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A New Approach to Solve Dynamical Fractional Model of Infertile Couples by The Finite Element Method

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Abstract: In this paper, we propose a dynamic model designed to predict the treatment outcomes of infertility in couples. To this end, five main groups of couples are considered: susceptible couples, patient couples, those undergoing treatment only, those receiving treatment with surgery, and cured couples. Our primary contribution lies in the formulation of a fractional-order model that captures the dynamics of infertile couples and identifies asymptotically stable disease-free equilibrium points. The main objective is to develop a novel numerical approach based on the finite element method to solve a nonlinear system of fractional differential equations. This method transforms the fractional system into a system of algebraic equations. Moreover, the proposed technique proves to be robust, efficient, and particularly well-suited to modeling infertility. To validate our model, we apply the proposed method and conclude with a comprehensive numerical analysis.

Keywords: Infertility modeling; fractional differential equations; Caputo derivative; finite element method; stability analysis; basic reproduction number; numerical simulation.

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1 Introduction

Infertility has emerged as a global health challenge in recent years, affecting millions of couples worldwide. This growing concern has triggered extensive research to unravel the mechanisms underlying infertility, a complex and multifactorial phenomenon. Clinically, infertility is defined as the inability of a couple to achieve natural conception after one year of regular unprotected intercourse. A wide range of causes can lead to infertility, including dysfunctions in the male or female reproductive systems, hormonal imbalances, as well as environmental and genetic factors. In addition to the physiological implications, infertility can lead to significant emotional and psychological distress for affected couples [18].

Despite its prevalence, mathematical modeling of infertility dynamics remains relatively scarce in the literature. A notable attempt by Ghiaschafezi and al. [10] introduced a model based on ordinary differential equations, though their results revealed the instability of the system. Motivated by the growing utility of fractional calculus in physics, engineering, and the life sciences [6], researchers have increasingly turned to fractional-order models as a powerful tool to describe memory effects and complex hereditary dynamics that cannot be captured by classical models. These models typically involve fractional derivatives of non-integer order [12, 16], which offer new avenues for accurately modeling and predicting the behavior of biological systems.

Recent advancements in fractional differential equations have focused on both analytical and numerical aspects. For instance, Li and Xu [17] proposed a time-space spectral method for solving fractional partial differential equations based on a weak formulation and detailed error analysis. Similarly, Ervin and Roop [7, 8] investigated fractional advection-dispersion equations using finite element techniques, accounting for non-local spatial dependencies. Adolfsson and al. [18, 17] explored fractional viscoelastic models through an efficient time-domain finite element method. Furthermore, Li and al. [1] established error bounds for finite element approximations of time-fractional differential equations. In this

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context, several recent works have further enriched the theoretical foundation of fractional differential systems. Djenaoui and Meftah [3] introduced new inequalities of the Maclaurin type for functions whose first derivatives are s-convex, contributing to the analytical tools relevant in fractional modeling. In a related direction, Hamoud and al. [13] addressed the existence and convergence of solutions to Caputo-type fractional Volterra integro-differential equations, strengthening the theoretical guarantees required for applying such models to real-world problems. Motivated by these developments, this paper investigates a fractional-order model for infertility dynamics. In particular, the literature highlights the growing importance of fractional models in representing complex biological phenomena. Several recent studies have demonstrated the advantages of fractional models over classical ones, particularly in cases where long-term behaviors or memory effects are significant. For example, fractional-order models allow for a more nuanced representation of infertility dynamics, where the interactions between different factors (such as hormonal imbalances, environmental influences, and genetic predispositions) may exhibit complex, nonlinear behaviors that are not captured by traditional models. This paper seeks to apply fractional derivatives to better understand the dynamics of infertility, offering a more robust tool for prediction and intervention. We adopt the framework introduced by Li and Xu [17], utilizing appropriate functional spaces and norms to formulate and analyze a weak variational formulation. The existence and uniqueness of the solution are established via the Lax-Milgram lemma [2].

This paper is organized as follows. In section 2, we present our proposed model given by a system of fractional differential equation. Then, in section 3, we consider the existence and uniqueness of the solution of the weak variational formulation and subsequently we develop the finite element method to obtain algebraic system. In section 4, the equilibrium points and basic reproduction number Ro are considered, followed by a detaild asymptotic stability analysis of our fractional model. To illustrate theoretical results, numerical simulations of the mathematical model are show cased in section 5. we then end our paper with conclusion in the last section.

2 A dynamical model of infertility couples

2.1 A classical model

The mathematical model presented in [10] describes the infertility of couples. For this purpose, all infertile couples at time t are represented by $N(t)$ and are divided into five groups. The first group includes couples susceptible to infertility-those who are unable to procreate during the first year of their marriage. Some of these couples may not have any problems, these patients are designated by $P(t)$. Those under treatment whitout surgery are presented by $T_1(t)$, while patients under treatment in which at least one of them is operated on are represented by $T_2(t)$ and cured couples are represented by $C(t)$. We therefore have: $N(t) = S(t) + P(t) + T_1(t) + T_2(t) + C(t)$. This model is named SPT_1T_2C and illustrated by the following diagram:

This model is formulated as follow :

$$\begin{cases} S'(t) & = \lambda N(t) - \mu S(t) - \beta\left[\delta_1 T_1(t) + \delta_2 T_2(t)\right] S(t), \\ P'(t) & = \beta\left[\delta_1 T_1(t) + \delta_2 T_2(t)\right] S(t) - (\mu + \gamma + \eta) P(t), \\ T_1'(t) & = v\gamma P(t) - (\mu + \delta_1) T_1(t), \\ T_2'(t) & = k(1-h)\delta_1 T_1(t) + (1-v)\gamma P(t) - (\mu + \delta_2) T_2(t), \\ C'(t) & = h\delta_1 T_1(t) + r\eta P(t) + \delta_2 T_2(t) - \mu C(t). \end{cases} \tag{1}$$

with,

- β, The magnitude of the disease that recurs in each infertile couple.
- μ, Infetile couples who abandon treatment.
- λ, Couples entering treatment.
- γ, Represents the rate of curred couples in compartment P(t).
- δ_1 and δ_2, The rates of couples under drug treatment and whitout surgery at time t and couples under surgical treatment respectively.
- v, The recovery rate among couples under treatment whitout surgery.
- r, The recovery rate in ηP.
- η, The number of patients who do not receive treatment.
- k, The recovery rate in group $(1-h)\delta_1 T_1$.
- h, The recovery rate in group $\delta_1 T_1$.

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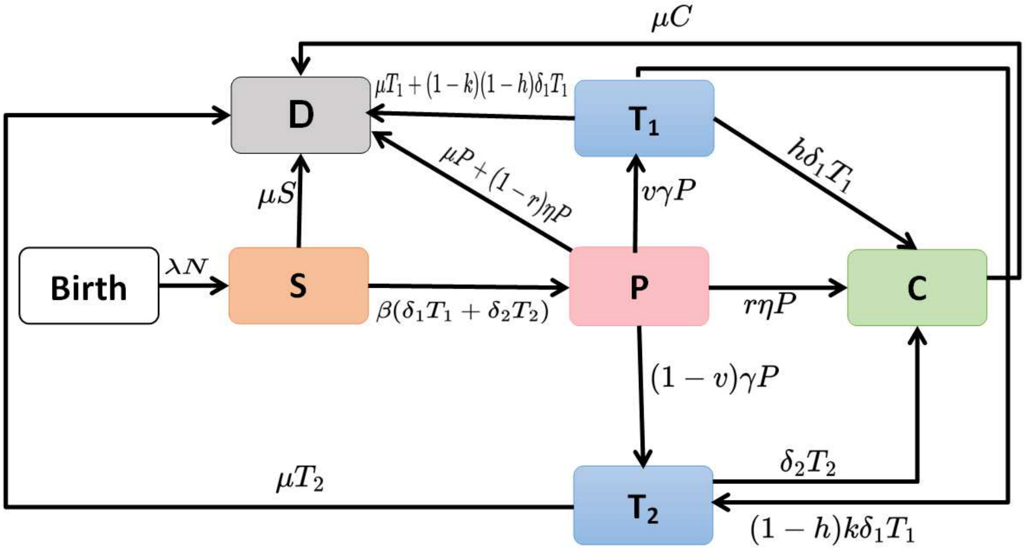

Fig. 1: Illustrative diagram SPT_1T_2C of infertility couples.

