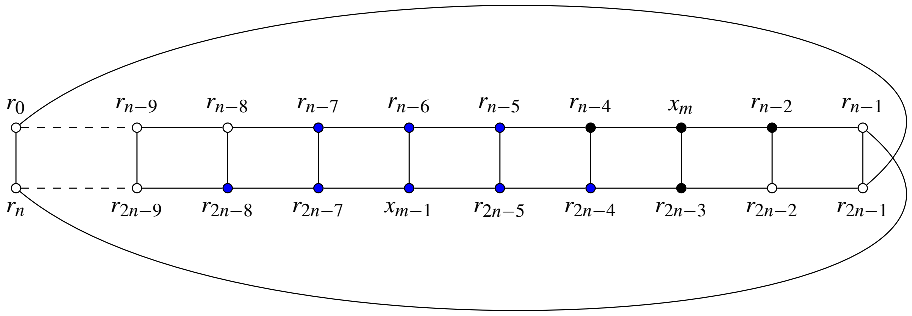


Fig. 4: $H(n,n) : N_1[x_m] \cup N_2[x_{m-1}]$

Condition (I) $\bigcup_{1\le i\le j} N_i[x_{m+1-i}]$ contains all the vertices except r_{n-1}, r_{2n-2}, r_{2n-1}, and the vertices below,

$$\begin{array}{l} r_0,\ \ r_1,\ \ \cdots,\ \ r_{t_j-1},\ \ r_{t_j},\\ r_n,\ r_{n+1},\ \cdots,\ r_{n+t_j-1}. \end{array}$$

Condition (II) $\bigcup_{1\le i\le j} N_i[x_{m+1-i}]$ contains all the vertices except r_{n-1}, r_{2n-2}, r_{2n-1}, and the vertices below,

$$\begin{array}{l} r_0,\ \ r_1,\ \ \cdots,\ \ r_{t_j-1},\\ r_n,\ r_{n+1},\ \cdots,\ r_{n+t_j-1},\ r_{n+t_j}. \end{array}$$

If Condition (I) holds, we set $x_{m-j} = r_{t_j-j-1}$. Since

$$\begin{aligned} d(r_{t_j}, r_{t_j-j-1}) &= j+1 = d(r_{t_j-2j-2}, r_{t_j-j-1}),\\ d(r_{n+t_j-j-1}, r_{t_j-j-1}) &= 1, \text{ and}\\ d(r_{n+t_j-1}, r_{n+t_j-j-1}) &= j = d(r_{n+t_j-2j-1}, r_{n+t_j-j-1}), \end{aligned}$$

we see that $|N_{j+1}[x_{m-j}]| = (2(j+1)+1)+(2j+1) = 4j+4$ and $N_{j+1}[x_{m-j}]$ contains all the vertices r_i's where $t_j-2j-2 \le i \le t_j$ and $n+t_j-2j-1 \le i \le n+t_j-1$. Therefore, $\bigcup_{1\le i\le j+1} N_i[x_{m+1-i}]$ contains all the vertices except r_{n-1}, r_{2n-2}, r_{2n-1}, and the vertices below,

$$\begin{array}{l} r_0,\ \ r_1,\ \ \cdots,\ \ r_{t_j-2j-3},\\ r_n,\ r_{n+1},\ \cdots,\ r_{n+t_j-2j-3},\ r_{n+t_j-2j-2}. \end{array}$$

If Condition (II) holds, we set $x_{m-j} = r_{n+t_j-j-1}$. Similarly, $|N_{j+1}[x_{m-j}]| = 4j+4$ and $N_{j+1}[x_{m-j}]$ contains all the vertices r_i's where $t_j-2j-1 \le i \le t_j-1$ and $n+t_j-2j-2 \le i \le n+t_j$. Therefore, $\bigcup_{1\le i\le j+1} N_i[x_{m+1-i}]$ contains all the vertices except r_{n-1}, r_{2n-2}, r_{2n-1}, and the vertices below,

$$\begin{array}{l} r_0,\ \ r_1,\ \ \cdots,\ \ r_{t_j-2j-3},\ \ r_{t_j-2j-2},\\ r_n,\ r_{n+1},\ \cdots,\ r_{n+t_j-2j-3}. \end{array}$$

Now, we have chosen $x_2, x_3, \ldots, x_m$.

To choose x_1, we consider the unburned vertices by $\{x_2, x_2, x_3, \ldots, x_m\}$ after $m+1$ round of burning, that is conditions (III) or (IV) as below. First, note that, by construction, for $1 \le i, i' \le m-1$, $N_i[x_{m+1-i}] \cap N_{i'}[x_{m+1-i'}] = \emptyset$ for $i \ne i'$ and $|N_i[x_{m+1-i}]| = 4i$. So,

$$\left|\bigcup_{1\le i\le m-1} N_i[x_{m+1-i}]\right| = \sum_{1\le i\le m-1} 4i = 2(m-1)m = 2m^2-2m.$$

The number of vertices that are not contained in $\bigcup_{1\le i\le m-1} N_i[x_{m+1-i}]$ is $L = 2n-(2m^2-2m)$ and

$$2m+2 = 2(m^2+1)-2m^2+2m \le L \le 2(m^2+m)-2m^2+2m = 4m.$$

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After the selection of $x_2, x_3, \ldots, x_m$, there are two possibilities:
(III) $\bigcup_{1\le i\le m-1} N_i[x_{m+1-i}]$ contains all the vertices except vertices below,

$$r_{2n-2},\ r_{2n-1},\ r_0,\ r_1,\ \cdots,\ r_{t_{m-1}-1},\ r_{t_{m-1}},$$
$$r_{n-1},\ r_n,\ r_{n+1},\ \cdots,\ r_{n+t_{m-1}-1}.$$

(IV) $\bigcup_{1\le i\le m-1} N_i[x_{m+1-i}]$ contains all the vertices except the vertices below,

$$r_{2n-2},\ r_{2n-1},\ r_0,\ r_1,\ \cdots,\ r_{t_{m-1}-1},$$
$$r_{n-1},\ r_n,\ r_{n+1},\ \cdots,\ r_{n+t_{m-1}-1},\ r_{n+t_{m-1}}.$$

We will assume that (III) holds. The case (IV) is similar. The number of vertices that are left is $L = 2(t_{m-1}+2)$. Therefore, $2(t_{m-1}+2) \ge 2m+2$, i.e., $t_{m-1} \ge m-1 > 0$ (recall that $m \ge 2$), and $2(t_{m-1}+2) \le 4m$, i.e., $t_{m-1}+2 \le 2m$. Consider the path of length $t_{m-1}+3$ in G:

$$P_0 = r_{2n-2}, r_{2n-1}, r_0, r_1, \ldots, r_{t_{m-1}-1}, r_{t_{m-1}}.$$

Since $t_{m-1}+3 \le 2m+1$, we can choose a vertex $x_1 = r_{t_m}$ along the path P_0 so that $d_{P_0}(x_1, r_{t_{m-1}}) \le m$ and $d_{P_0}(x_1, r_{2n-2}) \le m$. Here, $t_m \in \{2n-2, 2n-1, 0, 1, \ldots, t_{m-1}\}$. Consider the path of length $t_{m-1}+1$ in G:

$$P_1 = r_{n-1}, r_n, r_{n+1}, \ldots, r_{n+t_{m-1}-1}.$$

Since r_{t_m} is adjacent to r_{t_m+n}, $d_{P_1}(r_{n+t_m}, r_{n+t_{m-1}-1}) \le m-1$ and $d(r_{n+t_m}, r_{n-1}) \le m-1$, we see that $N_m[x_1]$ contains the remaining vertices.

Lastly, we can choose x_{m+1} to be any unburned vertex. Hence, $b(G) \le m+1$. This completes the proof of the theorem. □

3 Burning number of $H(n,k)$ of girth 6

3.1 Burning number of $H(2k,k)$ for $3 \le k \le 15$

Theorem 5. *For every odd natural number $3 \le k \le 13$,*

$$b(H(2k,k)) = \begin{cases} 3 & \text{if } k = 3; \\ 4 & \text{if } k = 5,7; \\ 5 & \text{if } k = 9,11,13 \end{cases}$$

Proof. If $k = 3$, then by Theorem 4, we have $b(H(6,3)) = 3$.

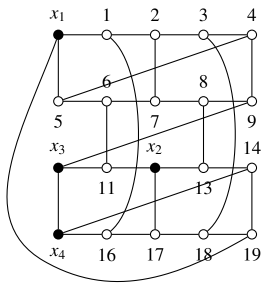
Now, let $G = H(2k,k)$ for $5 \le k \le 13$ and $(x_1, x_2, \ldots x_b)$ be the burning sequence of G. It is observed that for each burning source x_i, the maximum newly burned sequence that can be burned by each x_i is at most

$$\overbrace{\{1,3,6,9,9,8,8,8\ldots\}}^{b-i+1 \text{ times}}.$$

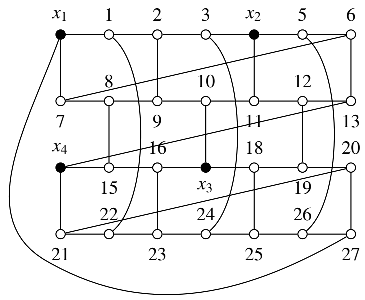
Then, in each G, the burning sequence (x_1, x_2, x_3, x_4) can burn at most $(1+3+6+9)+(1+3+6)+(1+3)+1 = 34$ vertices; the burning sequence (x_1, x_2, x_3) can burn at most 15 vertices; and the burning sequence (x_1, x_2) can burn at most 5 vertices. Since $H(2k,k)$ contains $4k$ vertices, we can easily see that

$$b(H(2k,k)) \ge \begin{cases} 4 & \text{if } k = 5,7; \\ 5 & \text{if } k = 9,11,13 \end{cases}$$