2.2 The fractional order model

The new proposed fractional mathematical model is based upon the ordinary model. The model of a fractional differential equation provides much more accurate results than the classical model,which is based on first derivatives. This is the main advantage of fractional differentiation, compared to classical integer models. Thus we replace the first order derivatives in model (1) by the fractional derivative of order α in the sense of caputo with $0 < \alpha < 1$. The choice of the Caputo derivative is particularly important in this context. The Caputo derivative was selected because it allows for more natural initial conditions, which is crucial in biological modeling. Unlike the Riemann-Liouville derivative, which is defined over the entire solution interval and may be less intuitive for systems with meaningful initial conditions, the Caputo derivative is more appropriate for capturing the dynamics of biological systems, where initial states can significantly influence the behavior of the system. Additionally, the Caputo derivative has been widely used in biomedical and complex system modeling, which further justifies its selection for this study. Dimensional analysis indicates that the unit of the first member of these equations becomes $(\text{time})^{-\alpha}$, but the right-hand side is in $(\text{time})^{-1}$ [4, 6], so we must ensure that both sides of these equations have the same dimensions $(\text{time})^{-\alpha}$. Subsequently, our fractional model of non integer order takes into account the normalization of the variables of the total population N.

$$\mathbf{S} = \frac{S}{N},\ \mathbf{P} = \frac{P}{N},\ \mathbf{T}_1 = \frac{T_1}{N},\ \mathbf{T}_2 = \frac{T_2}{N},\ \mathbf{C} = \frac{C}{N},$$

The non linear fractional system is formulated as follows:

$$\begin{cases} {}^cD^{\alpha}\mathbf{S}(t) = \lambda^{\alpha} - \mu^{\alpha}\mathbf{S}(t) - \beta\left[(\delta_1 N)^{\alpha}\mathbf{T_1}(t) + (\delta_2 N)^{\alpha}\mathbf{T_2}(t)\right]\mathbf{S}(t), \\ {}^cD^{\alpha}\mathbf{P}(t) = \beta\left[(\delta_1 N)^{\alpha}\mathbf{T_1}(t) + (\delta_2 N)^{\alpha}\mathbf{T_2}(t)\right]\mathbf{S}(t) - (\mu^{\alpha} + \gamma^{\alpha} + \eta^{\alpha})\mathbf{P}(t), \\ {}^cD^{\alpha}\mathbf{T_1}(t) = v\gamma^{\alpha}\mathbf{P}(t) - (\mu^{\alpha} + \delta_1^{\alpha})\mathbf{T_1}(t), \\ {}^cD^{\alpha}\mathbf{T_2}(t) = k(1-h)\delta_1^{\alpha}\mathbf{T_1}(t) + (1-v)\gamma^{\alpha}\mathbf{P}(t) - (\mu^{\alpha} + \delta_2^{\alpha})\mathbf{T_2}(t), \\ {}^cD^{\alpha}\mathbf{C}(t) = h\delta_1^{\alpha}\mathbf{T_1}(t) + rn^{\alpha}\mathbf{P}(t) + \delta_2^{\alpha}\mathbf{T_2}(t) - \mu^{\alpha}\mathbf{C}(t). \end{cases} \tag{2}$$

with the initial conditions: $S(0), P(0), T_1(0), T_2(0), C(0)$.
Where ${}^cD^{\alpha}$ is the Caputo derivative and it is defined as :

$$ {}^cD^{\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{d\phi}{d\tau}(t-\tau)^{-\alpha}\,d\tau, t > 0, 0 < \alpha < 1 \tag{3}$$

For more details, we refer the reader to [11].

3 A finite element method

In this section, the finite element method for fractional differential equations of time is highlighted. The existence and uniqueness of the solution are proved by using the Lax-Milgram lemma [2]. To do that, we first illustrate the method in the general framework of fractional differential equations, then we apply it to each equation of the system.

3.1 Description of method

We will consider the following fractional differential equation

$$\begin{cases} {}^cD^{\alpha}u(t)+c^{\alpha}u(t)=f(t), t\in[0,T] \\ u(0)=0. \end{cases} \tag{4}$$

where $f(t)\in L^2\left(]0,T[\right)$ and ${}^cD^{\alpha}u(t)$ denotes the caputo fractional derivative. We multiply the equation of the problem by a function $v\in H_0^{\alpha/2}\left(]0,T[\right)$ [17] and we integrate over $]0,T[$. We obtain the following weak variational formulation:

$$\begin{cases} \text{Find } u\in H_0^{\alpha/2}\left(]0,T[\right) \text{ that} \\ \int_0^T D^{\alpha/2}u(t)D^{\alpha/2}v(t)+c^{\alpha}\int_0^T u(t)v(t)dt=\int_0^T f(t)v(t)dt, \quad \forall v\in H_0^{\alpha/2}\left(]0,T[\right) \end{cases} \tag{5}$$

The solution of this problem exists and is unique.

*Proof.*To verify if (5) has a unique solution, we will verify the conditions of Lax Milgram lemma. By using the Cauchy Schwartz inequality, we have

–Continuity of linear form L :

$$|L(v)|=\left|\int_0^T f(t)v(t)\right|\leq\|f\|_{L^2(]0,T[)}\|v\|_{L^2(]0,T[)}\leq M\|v\|_{H^{\alpha/2}(]0,T[)}.$$

–Continuity of bilinear form a :

$$\begin{aligned} |a(u,v)|&=\left|\int_0^T D^{\alpha/2}u(t)D^{\alpha/2}v(t)dt+\int_0^T c^{\alpha}u(t)v(t)dt\right| \\ &\leq\left\|D^{\alpha/2}u\right\|_{L^2(]0,T[)}\left\|D^{\alpha/2}v\right\|_{L^2(]0,T[)}+c^{\alpha}\|u\|_{L^2(]0,T[)}\|v\|_{L^2(]0,T[)} \\ &\leq\|u\|_{H_0^{\alpha/2}(]0,T[)}\|v\|_{H_0^{\alpha/2}(]0,T[)}+c^{\alpha}\|u\|_{H_0^{\alpha/2}(]0,T[)}\|v\|_{H_0^{\alpha/2}(]0,T[)} \\ &\leq(1+c^{\alpha})\|u\|_{H_0^{\alpha/2}(]0,T[)}\|v\|_{H_0^{\alpha/2}(]0,T[)}. \end{aligned}$$

–Coercivity of bilinear form a :

$$\begin{aligned} a(v,v)&=\int_0^T\left(D^{\alpha/2}v(t)\right)^2dt+c^{\alpha}\int_0^T v^2(t)dt \\ &=\left\|D^{\alpha/2}v\right\|^2_{L^2(]0,T[)}+c^{\alpha}\|v\|^2_{L^2(]0,T[)} \\ &\geq\min(1,c^{\alpha})\left(\|D^{\alpha/2}v(t)\|^2_{L^2}+\|v\|^2_{L^2}\right) \\ &\geq\min(1,c^{\alpha})\|v\|^2_{H^{\alpha/2}(]0,T[)} \\ &\geq\beta\min(1,c^{\alpha})\|v\|^2_{H_0^{\alpha/2}(]0,T[)},\beta>0. \end{aligned}$$

–Discretisation of interval

We choose the domain $\Omega=[0,1]$. In dimension 1 a mesh is simply made up of a points $(t_j)_{0\leq j\leq n+1}$ such that

$$t_0=0<t_1<\cdots<t_n<t_{n+1}=1$$

We have

$$t_j = jh_d, h_d = \frac{1}{n+1},\ 0 \le j \le n+1$$

In our work we are based on the finite element method P_1 [19] which is based on the discrete space of globally continuous and affine function on each mesh

$$X_h^1 = \left\{ v \in C([0,1]) / v|_{[t_j,t_{j+1}]} \in P_1 \text{ for } 0 \le j \le n \right\}. \tag{6}$$

We can represent the piecewise affine functions of X_h^1 using very simple functions called basis functions defined as follows :

$$\phi_j(t) = \begin{cases} \dfrac{t - t_{j-1}}{h_d} & , \quad t \in [t_{j-1}, t_j] \\ \dfrac{t_{j+1} - t}{h_d} & , \quad t \in [t_j, t_{j+1}] \\ 0, & \quad \text{else} \end{cases} \tag{7}$$