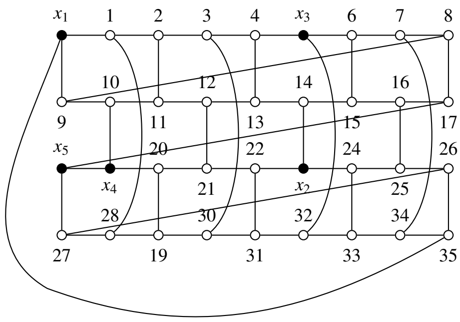
To show that equality holds for odd $5 \le k \le 13$, it is sufficient to show that there exists a burning sequence that can completely burn the graph. Here, by referring to Figure 5, we give the burning sources that can completely burn each of the graph. For simplicity, we label r_i as i in the figure. This completes the proof of the theorem. □



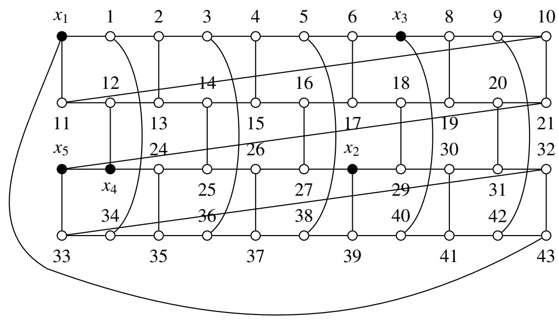
Burning of $b(H(10,5)) \leq 4$



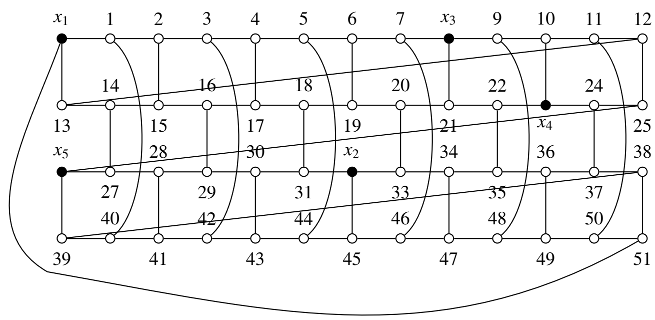
Burning of $b(H(14,7)) \leq 4$



Burning of $b(H(18,9)) \leq 5$



Burning of $b(H(22,11)) \leq 5$



Burning of $b(H(26,13)) \leq 5$

Fig. 5: Burning of $H(2k,k)$ for $5 \leq k \leq 13$

Theorem 6. *The burning number of $H(30,15)$ is 6.*

Proof. Let $G = H(30,15)$ with 60 vertices.

To show that $b(G) \geq 6$, we will show that G is not possible to be burned by 5 burning sources. Suppose $b(G) \leq 5$ and this implies that there exists a burning sequence $(x_1, x_2, x_3, x_4, x_5)$ to completely burn G. Since each of the vertices of G is vertex-transitive, without loss of generality, we consider $x_1 = r_0$. By refering to Figure 6 which depicts $H(30,15)$, it is easily to observed that x_1 can maximally burn 28 vertices, x_2 can maximally burn 19 vertices, x_3 can maximally burn 10 vertices and x_4 can maximally burn 4 vertices. In the rest of the proof, we will show that regardless the position of x_2, the set of burning sources $\{x_3, x_4, x_5\}$ are not able to completely the whole graph after 5 rounds.

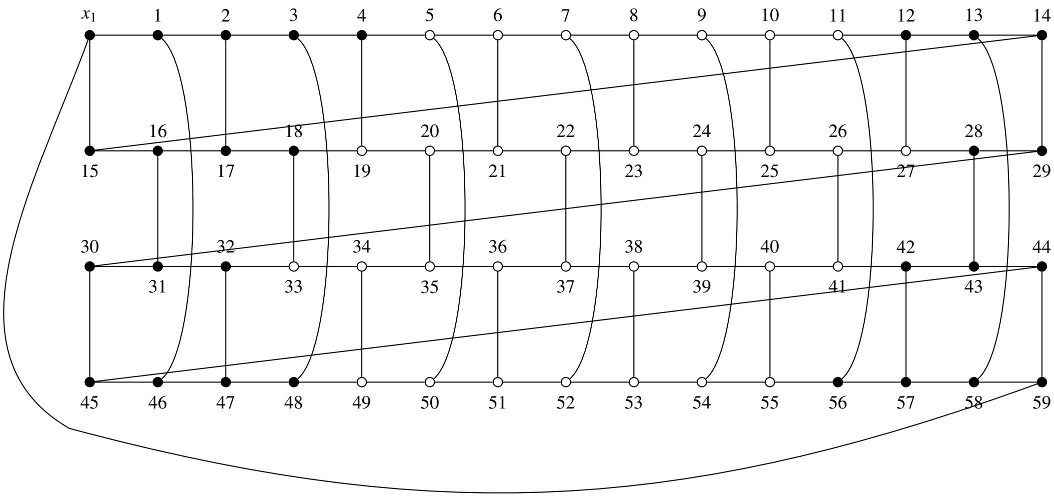


Fig. 6: The graph $H(30,15)$ with vertices labelled as $0,1,2,\ldots,59$. The burning source x_1 placed at 0 burns maximally the 28 filled vertices after 5 rounds.

To ease the reading, we refer the readers to Figure 6 and we will position x_2 in the graph. Here, we claim that $x_2 \notin P$ where $P = N_4[x_1] \cup \{5,10,11,19,20,25,26,27,33,34,35,40,41,49,50,55\}$. If x_2 is placed at $i \in P$, then $|N_3[x_2] \cap N_4[x_1]| \geq 3$. Consequently, by considering the number of vertices burned by x_1 and x_2 after 5 rounds, the number of vertices in $G - (N_4[x_1] \cup N_3[x_2])$ is at least $|G| - |N_4[x_1]| - |N_3[x_2]| + |N_3[x_2] \cap N_4[x_1]| \geq 60 - 28 - 19 + 3 = 16$. However, regardless the positions of x_3, x_4 and x_5 to be placed later, these three burning sources can only maximally burn 15 vertices. That is, $|N_2[x_3]| + |N_1[x_4]| + |N_0[x_5]| \leq 10 + 4 + 1 = 15$. This implies that the rest of the burning sources x_3, x_4 and x_5 are not able to completely burn G. This is a contradiction.

So, we may assume that $x_2 \in R$ where

$$R = V(G) - P = \{6,7,8,9,21,22,23,24,36,37,38,39,51,52,53,54\}.$$

We consider 4 cases.

Case 1. $x_2 \in \{6,9,21,39,51,54\}$. Then $|N_3[x_2] \cap N_4[x_1]| = 2$. Then number of vertices not burned by x_1 and x_2 is equal to $|V(G)| - |N_3[x_2]| - |N_4[x_1]| + |N_3[x_2] \cap N_4[x_1]| = 60 - 28 - 19 + 2 = 15$. This implies that each of the x_3, x_4 and x_5 must exactly burn the rest of these 15 vertices maximally, i.e., $N_2[x_3]$, $N_1[x_4]$ and $N_0[x_5]$ are disjoint sets. Hence x_3, x_4 and x_5 cannot be in $N_4[x_1]$. Besides that, $G - N_4[x_1] - N_3[x_2]$ cannot contain any disjoint subgraph of 2-path or a subgraph of at least 2 isolated vertices, otherwise these two subgraphs are not able to be burned maximally by x_3, x_4 and x_5. However, $G - N_4[x_1] - N_3[x_2]$ contains a subgraph of 2-path or a subgraph of at least 2 isolated vertices for any $x_2 \in \{6,9,21,39,51,54\}$. This is a contradiction.

Case 2. $x_2 \in \{7,53\}$. Then $|N_3[x_2] \cap N_4[x_1]| = 1$. Then regardless of the position of x_3, we have $|N_2[x_3] \cap (N_4[x_1] \cup N_3[x_2])| \geq 2$. Hence, the number of vertices in $G - (N_4[x_1] \cup N_3[x_2] \cup N_2[x_3])$ is at least $60 - 28 - 19 - 10 + 1 + 2 = 6$. However, x_4 and x_5 can only maximally burn a total of 5 vertices among the remaining 6 unburned vertices, a contradiction.

Case 3. $x_2 \in \{8,22,23,37,38,39,52\}$. Then $N_3[x_2] \cap N_4[x_1] = \emptyset$. Then regardless of the position of x_3, we have $|N_2[x_3] \cap (N_4[x_1] \cup N_3[x_2])| \geq 3$. Hence the number of vertices in $G - (N_4[x_1] \cup N_3[x_2] \cup N_2[x_3])$ is at least $60 - 28 - 19 - 10 + 3 = 6$. However, x_4 and x_5 can only maximally burn a total of 5 vertices among these 6 unburned vertices. Again, a contradiction.

Case 4. $x_2 \in \{24,36\}$ such that $N_3[x_2] \cap N_4[x_1] = \emptyset$. Then regardless of the position of x_3, at least one of the following occurs:

1. $|N_2[x_3] \cap (N_4[x_1] \cup N_3[x_2])| \geq 3$;
2. $|N_2[x_3] \cap (N_4[x_1] \cup N_3[x_2])| \geq 2$ and $G - (N_4[x_1] \cup N_3[x_2] \cup N_2[x_3])$ contains at least 3 disjoint subgraphs.

If the former occurs, then the number of vertices in $G - (N_4[x_1] \cup N_3[x_2] \cup N_2[x_3])$ is at least $60 - 28 - 19 - 10 + 3 = 6$. However, x_4 and x_5 can only maximally burn total of 5 vertices among these 6 unburned vertices, a contradiction.

If the latter occurs, the remaining vertices in $G - (N_4[x_1] \cup N_3[x_2] \cup N_2[x_3])$ is at least $60 - 28 - 19 - 10 + 2 = 5$ and these vertices must be burned maximally by x_4 and x_5. However, x_4 and x_5 are not able to completely burn the 3 disjoint subgraphs.

Following this, $b(G) \geq 6$. To show that equality holds, we can burn G using the burning sequence $(x_1,x_2,x_3,x_4,x_5,x_6) = (0,36,25,55,8,6)$. This completes the proof.