–Internal approximation

Let $V_h = H_0^{\alpha/2} \cap X_h^1 \left(V_h \subset H_0^{\alpha/2} \right)$ the approximate weak variational formulation becomes :

$$\begin{cases} \text{Find } U_h \in V_h \text{ such that} \\ a(u_h, v_h) = L(v_h), \forall v_h \in V_h \end{cases} \tag{8}$$

We set $v_h = (\phi_j)_{0 \le j \le n+1}$ and we write u_h in the bases

$$u_h(t) = \sum_{i=0}^{n+1} u_h(t_i)\, \phi_i(t), \forall t \in [0,1] \tag{9}$$

By replacing u_h and v_h in $a(u_h, v_h) = L(v_h)$, we obtain the following system,

$$K_h U_h = B_h \tag{10}$$

Such that,

•K_h represents the rigidity matrix,

$$(K_h)_{ij} = a(\phi_i, \phi_j), 0 \le i, j \le n+1 \tag{11}$$

•B_h the right-hand side,

$$(B_h)_j = L(\phi_j),\ 0 \le j \le n+1 \tag{12}$$

•U_h the vector of solutions,

$$U_h = \left(u_h^{(1)}, u_h^{(2)}, u_h^{(3)}, \ldots \right) \tag{13}$$

The coercivity of the bilinear form leads to the positive definite character of the matrix K_h and therefore its inversibility. Indeed for any vector $U_h \in \mathbb{R}^{N_h}$ we have.

$$k_h U_h U_h \ge \gamma \left\| \sum_i U_h^i \phi_i \right\|^2 \ge c\,|U_h|^2, c > 0 \tag{14}$$

Similarity the symmetry of $a(u,v)$ implies that of K_h.

3.2 The rigidity matrix and the right hand side

First, as the intersection of the basis function is often empty, we have most of the coefficients of (K_h) are zero and the matrix is tridiagonal. The non-zero coefficients are $(K_h)_{i,i-1}, (K_h)_{i,i}, (K_h)_{i,i+1}$.

Next, we present the caputo derivative of order $(\alpha/2)$

$${}^cD^{\alpha/2}\phi(t) = \frac{1}{\Gamma\left(n - \dfrac{\alpha}{2}\right)} \sum_{j=0}^{n+1} \int_{t_j}^{t_j+1} \frac{d\phi(\tau)}{d\tau}(\tau)(t-\tau)^{-\alpha/2} d\tau, \tag{15}$$

let us consider the properties of $\phi(t)$ [15], we have :
On $[t_j, t_{j+1}]$

$$
\begin{aligned}
{}^{c}D^{\alpha/2}\phi(t) &= \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)} \int_{t_j}^{t_j+1} \frac{d\phi(\tau)}{d\tau}(\tau)(t-\tau)^{-\alpha/2} d\tau \\
&= \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)} \left(\frac{-1}{h}\right) \int_{t_j}^{t_j+1} (t-\tau)^{-\alpha/2} d\tau \\
&= \frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)} \left(\frac{-1}{h}\right) \left[\frac{(t-\tau)^{1-\frac{\alpha}{2}}}{\left(1-\frac{\alpha}{2}\right)}\right]_{t_j}^{t_{j+1}},
\end{aligned}
$$

then

$$
{}^{c}D^{\alpha/2}\phi(t) = \frac{1}{h\Gamma\left(2-\frac{\alpha}{2}\right)} \left[(t-t_{j+1})^{1-\frac{\alpha}{2}} - (t-t_j)^{1-\frac{\alpha}{2}}\right]. \tag{16}
$$

On $[t_{j-1}, t_j]$

$$
{}^{c}D^{\alpha/2}\phi(t) = \frac{-1}{h\Gamma\left(2-\frac{\alpha}{2}\right)} \left[(t-t_j)^{1-\frac{\alpha}{2}} - (t-t_{j-1})^{1-\frac{\alpha}{2}}\right]. \tag{17}
$$

Each equation in (2) can be writing as $K_h^\alpha U_h = B_h^\alpha$.

3.3 The matrices and right-hand sides of our fractional model

Now, we apply the method to the system (2) and the trapezoid quadrature formula for complicated functions to integrate, we obtain the following matrices and right-hand sides.

$$
K_S^\alpha = \begin{cases}
\left(K_S^\alpha\right)_{0,0} \simeq \dfrac{1}{2h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H + \left[\beta(\delta_1 N)^\alpha T_1^0 + \beta(\delta_2 N)^\alpha T_2^0 + \mu^\alpha\right]\dfrac{h_d}{2}, \\
\left(K_S^\alpha\right)_{i,i} \simeq \dfrac{1}{h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H, \qquad \text{for } i=\overline{1,n} \\
\left(K_S^\alpha\right)_{i,i+1} \simeq \dfrac{-1}{h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H, \qquad \text{for } i=\overline{0,n} \\
\left(K_S^\alpha\right)_{i,i-1} \simeq \dfrac{-1}{h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H, \qquad \text{for } i=\overline{1,n+1} \\
(K_S)_{n+1,n+1}^\alpha \simeq \dfrac{1}{2h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H + \left[\beta(\delta_1 N)^\alpha T_1^{n+1} + \beta(\delta_2 N)^\alpha T_2^{n+1} + \mu^\alpha\right]\dfrac{h_d}{2},
\end{cases}
$$

and

$$
B_S^\alpha = \begin{cases}
\left(B_S^\alpha\right)_0 \simeq \frac{\lambda^\alpha h_d}{2} + S^0, \\
\left(B_S^\alpha\right)_i \simeq \lambda^\alpha h_d, \text{ for } i=\overline{1,n} \\
\left(B_S^\alpha\right)_{n+1} \simeq \frac{\lambda^\alpha h_d}{2} + S^{n+1}.
\end{cases}
$$

$$
K_P^\alpha = \begin{cases}
(K_P^\alpha)_{0,0} = (K_P^\alpha)_{n+1,n+1} \simeq \dfrac{1}{2h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)\dfrac{h_d}{2}, \\
(K_P^\alpha)_{i,i} \simeq \dfrac{1}{2h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H + (\mu^\alpha + \gamma^\alpha + \eta^\alpha) h_d, \qquad \text{for } i=\overline{1,n} \\
(K_P^\alpha)_{i,i+1} = \left(K_S^\alpha\right)_{i,i+1}, \qquad \text{for } i=\overline{0,n} \\
(K_P^\alpha)_{i,i-1} = \left(K_S^\alpha\right)_{i,i-1}, \qquad \text{for } i=\overline{1,n+1}
\end{cases}
$$

and

$$B_P^{\alpha} = \begin{cases} (B_P^{\alpha})_0 \simeq \left(\beta\left(\delta_1 N\right)^{\alpha} S^0 T_1^0 + \beta\left(\delta_2 N\right)^{\alpha} S^0 T_2^0\right)\dfrac{h_d}{2}, \\ (B_P^{\alpha})_i \simeq \left(\beta\left(\delta_1 N\right)^{\alpha} S^i T_1^i + \beta\left(\delta_2 N\right)^{\alpha} S^i T_2^i\right) h_d, \text{ for } i = \overline{1,n} \\ (B_P^{\alpha})_{n+1} \simeq \left(\beta\left(\delta_1 N\right)^{\alpha} S^{n+1} T_1^{n+1} \beta\left(\delta_2 N\right)^{\alpha} S^{n+1} T^{n+1}\right)\dfrac{h_d}{2}. \end{cases}$$

$$K_{T_1}^{\alpha} = \begin{cases} \left(K_{T_1}^{\alpha}\right)_{0,0} = \left(K_{T_1}^{\alpha}\right)_{n+1,n+1} \simeq \dfrac{1}{2h_d\Gamma\left(2-\dfrac{\alpha}{2}\right)^2} H + \left(\mu^{\alpha} + \delta_1^{\alpha}\right)\dfrac{h_d}{2}, \\ \left(K_{T_1}^{\alpha}\right)_{i,i} \simeq \dfrac{1}{h_d\Gamma\left(2-\dfrac{\alpha}{2}\right)^2} H + \left(\mu^{\alpha} + \delta_1^{\alpha}\right) h_d, \text{ for } i = \overline{1,n} \\ \left(K_{T_1}^{\alpha}\right)_{i,i+1} = \left(K_S^{\alpha}\right)_{i,i+1}, \text{ for } i = \overline{0,n} \\ \left(K_{T_1}^{\alpha}\right)_{i,i-1} = \left(K_S^{\alpha}\right)_{i,i-1}, \text{ for } i = \overline{1,n+1} \end{cases}$$