□

3.2 Burning number of $H(2k,k)$ for $k \geq 17$

Lemma 4. *If $k \geq 17$ is an odd integer, then*

$$b((H(2k,k)) \geq \sqrt{k+0.5}+1.$$

Proof. Let $G = H(2k,k)$ and $(x_1,x_2,\ldots x_b)$ be a burning sequence of G. It is easily observed that for each burning source x_i, the maximum newly burned sequence by each x_i is

$$\overbrace{\{1,3,6,9,9,8,8,8\ldots\}}^{b-i+1 \text{ times}}.$$

If $b \leq 5$, then $\{x_1,x_2,x_3,x_4,x_5\}$ can burn at most 62 vertices. So, for $k \geq 17$, $H(2k,k)$ contains at least 68 vertices and requires at least 6 burning sources to burn the graph.

Hence the maximum number of vertices burned by $(x_1,x_2,\ldots x_b)$ where $b \geq 6$ is

$$\begin{aligned} & b+3(b-1)+6(b-2)+9(b-3)+9(b-4)+8\sum_{r=1}^{b-5} r \\ =& 28b-78+4(b-5)(b-4) \\ =& 4b^2-8b+2. \end{aligned}$$

In order to completely burn G, we must have $4b^2-8b+2 \geq 4k$ and this gives $b \geq \sqrt{k+0.5}+1$.

□

Theorem 7. *Let $k \geq 17$ be an odd integer. If m is the largest positive integer such that $m^2 < k$, then*

$$b((H(2k,k)) = \begin{cases} m+2, & \text{if } m^2 < k \leq (m+1)^2-2; \\ m+2 \ \text{ or } \ m+3, & \text{if } (m+1)^2-1 \leq k \leq (m+1)^2 \end{cases}$$

Proof. By the choice of m, $m^2 < k \leq (m+1)^2$. Let $G = H(2k,k)$ and $b(G) = b$. By Lemma 4,

$$b \geq \sqrt{k+0.5}+1 \geq \sqrt{m^2+1+0.5}+1 > m+1.$$

Thus, $b \geq m+2$. To show that equality holds and $b = m+3$ for some cases, in the rest of the proof, we first assign $(x_1,x_2,\ldots,x_{m+2})$ in G to be burned in $m+2$ rounds. Then, for some cases, we show that $m+2$ burning sources may not sufficient but it is burnable in $m+3$ rounds.

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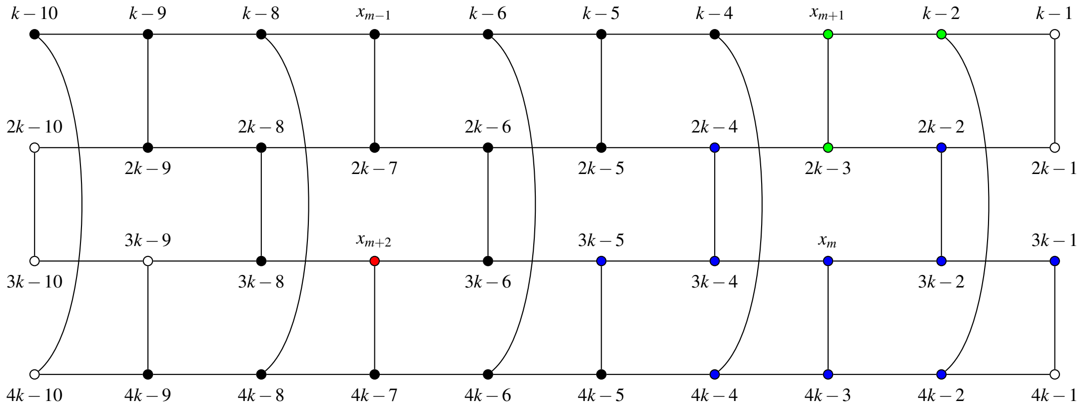

Fig. 7: $H(2k,k) : N_0[x_{m+2}] \cup N_1[x_{m+1}] \cup N_2[x_m] \cup N_3[x_{m-1}]$

Since $k \geq 17$, then $m \geq 4$. For convenience, the vertices in G will be labelled as $0, 1, \ldots, 4k-1$. Note that i is adjacent to $i+1$ for all i (the integers are taken modulo $4k$). So, the vertex $4k-1$ is adjacent to 0. Furthermore, if i is even, then i is adjacent to $i+k$ and if i is odd, then i is adjacent to $i+3k$ (again, the integers are taken modulo $4k$).

To burn G in $m+2$ rounds, we let $x_{m+2} = 3k-7$, $x_{m+1} = k-3$, $x_m = 3k-3$, $x_{m-1} = k-7$ (see Figure 7).

Clearly, $|N_0[x_{m+2}] \cup N_1[x_{m+1}] \cup N_2[x_m] \cup N_3[x_{m-1}]| = 33$ and $\bigcup_{0 \leq i \leq 3} N_i[x_{m+2-i}]$ contains all vertices except $k-1, 2k-1, 4k-1$ and the vertices below:

$$\begin{array}{l} 0, \quad 1, \quad \cdots, \; k-11, \\ k, \; k+1, \; \cdots, \; 2k-11, \; 2k-10, \\ 2k, \; 2k+1, \; \cdots, \; 3k-11, \; 3k-10, \; 3k-9 \\ 3k, \; 3k+1, \; \cdots, \; 4k-11, \; 4k-10. \end{array}$$

Before we proceed to choose x_{m+2-i} for $4 \leq i \leq m$, we present the following two observations:

Observation II. Note that if $i \geq 4$ and $i \leq j \leq k-1-i$, then $N_i[j]$ contains only the vertices below:

$$\begin{array}{c} j-i, \; j-i+1, \; j-i+2, \; \cdots, \; j-1, \; j, \; j+1, \; \cdots, \; j+i-2, \; j+i-1, \; j+i \\ k+j-i+1, \; k+j-i+2, \; \cdots, \; k+j-1, \; k+j, \; k+j+1, \; \cdots, \; k+j+i-2, \; k+j+i-1, \\ 2k+j-i+2, \; \cdots, \; 2k+j-1, \; 2k+j, \; 2k+j+1, \; \cdots, \; 2k+j+i-2, \\ 3k+j-i+1, \; 3k+j-i+2, \; \cdots, \; 3k+j-1, \; 3k+j, \; 3k+j+1, \; \cdots, \; 3k+j+i-2, \; 3k+j+i-1. \end{array}$$

See Figure 8 for the case j is odd and Figure 9 for the case j is even. In either case, $|N_i[j]| = 8i-4$.

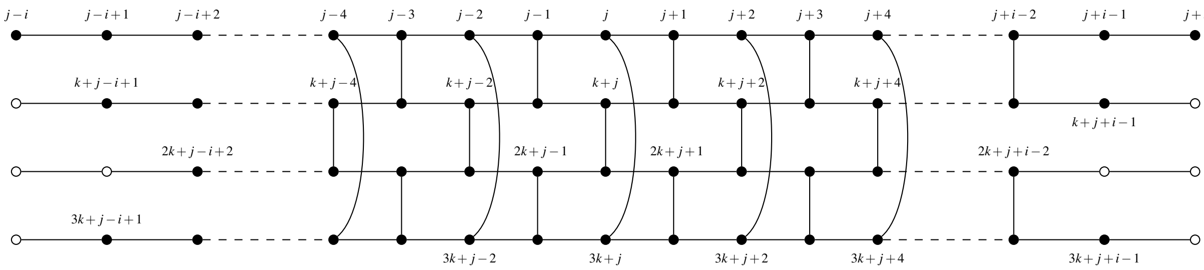

Fig. 8: $H(2k,k) : N_i[j]$ and j is odd

Observation I. Similarly, if $i \geq 4$ and $i \leq j \leq k-1-i$, then $N_i[2k+j]$ contains only the vertices below:

$$\begin{array}{c} j-i+2, \; \cdots, \; j-1, \; j, \; j+1, \; \cdots, \; j+i-2, \\ k+j-i+1, \; k+j-i+2, \; \cdots, \; k+j-1, \; k+j, \; k+j+1, \; \cdots, \; k+j+i-2, \; k+j+i-1, \\ 2k+j-i, \; 2k+j-i+1, \; 2k+j-i+2, \; \cdots, \; 2k+j-1, \; 2k+j, \; 2k+j+1, \; \cdots, \; 2k+j+i-2, \; 2k+j+i-1, \; 2k+j+i \\ 3k+j-i+1, \; 3k+j-i+2, \; \cdots, \; 3k+j-1, \; 3k+j, \; 3k+j+1, \; \cdots, \; 3k+j+i-2, \; 3k+j+i-1, \end{array}$$

and $|N_i[2k+j]| = 8i-4$.

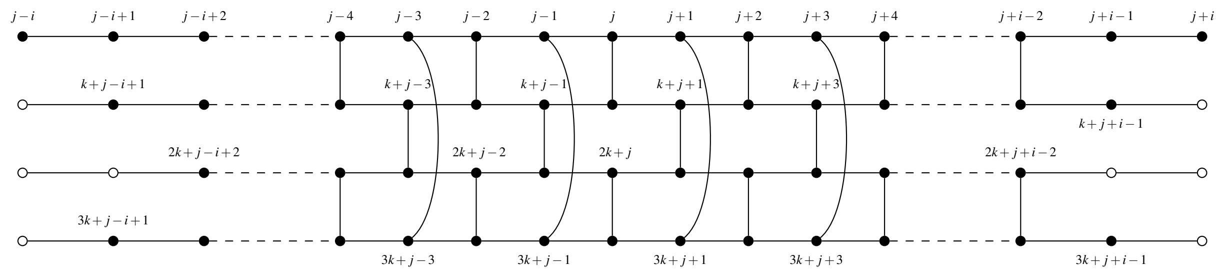

Fig. 9: $H(2k,k) : N_i[j]$ and j is even

Now, we are ready to choose the set of burning sources $(x_{m-2}, x_{m-3}, \ldots, x_2)$.

Here, following the fixed $x_{m+2}, x_{m+1}, x_m, x_{m-1}$, then either Condition (I) or (II) (see below in this proof) holds after each x_{m+2-j} has been chosen for $3 \le j \le i$ and $3 \le i \le m-1$. That is, for instance, let $j = 3$. After selecting x_{m-1}, one of the conditions (I) or (II) holds. Subsequently, x_{m-2} is chosen based on whether Condition (I) or Condition (II) applies. Once x_{m-2} is determined, again either Condition (I) or (II) holds, and x_{m-3} is then selected again accordingly. This process continues until x_3 (i.e. $j = m-1$) is chosen, and finally, x_2 is determined based on whether Condition (I) or (II) holds.