and

$$B_{T_1}^{\alpha} = \begin{cases} \left(B_{T_1}^{\alpha}\right)_0 \simeq \nu\gamma^{\alpha} P^0 \dfrac{h_d}{2}, \\ \left(B_{T_1}^{\alpha}\right)_i \simeq \nu\gamma^{\alpha} P^i h_d, \text{ for } i = \overline{1,n} \\ \left(B_{T_1}^{\alpha}\right)_{n+1} \simeq \nu\gamma^{\alpha} P^{n+1} \dfrac{h_d}{2}. \end{cases}$$

$$K_{T_2}^{\alpha} = \begin{cases} \left(K_{T_2}^{\alpha}\right)_{0,0} = \left(K_{T_2}^{\alpha}\right)_{n+1,n+1} \simeq \dfrac{1}{2h_d\Gamma\left(2-\dfrac{\alpha}{2}\right)^2} H + \left(\mu^{\alpha} + \delta_2^{\alpha}\right)\dfrac{h_d}{2}, \\ \left(K_{T_2}^{\alpha}\right)_{i,i} \simeq \dfrac{1}{2h_d\Gamma\left(2-\dfrac{\alpha}{2}\right)^2} H + \left(\mu^{\alpha} + \delta_2^{\alpha}\right) h_d, \text{ for } i = \overline{1,n} \\ \left(K_{T_2}^{\alpha}\right)_{i,i+1} = \left(K_S^{\alpha}\right)_{i,i+1}, \text{ for } i = \overline{0,n} \\ \left(K_{T_2}^{\alpha}\right)_{i,i-1} = \left(K_S^{\alpha}\right)_{i,i-1}, \text{ for } i = \overline{1,n+1} \end{cases}$$

and

$$B_{T_2}^{\alpha} = \begin{cases} \left(B_{T_2}^{\alpha}\right)_0 \simeq \dfrac{h_d}{2}\left(k(1-h)\delta_1^{\alpha} T_1^0 + (1-\nu)\gamma^{\alpha} P^0\right), \\ \left(B_{T_2}^{\alpha}\right)_i \simeq h_d\left(k(1-h)\delta_1^{\alpha} T_1^i + (1-\nu)\gamma^{\alpha} P^i\right), \text{ for } = \overline{1,n} \\ \left(B_{T_2}^{\alpha}\right)_{n+1} \simeq \dfrac{h_d}{2}\left(k(1-h)\delta_1^{\alpha} T_1^{n+1} + (1-\nu)\gamma^{\alpha} P^{n+1}\right). \end{cases}$$

$$K_C^{\alpha} = \begin{cases} (K_C\alpha)_{0,0} = (K_c^{\alpha})_{n+1,n+1} \simeq \dfrac{1}{2h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H + \dfrac{h_d}{2}\mu^{\alpha}, \\ (K_C^{\alpha})_{i,i} \simeq \dfrac{1}{h_d\Gamma\left(2-\frac{\alpha}{2}\right)^2} H + h_d\mu^{\alpha}, \text{ for } i = \overline{1,n} \\ (K_C^{\alpha})_{i,i+1} = \left(K_S^{\alpha}\right)_{i,i+1}, \text{ for } i = \overline{0,n} \\ (K_C^{\alpha})_{i,i-1} = \left(K_S^{\alpha}\right)_{i,i-1}, \text{ for } i = \overline{1,n+1} \end{cases}$$

and

$$B_C^{\alpha} = \begin{cases} \left(B_C^{\alpha}\right)_0 \simeq \dfrac{h_d}{2}\left(h\delta_1^{\alpha} T_1^0 + r\eta^{\alpha} P^0 + \delta_2^{\alpha} T_2^0\right), \\ \left(B_C^{\alpha}\right)_i \simeq h_d\left(h\delta_1^{\alpha} T_1^i + r\eta^{\alpha} P^i + \delta_2^{\alpha} T_2^i\right), \text{ for } i = \overline{1,n} \\ \left(B_C^{\alpha}\right)_{n+1} \simeq \dfrac{h_d}{2}\left(h\delta_1^{\alpha} T_1^{n+1} + r\eta^{\alpha} P^{n+1} + \delta_2^{\alpha} T_2^{n+1}\right). \end{cases}$$

where,

$$\begin{cases} H = \left(h_d^{1-\frac{\alpha}{2}}\right)^2 + \left((-h)^{1-\frac{\alpha}{2}}\right)^2 \\ \left(S^i, P^i, T_1^i, T_2^i, C^i\right) = \left(S(t_i), P(t_i), T_2(t_i), T_2(t_i), C(t_i)\right), \text{ for } i = \overline{0,n+1} \end{cases}$$

To find the solutions (S, P, T_1, T_2, C), we solve respectively the following systems

$$\begin{cases} K_S^\alpha S = B_S^\alpha, \\ K_P^\alpha P = B_P^\alpha, \\ K_{T_1}^\alpha T_1 = B_{T_1}^\alpha, \\ K_{T_2}^\alpha T_2 = B_{T_2}^\alpha, \\ K_C^\alpha C = B_C^\alpha. \end{cases}$$

4 Analysis of the model

4.1 Basic reproduction member and Equilibrium points

In this section, the equilibrium points for (2) is reviewed. First we calculate the basic reproduction number of the model using the next generation approach which is defined as

$$R_0 = \rho\left(FV^{-1}\right)$$

where $\rho\left(FV^{-1}\right)$ denotes the spectral radius of a Matrix $\left(FV^{-1}\right)$. [5]

$$\begin{pmatrix} {}^cD^\alpha P(t) \\ {}^cD^\alpha T_1(t) \\ {}^cD^\alpha T_2(t) \end{pmatrix} = \begin{pmatrix} \beta\left(\delta_1 N\right)^\alpha S_0 + \beta\left(\delta_2 N\right)^\alpha S_0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \left(\mu^\alpha + \gamma^\alpha + \eta^\alpha\right) P \\ \left(\mu^\alpha + \delta_1^\alpha\right) T_1 - v\gamma^\alpha P \\ k(h-1)\delta_1^\alpha T_1 + (v-1)\gamma^\alpha P + \left(\mu^\alpha + \delta_2^\alpha\right) T_2 \end{pmatrix}$$
$$= f\left(P, T_1, T_2\right) - v_R\left(P, T_1, T_2\right)$$

there for,

$$F = \begin{pmatrix} 0 & \beta\left(\delta_1 N\right)^\alpha S_0 & \beta\left(\delta_2 N\right)^\alpha S_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$V = \begin{pmatrix} \mu^\alpha + \gamma^\alpha + \eta^\alpha & 0 & 0 \\ -v\gamma^\alpha & \mu^\alpha + \delta_1^\alpha & 0 \\ (v-1)\gamma^\alpha & k(h-1)\delta_1^\alpha & \mu^\alpha + \delta_2^\alpha \end{pmatrix}$$

with F and V are Jacobian matrices of the matrices f and v_R respectively.
The matrix $\left(FV^{-1}\right)$ is given by

$$FV^{-1} = \begin{pmatrix} A & B & \dfrac{\beta\left(\delta_2 N\right)^\alpha S_0}{\mu^\alpha + \delta_2^\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with,

$$A = \frac{\beta S_0 \gamma^\alpha\left[\left(N\delta_1\right)^\alpha v\left(\mu^\alpha + \delta_2^\alpha\right) + \left(N\delta_2\right)^\alpha vk(1-h)\delta_1^\alpha + \left(N\delta_2\right)^\alpha(1-v)\left(\mu^\alpha + \delta_1\right)\right]}{\left(\mu^\alpha + \gamma^\alpha + \eta^\alpha\right)\left(\mu^\alpha + \delta_1^\alpha\right)\left(\mu^\alpha + \delta_2^\alpha\right)},$$

and $$B = \frac{BS_0\left(N\delta_1\right)^\alpha\left(\mu^\alpha + \delta_2^\alpha\right) + \beta S_0\left(N\delta_2\right)^\alpha k(1-h)\delta_1^\alpha}{\left(\mu^\alpha + \delta_1^\alpha\right)\left(\mu^\alpha + \delta_2^\alpha\right)}.$$

The eigenvalues of this matrix are : $\lambda_1 = A$ and $\lambda_2 = 0$. The basic reproduction rate R_0 is the spectral radius of the matrix of this model is given by:

$$R_0 = \frac{\beta S_0 \gamma^\alpha\left[(N\delta_1)^\alpha v(\mu^\alpha + \delta_2^\alpha) + (N\delta_2)^\alpha vk(1-h)\delta_1^\alpha + (N\delta_2)^\alpha(1-v)(\mu^\alpha + \delta_1^\alpha)\right]}{(\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha)}. \tag{18}$$

Now we develop the calculus to find the positive stationary solutions as well as the conditions for their existence.
Let an invariant set of the system (2)

$$\Omega = \left\{(S, P, T_1, T_2, C) \in \mathbb{R}_+^*\right\}$$

Theorem 1. *The model admits two equilibrium points in Ω for all strictly positive parameters*

1. E_0 is the equilibrium points without disease (DFE), this point is given by

$$E_0 = \left(\left(\frac{\lambda}{\mu}\right)^{\alpha}, 0, 0, 0, 0 \right)^{\top}$$

2. E_1 is the equilibrium point with disease (DEE), this point is given by $E_1 = (S^*, P^*, T_1^*, T_2^*, C^*)^{\top}$ with,

$$\begin{cases} S^* = \frac{\lambda^{\alpha}}{\mu^{\alpha} R_0} \\ P^* = \frac{\lambda^{\alpha}}{R_0(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})}(R_0-1) \\ T_1^* = \frac{\lambda^{\alpha} v\gamma^{\alpha}}{R_0(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})\left(\mu^{\alpha}+\delta_1^{\alpha}\right)}(R_0-1) \\ T_2^* = \frac{\lambda^{\alpha}\left(v\gamma^{\alpha}\delta_1^{\alpha}(1-h)k+(1-v)\gamma^{\alpha}\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\right)}{R_0(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\left(\mu^{\alpha}+\delta_2^{\alpha}\right)}(R_0-1) \\ C^* = \frac{\lambda^{\alpha}\left[v\gamma^{\alpha}\delta_1^{\alpha}(1-h)k\delta_2^{\alpha}+(1-v)\gamma^{\alpha}\delta_2^{\alpha}\left(\mu^{\alpha}+\delta_1^{\alpha}\right)+hv\gamma^{\alpha}\delta_1^{\alpha}\left(\mu^{\alpha}+\delta_2^{\alpha}\right)+r\eta^{\alpha}\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\left(\mu^{\alpha}+\delta_2^{\alpha}\right)\right]}{R_0\mu^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\left(\mu^{\alpha}+\delta_2^{\alpha}\right)}(R_0-1) \end{cases}$$

Proof. To determine the equilibrium points of the system (2), we solve the following system

$$\begin{cases} \lambda^{\alpha} - \mu^{\alpha}\mathbf{S}(\mathbf{t}) - \beta\left[(\delta_1 N)^{\alpha}\mathbf{T_1}(\mathbf{t}) + (\delta_2 N)^{\alpha}\mathbf{T_2}(\mathbf{t})\right]\mathbf{S}(\mathbf{t}) = 0, & (19) \\ \beta\left[(\delta_1 N)^{\alpha}\mathbf{T_1}(\mathbf{t}) + (\delta_2 N)^{\alpha}\mathbf{T_2}(\mathbf{t})\right]\mathbf{S}(\mathbf{t}) - (\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})\mathbf{P}(\mathbf{t}) = 0, & (20) \\ v\gamma^{\alpha}\mathbf{P}(\mathbf{t}) - (\mu^{\alpha}+\delta_1^{\alpha})\mathbf{T_1}(\mathbf{t}) = 0, & (21) \\ k(1-h)\delta_1^{\alpha}\mathbf{T_1}(\mathbf{t}) + (1-v)\gamma^{\alpha}\mathbf{P}(\mathbf{t}) - (\mu^{\alpha}+\delta_2^{\alpha})\mathbf{T_2}(\mathbf{t}) = 0, & (22) \\ h\delta_1^{\alpha}\mathbf{T_1}(\mathbf{t}) + r\eta^{\alpha}\mathbf{P}(\mathbf{t}) + \delta_2^{\alpha}\mathbf{T_2}(\mathbf{t}) - \mu^{\alpha}\mathbf{C}(\mathbf{t}) = 0. & (23) \end{cases}$$

According to the equation (19) and (21), we get

$$S(t) = \frac{\lambda^{\alpha}}{\mu^{\alpha} + \beta\left((N\delta_1)^{\alpha}\mathbf{T}_1 + (N\delta_2)^{\alpha}\mathbf{T}_2\right)}$$

$$\mathbf{T}_1 = \frac{v\gamma^{\alpha}}{\mu^{\alpha}+\delta_1^{\alpha}}\mathbf{P}$$

By remplacing the expression of T_1 in equation (22),

$$\mathbf{T}_2 = \left(\frac{k(1-h)\delta_1^{\alpha}v\gamma^{\alpha} + (1-v)\gamma^{\alpha}(\mu^{\alpha}+\delta_1^{\alpha})}{(\mu^{\alpha}+\delta_1^{\sigma})(\mu^{\alpha}+\delta_2^{\alpha})}\right)\mathbf{P},$$

and T_1, T_2 in equation (23) after in equation (20), we have

$$\mathbf{C} = \frac{1}{\mu^{\alpha}}\left(\frac{h\delta_1^{\alpha}v\gamma^{\alpha}}{\mu^{\alpha}+\delta_1^{\alpha}} + r\eta^{\alpha} + \frac{k(1-h)\delta_1^{\alpha}v\gamma^{\alpha} + (1-v)\gamma^{\alpha}(\mu^{\alpha}+\delta_1^{\alpha})}{(\mu^{\alpha}+\delta_1^{\alpha})(\mu^{\alpha}+\delta_2^{\alpha})}\right)\mathbf{P}$$

$$\left(\beta\left((N\delta_1)^{\alpha}\left(\frac{v\gamma^{\alpha}}{\mu^{\alpha}+\delta_1^{\alpha}}\right) + (N\delta_2)^{\alpha}\left(\frac{k(1-h)\delta_1^{\alpha}v\gamma^{\alpha}+(1-v)\gamma^{\alpha}\left(\mu^{\alpha}+\delta_1^{\alpha}\right)}{\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\left(\mu^{\alpha}+\delta_2^{\delta_2^{\alpha}}\right)}\right)\right)\mathbf{S} - (\mu^{\alpha}+\eta^{\alpha}+\gamma^{\alpha})\right)\mathbf{P} = 0$$

here we have two cases:

case 1 :if $P=0$

The equation (19) implies

$$S_0=\left(\frac{\lambda}{\mu}\right)^{\alpha}$$

hence the first point equilibrium

$$E_0=\left(\left(\frac{\lambda}{\mu}\right)^{\alpha},0,0,0,0\right)^{T}$$

case 2 :if $P=1$

$$\beta\left((N\delta_1)^{\alpha}\left(\frac{v\gamma^{\alpha}}{\mu^{\alpha}+\delta_1^{\alpha}}\right)+(N\delta_2)^{\alpha}\left(\frac{k(1-h)\delta_1^{\alpha}v\gamma^{\alpha}+(1-v)\gamma^{\alpha}(\mu^{\alpha}+\delta_1^{\alpha})}{(\mu^{\alpha}+\delta_1^{\alpha})(\mu^{\alpha}+\delta_2^{\alpha})}\right)\right)\mathbf{S}-(\mu^{\alpha}+\eta^{\alpha}+\gamma^{\alpha})=0$$

$$\mathrm{S}=\frac{(\mu^{\alpha}+\eta^{\alpha}+\gamma^{\alpha})(\mu^{\alpha}+\delta_1^{\alpha})(\mu^{\alpha}+\delta_2^{\alpha})}{\beta(N\delta_1)^{\alpha}v\gamma^{\alpha}(\mu^{\alpha}+\delta_2^{\alpha})+\beta(N\delta_2)^{\alpha}k(1-h)\delta_1^{\alpha}v\gamma^{\alpha}+\beta(N\delta_2)^{\alpha}(1-v)\gamma^{\alpha}(\mu^{\alpha}+\delta_1^{\alpha})}$$

therefore

$$\mathbf{S}=\frac{\lambda^{\alpha}}{\mu^{\alpha}R_0}$$

The equation (19) involves

$$\frac{\mu^{\alpha}}{\beta(N\delta_2)^{\alpha}}(R_0-1)-\left(\frac{\delta_1}{\delta_2}\right)^{\alpha}\mathbf{T}_1=\mathbf{T}_2$$

equations (21) and (23) implie

$$\mathbf{P}=\frac{(\mu^{\alpha}+\delta_1^{\alpha})}{v\gamma^{\alpha}}\mathbf{T}_1$$

and

$$\mathbf{C}=\left(\frac{h\delta_1^{\alpha}}{\mu^{\alpha}}+\frac{r\eta^{\alpha}(\mu^{\alpha}+\delta_1^{\alpha})}{\mu^{\alpha}v\gamma^{\alpha}}-\frac{\delta_1^{\alpha}}{\mu^{\alpha}}\right)\mathbf{T}_1+\frac{\delta_2^{\alpha}}{\beta(N\delta_2)^{\alpha}}(R_0-1)$$