Condition (I) $\bigcup_{0 \le j \le i} N_j[x_{m+2-j}]$ contains all vertices except $k-1, 2k-1, 4k-1$ and the vertices below:

$$\begin{array}{l} 0,\quad 1,\quad \cdots,\quad t_i-2, \\ k,\ \ k+1,\ \cdots,\ k+t_i-2,\ \ k+t_i-1, \\ 2k,\ 2k+1,\ \cdots,\ 2k+t_i-2,\ 2k+t_i-1,\ 2k+t_i \\ 3k,\ 3k+1,\ \cdots,\ 3k+t_i-2,\ 3k+t_i-1. \end{array}$$

Condition (II) $\bigcup_{0 \le j \le i} N_j[x_{m+2-j}]$ contains all vertices except $k-1, 2k-1, 4k-1$ and the vertices below:

$$\begin{array}{l} 0,\quad 1,\quad \cdots,\quad t_i-2,\qquad t_i-1,\quad t_i \\ k,\ \ k+1,\ \cdots,\ k+t_i-2,\ \ k+t_i-1, \\ 2k,\ 2k+1,\ \cdots,\ 2k+t_i-2, \\ 3k,\ 3k+1,\ \cdots,\ 3k+t_i-2,\ 3k+t_i-1. \end{array}$$

If Condition (I) holds, we set $x_{m+1-i} = 2k + t_i - i - 1$. By Observation I, $N_{i+1}[x_{m+1-i}] \cap \bigcup_{0 \le j \le i} N_j[x_{m+2-j}] = \emptyset$ and $\bigcup_{0 \le j \le i+1} N_j[x_{m+2-j}]$ contains all vertices except $k-1, 2k-1, 4k-1$ and the vertices below:

$$\begin{array}{l} 0,\quad 1,\quad \cdots,\quad t_{i+1}-2,\qquad t_{i+1}-1,\quad t_{i+1} \\ k,\ \ k+1,\ \cdots,\ k+t_{i+1}-2,\ \ k+t_{i+1}-1, \\ 2k,\ 2k+1,\ \cdots,\ 2k+t_{i+1}-2, \\ 3k,\ 3k+1,\ \cdots,\ 3k+t_{i+1}-2,\ 3k+t_{i+1}-1. \end{array}$$

Here, $t_{i+1} = t_i - 2i - 1$.

If Condition (II) holds, we set $x_{m+1-i} = t_i - i - 1$. By Observation II, $N_{i+1}[x_{m+1-i}] \cap \bigcup_{0 \le j \le i} N_j[x_{m+2-j}] = \emptyset$ and $\bigcup_{0 \le j \le i+1} N_j[x_{m+2-j}]$ contains all vertices except $k-1, 2k-1, 4k-1$ and the vertices below:

$$\begin{array}{l} 0,\quad 1,\quad \cdots,\quad t_{i+1}-2, \\ k,\ \ k+1,\ \cdots,\ k+t_{i+1}-2,\ \ k+t_{i+1}-1, \\ 2k,\ 2k+1,\ \cdots,\ 2k+t_{i+1}-2,\ 2k+t_{i+1}-1,\ 2k+t_{i+1} \\ 3k,\ 3k+1,\ \cdots,\ 3k+t_{i+1}-2,\ 3k+t_{i+1}-1. \end{array}$$

Here, $t_{i+1} = t_i - 2i - 1$.

After the selection of x_{m+2-j} for $0 \le j \le m$, there are two possibilities:

(III) $\bigcup_{0 \le j \le m} N_j[x_{m+2-j}]$ contains all vertices except the vertices below:

$$\begin{array}{l} 4k-1,\ 0,\quad 1,\quad \cdots,\quad t_m-2, \\ k-1,\ \ k,\ \ k+1,\ \cdots,\ k+t_m-2,\ \ k+t_m-1, \\ 2k-1,\ 2k,\ 2k+1,\ \cdots,\ 2k+t_m-2,\ 2k+t_m-1,\ 2k+t_m \\ \qquad 3k,\ 3k+1,\ \cdots,\ 3k+t_m-2,\ 3k+t_m-1. \end{array}$$

(IV) $\bigcup_{0\le j\le m} N_j[x_{m+2-j}]$ contains all vertices except $k-1, 2k-1, 4k-1$ and the vertices below:

$$\begin{array}{llllll} 4k-1,\ 0, & 1, & \cdots, & t_m-2, & t_m-1, & t_m \\ k-1,\ k, & k+1, & \cdots, & k+t_m-2, & k+t_m-1, & \\ 2k-1,\ 2k, & 2k+1, & \cdots, & 2k+t_m-2, & & \\ \quad\ 3k, & 3k+1, & \cdots, & 3k+t_m-2, & 3k+t_m-1. & \end{array}$$

Let L be the number of vertices not contained in $\bigcup_{0\le j\le m} N_j[x_{m+2-j}]$. Then $L = 4t_m+3$.