The equation (22) gives

$$k(1-h)\delta_1^{\alpha}\mathbf{T}_1+\frac{(1-v)(\mu^{\alpha}+\delta_1^{\alpha})\gamma^{\alpha}}{v\gamma^{\alpha}}\mathbf{T}_1+(\mu^{\alpha}+\delta_2^{\alpha})\left[\left(\frac{\delta_1}{\delta_2}\right)^{\alpha}\mathbf{T}_1-\frac{\mu^{\alpha}}{\beta(N\delta_2)^{\alpha}}(R_0-1)\right]=0$$

$$\mathbf{T}_1=\frac{\lambda^{\alpha}v\gamma^{\alpha}}{R_0(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})(\mu^{\alpha}+\delta_1^{\alpha})}(R_0-1),$$

$$\mathbf{T}_2=\frac{\lambda^{\alpha}[v\gamma^{\alpha}\delta_1^{\alpha}(1-h)k+(1-v)\gamma^{\alpha}(\mu^{\alpha}+\delta_1^{\alpha})]}{R_0(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})(\mu^{\alpha}+\delta_1^{\alpha})(\mu^{\alpha}+\delta_2^{\alpha})}(R_0-1)$$

$$\mathbf{P}=\frac{\lambda^{\alpha}}{R_0(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})}(R_0-1)$$

Finally, C is given by the equation (23)

$$\mathbf{C}=\frac{\lambda^{\alpha}\left[v\gamma^{\alpha}\delta_1^{\alpha}(1-h)k\delta_2^{\alpha}+(1-v)\gamma^{\alpha}\delta_2^{\alpha}\left(\mu^{\alpha}+\delta_1^{\alpha}\right)+hv\gamma^{\alpha}\delta_1^{\alpha}\left(\mu^{\alpha}+\delta_2^{\alpha}\right)+r\eta^{\alpha}\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\left(\mu^{\alpha}+\delta_2^{\alpha}\right)\right]}{R_0\mu^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\eta^{\alpha})\left(\mu^{\alpha}+\delta_1^{\alpha}\right)\left(\mu^{\alpha}+\delta_2^{\alpha}\right)}(R_0-1)$$

4.2 Stability analysis of the equilibrium points

In this subsection we study the stability of the equilibrium points. We will calculate the Jacobian matrix of the system which is given by

$$J(S,P,T_1,T_2,C) = \begin{pmatrix} -\mu^\alpha - \beta\left((\delta_1 N)^\alpha T_1 + (\delta_2 N)^\alpha T_2\right) & 0 & -\beta(\delta_1 N)^\alpha S & -\beta(\delta_2 N)^\alpha S & 0 \\ \beta\left((\delta_1 N)^\alpha T_1 + (\delta_2 N)^\alpha T_2\right) & -(\mu^\alpha + \gamma^\alpha + \eta^\alpha) & \beta(\delta_1 N)^\alpha S & \beta(\delta_2 N)^\alpha S & 0 \\ 0 & v\gamma^\alpha & -(\mu^\alpha + \delta_1^\alpha) & 0 & 0 \\ 0 & (1-v)\gamma^\alpha & k(1-v)\delta_1^\alpha & -(\mu^\alpha + \delta_2^\alpha) & 0 \\ 0 & r\eta^\alpha & h\delta_1^\alpha & \delta_2^\alpha & -\mu^\alpha \end{pmatrix}$$

Theorem 2.*The disease free equilibrium point E_0 is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.*

To prove the stability of E_0, linearization of the equations of the dynamic system is determined by the Jacobian matrix.

*Proof.*Evaluating the Jacobian matrix at the DFE, We obtain

$$J(DFE) = \begin{pmatrix} -\mu^\alpha & 0 & -A & -B & 0 \\ 0 & -C & A & B & 0 \\ 0 & v\gamma^\alpha & -D & 0 & 0 \\ 0 & G & E & -F & 0 \\ 0 & r\eta^\alpha & h\delta_1^\alpha & \delta_2^\alpha & -\mu^\alpha \end{pmatrix}$$

with,

$$\begin{aligned} A &= \beta(\delta_1 N)^\alpha \left(\frac{\lambda}{\mu}\right)^\alpha \\ B &= \beta(\delta_2 N)^\alpha \left(\frac{\lambda}{\mu}\right)^\alpha \\ C &= \mu^\alpha + \gamma^\alpha + \eta^\alpha \\ D &= \mu^\alpha + \delta_1^\alpha \\ E &= (1-h)k\delta_1^\alpha \\ F &= \mu^\alpha + \delta_2^\alpha \\ G &= (1-v)\gamma^\alpha \end{aligned}$$

Stability at this point can be noted from the value of the eigen at matrix Jacobian. Based on the Routh Hurwitz criterion [14], if all the parts of the value of real eigen Jacobian matrix are negative, E_0 is locally asymptotically stable.

The characteristic polynomial is given by

$$q(y) = (y+\mu^\alpha)^2\left(a_0 y^3 + a_1 y^2 + a_2 y + a_3\right),$$

then the eigenvalues $y_1 = y_2 = -\mu^\alpha$.

The remaining eigenvalues correspond to the roots of the following polynomial :

$$q(y) = a_0 y^3 + a_1 y^2 + a_2 y + a_3$$

Where,

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 3\mu^\alpha + \eta^\alpha + \gamma^\alpha + \delta_1^\alpha + \delta_2^\alpha \\ a_2 &= \beta\gamma^\alpha\left(\frac{\lambda}{\mu}\right)^\alpha\left((\delta_2 N)^\alpha v - (\delta_1 N)^\alpha v - (\delta_2 N)^\alpha\right) + (\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha) \\ &\quad + (\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \eta^\alpha) + (\mu^\alpha + \gamma^\alpha)(\mu^\alpha + \delta_2^\alpha) + (\gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_2^\alpha) \\ a_3 &= (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha)(\mu + \delta_2^\alpha)(1 - R_0) \end{aligned}$$

According to Routh-Hurwitz stability criterion, so the roots have negative real parts if and only if all principal minors are strictly positive, that is :

$$
\begin{aligned}
\Delta_1 &= a_1 \\
&= 3\mu^\alpha + \eta^\alpha + \gamma^\alpha + \delta_1^\alpha + \delta_2^\alpha > 0 \\
\Delta_2 &= a_1 \cdot a_2 - a_3 \\
&= (3\mu^\alpha + \eta^\alpha + \gamma^\alpha + \delta_1^\alpha + \delta_2^\alpha)\left(\beta\gamma^\alpha \left(\frac{\lambda}{\mu}\right)^\alpha ((\delta_2 N)^\alpha v - (\delta_1 N)^\alpha v - (\delta_2 N)^\alpha)\right. \\
&\quad + (\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha) + (\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \eta^\alpha) + (\mu^\alpha + \gamma^\alpha)(\mu^\alpha + \delta_2^\alpha) \\
&\quad \left. + (\gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_2^\alpha)\right) - (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha)(1 - R_0) > 0 \\
\Delta_3 &= a_3 \cdot \Delta_2 > 0
\end{aligned}
$$

when $R_0 < 1$.

Theorem 3.*The endemic equilibrium point E_1 is locally asymptotically stable if $R_0 > 1$ and unstable if $R_0 < 1$.*

As in Theorem 2, the Jacobian matrix is evaluated at the equilibrium point, and the same procedure is followed to analyze the stability of the endemic equilibrium

*Proof.*Evaluating the Jacobian matrix at (EE), we obtain

$$
J(EE) = \begin{pmatrix}
-\mu^\alpha R_0 & 0 & -\frac{A}{R_0} & -\frac{B}{R_0} & 0 \\
\mu^\alpha (R_0 - 1) & -C & \frac{A}{R_0} & \frac{B}{R_0} & 0 \\
0 & v\gamma^\alpha & -D & 0 & 0 \\
0 & G & E & -F & 0 \\
0 & r\eta^\alpha & h\delta_1^\alpha & \delta_2^\alpha & -\mu^\alpha
\end{pmatrix}
$$

The caracteristic polynomial is given by

$$
p(y) = (y + \mu^\alpha)\left(a_0 y^4 + a_1 y^3 + a_2 y^2 + a_3 y + a_4\right)
$$

then $y = -\mu^\alpha$ is only one eigenvalue. By the following polynomial we remain eigenvalues :

$$
p(y) = a_0 y^4 + a_1 y^3 + a_2 y^2 + a_3 y + a_4
$$

with,

$$
\begin{aligned}
a_0 =& 1 \\
a_1 =& 3\mu^\alpha + \eta^\alpha + \gamma^\alpha + \delta_1^\alpha + \delta_2^\alpha + \lambda^\alpha R_0 \\
a_2 =& \frac{1}{R_0}\left(\beta (\delta_1 N)^\alpha \left(\frac{\lambda}{\mu}\right)^\alpha v\gamma^\alpha - \beta (\delta_2 N)^\alpha \left(\frac{\lambda}{\mu}\right)^\alpha (1 - v)\gamma^\alpha\right) \\
& + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha) + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_2^\alpha) + (\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha) \\
& + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)\lambda^\alpha R_0 + (\mu^\alpha + \delta_1^\alpha)\lambda^\alpha R_0 + (\mu^\alpha + \delta_1^\alpha)\lambda^\alpha R_0 \\
a_3 =& \frac{1}{R_0}\left(\frac{\lambda}{\mu}\right)^\alpha (-\beta (\delta_1 N)^\alpha ((\mu^\alpha + \delta_2^\alpha) v\gamma^\alpha - v\mu^\alpha\gamma^\alpha (R_0 - 1) + v\gamma^\alpha\lambda^\alpha R_0) \\
& - \beta (\delta_2 N)^\alpha ((1 - h)kv\delta_1^\alpha\gamma^\alpha + (1 - v)\gamma^\alpha (\mu^\alpha + \delta_1^\alpha) - \mu^\alpha\gamma^\alpha (1 - v)(R_0 - 1) + \gamma^\alpha\lambda^\alpha (1 - v)R_0)) \\
& + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha) + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha)\lambda^\alpha R_0 \\
& + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_2^\alpha)\lambda^\alpha R_0 + (\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha)\lambda^\alpha R_0 \\
a_4 =& \frac{1}{R_0}\left(\frac{\lambda}{\mu}\right)^\alpha (\beta (\delta_1 N)^\alpha ((\mu^\alpha + \delta_2^\alpha) v\gamma^\alpha\mu^\alpha (R_0 - 1) - (\mu^\alpha + \delta_2^\alpha) v\gamma^\alpha\lambda^\alpha R_0^2) \\
& + \beta (\delta_2 N)^\alpha ((\mu^\alpha + \delta_1^\alpha)(1 - v)\gamma^\alpha\mu^\alpha (R_0 - 1) + (1 - h)kv\delta_1^\alpha\gamma^\alpha\lambda^\alpha\mu^\alpha (R_0 - 1) \\
& - (\mu^\alpha + \delta_1^\alpha)(1 - v)\gamma^\alpha\lambda^\alpha R_0^2)) - (1 - h)k\delta_1^\alpha v\gamma^\alpha R_0^2) + (\mu^\alpha + \gamma^\alpha + \eta^\alpha)(\mu^\alpha + \delta_1^\alpha)(\mu^\alpha + \delta_2^\alpha)\lambda^\alpha R_0
\end{aligned}
$$

So by the Routh-Hurwitz criterion, we show that E_1 is locally asymptotically stable if the following conditions are verified

$$
\begin{aligned}
\Delta_1 &= a_1\\
&= 3\mu^\alpha+\eta^\alpha+\gamma^\alpha+\delta_1^\alpha+\delta_2^\alpha+\lambda^\alpha R_0>0\\
\Delta_2 &= a_4\\
&= \frac{1}{R_0}\left(\frac{\lambda}{\mu}\right)^\alpha(\beta\,(\delta_1 N)^\alpha\,((\mu^\alpha+\delta_2^\alpha)\,v\gamma^\alpha\mu^\alpha\,(R_0-1)-(\mu^\alpha+\delta_2^\alpha)\,v\gamma^\alpha\lambda^\alpha R_0^2)\\
&+\beta\,(\delta_2 N)^\alpha\,((\mu^\alpha+\delta_1^\alpha)\,(1-v)\,\gamma^\alpha\mu^\alpha\,(R_0-1)+(1-h)kv\delta_1^\alpha\gamma^\alpha\lambda^\alpha\mu^\alpha\,(R_0-1)\\
&-(\mu^\alpha+\delta_1^\alpha)\,(1-v)\,\gamma^\alpha\lambda^\alpha R_0^2))-(1-h)k\delta_1^\alpha v\gamma^\alpha R_0^2)+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,(\mu^\alpha+\delta_1^\alpha)\,(\mu^\alpha+\delta_2^\alpha)\,\lambda^\alpha R_0>0
\end{aligned}
$$

$$
\begin{aligned}
\Delta_3 &= a_1a_2-a_3\\
&= (3\mu^\alpha+\eta^\alpha+\gamma^\alpha+\delta_1^\alpha+\delta_2^\alpha+\lambda^\alpha R_0)(\frac{1}{R_0}\left(\beta\,(\delta_1 N)^\alpha\left(\frac{\lambda}{\mu}\right)^\alpha v\gamma^\alpha-\beta\,(\delta_2 N)^\alpha\left(\frac{\lambda}{\mu}\right)^\alpha(1-v)\,\gamma^\alpha\right)\\
&+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,(\mu^\alpha+\delta_1^\alpha)+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,(\mu^\alpha+\delta_2^\alpha)+(\mu^\alpha+\delta_1^\alpha)\,(\mu^\alpha+\delta_2^\alpha)\\
&+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,\lambda^\alpha R_0+(\mu^\alpha+\delta_1^\alpha)\,\lambda^\alpha R_0+(\mu^\alpha+\delta_1^\alpha)\,\lambda^\alpha R_0)\\
&-\frac{1}{R_0}\left(\frac{\lambda}{\mu}\right)^\alpha(-\beta\,(\delta_1 N)^\alpha\,((\mu^\alpha+\delta_2^\alpha)\,v\gamma^\alpha-v\mu^\alpha\gamma^\alpha\,(R_0-1)+v\gamma^\alpha\lambda^\alpha R_0)\\
&-\beta\,(\delta_2 N)^\alpha\,((1-h)kv\delta_1^\alpha\gamma^\alpha+(1-v)\,\gamma^\alpha\,(\mu^\alpha+\delta_1^\alpha)-\mu^\alpha\gamma^\alpha\,(1-v)\,(R_0-1)+\gamma^\alpha\lambda^\alpha\,(1-v)\,R_0))\\
&+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,(\mu^\alpha+\delta_1^\alpha)\,(\mu^\alpha+\delta_2^\alpha)+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,(\mu^\alpha+\delta_1^\alpha)\,\lambda^\alpha R_0\\
&+(\mu^\alpha+\gamma^\alpha+\eta^\alpha)\,(\mu^\alpha+\delta_2^\alpha)\,\lambda^\alpha R_0+(\mu^\alpha+\delta_1^\alpha)\,(\mu^\alpha+\delta_2^\alpha)\,\lambda^\alpha R_0>0\\
\Delta_4 &= a_1a_2a_3-a_1^2a_4-a_3^2>0
\end{aligned}
$$

when $R_0>1$.

5 Numerical simulations

In this section, we perform numerical simulations of the proposed fractional model to illustrate our theoretical results and demonstrate its effectiveness and feasibility.