By construction, $N_{j'}[x_{m+2-j'}] \cap N_j[x_{m+2-j}] = \emptyset$ and $N_j[x_{m+2-j}] \cap \bigcup_{0\le i\le 3} N_i[x_{m+2-i}] = \emptyset$ for $j, j' \ge 4$ and $j \ne j'$. Therefore,

$$\begin{aligned} \left|\bigcup_{0\le j\le m} N_j[x_{m+2-j}]\right| &= \sum_{4\le j\le m} |N_j[x_{m+2-j}]| + \left|\bigcup_{0\le i\le 3} N_i[x_{m+2-i}]\right| \\ &= \sum_{4\le j\le m} (8j-4) + 33 = 4m^2 - 3. \end{aligned}$$

So, $L = 4k-4m^2+3$ and $4t_m+3 = 4k-4m^2+3$, i.e., $t_m = k-m^2$. This implies that $t_m \ge (m^2+1)-m^2 \ge 1$.

We will only consider case (III). The case (IV) is similar.

Suppose $k \le (m+1)^2-2$. Then,

$$t_m \le k-m^2 \le (m+1)^2-2-m^2 = 2m-1.$$

Now, consider the path P_1 of length t_m+2,

$$P_1 = 2k-1, 2k, 2k+1, \ldots, 2k+t_m-2, 2k+t_m-1, 2k+t_m.$$

Since $t_m+2 \le 2m+1$, we may choose a vertex $2k+j_0$ on the path P_1 such that $d_{P_1}(2k+j_0, 2k+t_m) \le m+1$ and $d_{P_1}(2k+j_0, 2k-1) \le m-1$. Using Observations I and II and the fact that $m+1 \ge 5$, we see that $N_{m+2}[x_1]$ contains all the remaining vertices. Hence, $b = m+2$.

Suppose $(m+1)^2-1 \le k \le (m+1)^2$. Then,

$$t_m \le k-m^2 \le (m+1)^2-m^2 = 2m+1.$$

Now, consider the path P_1 of length t_m+2,

$$P_1 = 2k-1, 2k, 2k+1, \ldots, 2k+t_m-2, 2k+t_m-1, 2k+t_m.$$

Since $t_m+2 \le 2m+3$, we may choose a vertex $2k+j_0$ on the path P_1 such that $d_{P_1}(2k+j_0, 2k+t_m) \le m+2$ and $d_{P_1}(2k+j_0, 2k-1) \le m$. Using Observations I and II and the fact that $m+1 \ge 5$, we see that $N_{m+3}[x_1]$ contains all the remaining vertices. However, $N_{m+2}[x_1]$ may not contain the vertices $2k+t_m$, $k+t_m-1$, t_m-2, $3k+t_m-1$ and $4k-1$. Hence, $b \le m+3$.

This completes the proof of the theorem.

□

3.3 Concluding Remark

Recall that the generalized Heawood graph $H(n,k)$ is a cubic bipartite graph. Here, we briefly compare the burning number of $H(n,k)$ with that of other well-known cubic graphs, such as the generalized Petersen graph $P(n,k)$ and 3-regular circulant graph $C(n;1,n/2)$. These families of graphs differ in structure, symmetry, and girth, which affect their burning behavior.

Table 1 summarizes the exact values of the burning number for some generalized Heawood graph which are shown in this paper. For the generalized Heawood graphs, Theorems 3, 4, and 7 show that their burning numbers depend on the largest integer m such that $m^2 < k$. In general, the burning number of these graphs is either $m+1, m+2$, or $m+3$, indicating that it grows roughly on the order of $\sqrt{k}$. For example, the family $H(2k,k)$ has about $4k$ vertices, and its burning number is approximately $\frac{1}{2}\sqrt{N}$, where N is the number of vertices.

For the generalized Petersen graphs [22], the burning number satisfies the bounds

$$\left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil \leq b(P(n,k)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

This range shows that while the lower bound grows like $\sqrt{n/k}$, the upper bound includes an additive term of $\lfloor k/2 \rfloor$, which can be significant when k increases. As a result, the burning number of $P(n,k)$ can be larger than that of the Heawood graphs, especially for moderate values of k.

For the 3-regular circulant graphs $C(n;1,n/2)$ [7], the burning number is given exactly by

$$b(C(n;1,n/2)) = \left\lceil \frac{1+\sqrt{2n+1}}{2} \right\rceil.$$

This value grows approximately as $0.707\sqrt{N}$, where $N = n$ is the number of vertices. In comparison, the generalized Heawood graphs such as $H(2k,k)$ have smaller constants (around $0.5\sqrt{N}$), suggesting they burn more efficiently.

In comparison, the burning numbers of the generalized Heawood graphs tend to be relatively smaller than those of most known cubic families of comparable order, likely due to their high symmetry and bipartite configuration that facilitate more efficient burning. However, the burning number of $H(n,k)$ for $n \geq k+2$ and $k \geq 5$, corresponding to graphs of girth 6, remains an open problem for future investigation.

Table 1: Burning numbers of Generalized Heawood graphs for several small values of $k \leq 15$

Generalized Heawood graph	Burning number
$H(6,5), H(6,3), H(10,5)$	3
$H(8,7), H(10,9), H(12,11), H(14,7)$	4
$H(14,13), H(16,15), H(18,9), H(22,11), H(26,13)$	5
$H(30,15)$	6

Declarations

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