5.1 Theorical results of finite elements method of the model of infertility couples

In this subsection, we numerically confirm the theorical results obtained in section 3. We consider the following parameter values which are summarized by the following table. [10]

Parametres	Description	Values
N	Size population	1000
λ	Couples entering treatment	0.28
η	The number of patients who do not receive treatment	0.15
r	The recovery rate in ηP	0.2
μ	Infetile couples who abandon treatment	0.15
β	The magnitude of the disease that recurs in each infertile couple	0.02
δ_1	The rates of couples under drug treatment and whitout surgery	0.1
δ_2	The rates of couples under surgical treatment	0.08
γ	Represents the rate of curred couples in compartment P(t)	0.5
v	The recovery rate among couples under treatment whitout surgery	0.6
h	The recovery rate in group $\delta_1 T_1$	0.75
k	The recovery rate in group $(1-h)\,\delta_1 T_1$	0.2

Table 1: Epidemiological data

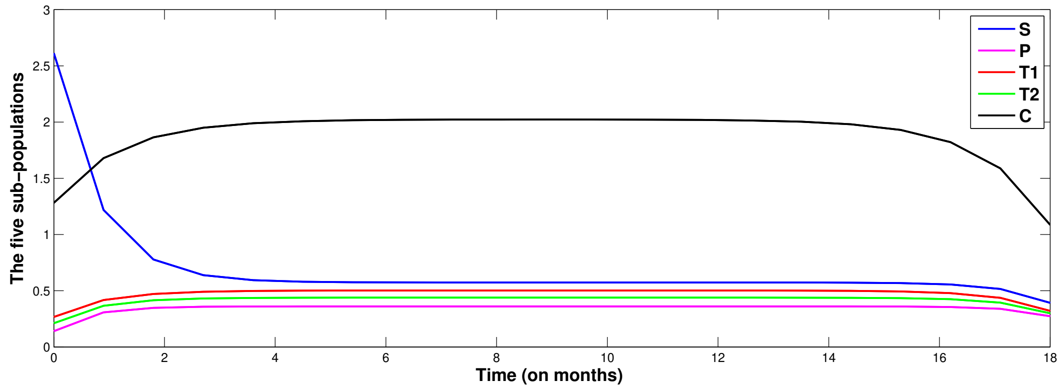

Fig. 2: Numerical simulation of model with $\alpha = 0.5$.

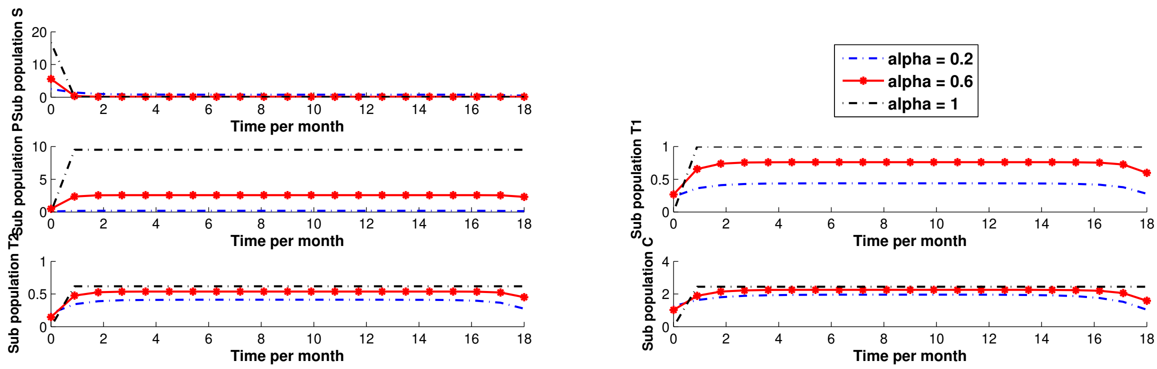

Fig. 3: Numerical simulation for different values of α.

with the initial conditions : $S(0) = 1000, P(0) = 0, T_1(0) = 0, T_2(0) = 0, C(0) = 0$.

Figure (2) illustrates the temporal dynamics of the five subpopulations over a period of 18 months. A sharp decrease in the number of couples susceptible to infertility S is observed during the first month, while the number of recovered couples C increases rapidly and reaches a plateau shortly thereafter. This reflects the effectiveness of early interventions. Following medical treatment — either pharmacological T_1 or surgical T_2 — both treated populations rise and then stabilize after approximately six months. It is also noted that surgically treated couples T_2 exhibit a slightly higher recovery rate, consistent with clinical assumptions. Although the system appears to stabilize from month 2 onward, a very slow decrease in some variables (notably C) is observed after long periods. This behavior does not contradict the theoretical stability of the system, which admits equilibrium points. Similar dynamics are observed for all fractional orders $0 < \alpha < 1$ as shown in Figure (3).

The simulation of our model (2) for different values of α clearly shows that the data in Table 1, estimated in [10], are consistent with real-world observations. It is clear that all the numerical solutions corresponding to fractional orders better reflect the actual data than the classical solution with ($\alpha = 1$).

To further validate the effectiveness of our fractional model and the finite element method, we considered several epidemiological scenarios based on the parameter sets provided in Table 2 of [10]. The corresponding numerical simulations are shown in Figure (4).

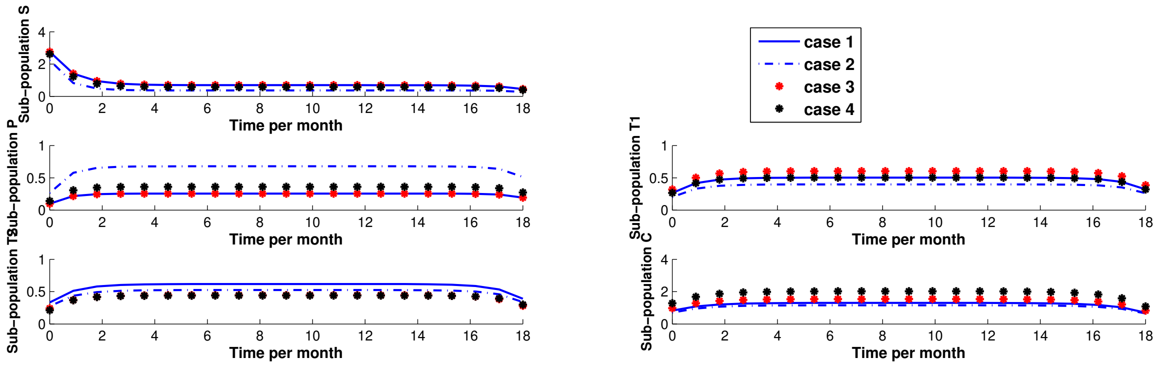

Fig. 4: Numerical simulation for different parameters value.

Cases \ Parameters	μ	β	δ_1	δ_2	γ	v	h	k
Case 1	0.15	0.02	0.1	0.08	0.5	0.5	0.46	0.355
Case 2	0.2	0.05	0.12	0.08	0.4	0.5	0.45	0.35
Case 3	0.15	0.02	0.1	0.08	0.5	0.6	0.75	0.2
Case 4	0.15	0.02	0.1	0.28	0.5	0.5	0.56	0.35

Table 2: Different parameters value.

As shown in Figure (4), the fractional-order solutions obtained with the finite element method remain coherent across all the parameter scenarios considered. In each case, the model is able to reproduce the expected epidemiological trends, which confirms the reliability and flexibility of the proposed fractional approach. These results clearly show the benefit of using a fractional model instead of the classical one. We can also observe realistic behaviors in the trajectories, such as convergence to equilibrium or sustained oscillations, depending on the scenario. This shows that the model can capture more complex dynamics in a way that better reflects real data.

5.2 The stability of equilibrium points

Now, let us confirm the theoretical results obtained in section 4. To simulate the convergence of the two points. The epidemiological data are presented in Table 3, with the initial conditions $S(0) = 15, P(0) = 30, T_1(0) = 27, T_2(0) = 12, C(0) = 20$;

Parametres	D F E	E E
λ	0.28	0.28
η	0.15	0.15
r	0.2	0.2
μ	0.15	0.2
β	0.02	0.05
δ_1	0.1	0.12
δ_2	0.08	0.08
γ	0.5	0.4
v	0.6	0.5
h	0.75	0.45
k	0.2	0.35

Table 3: Parameters for disease-free and disease-endemic equilibrium.

Figure (5) illustrates the existence and stability conditions stated in Theorem 2, with $R_0 = 0.6486$, which is less than 1. This condition satisfies the stability criterion for the disease-free equilibrium (DFE), indicating that the system (2) has

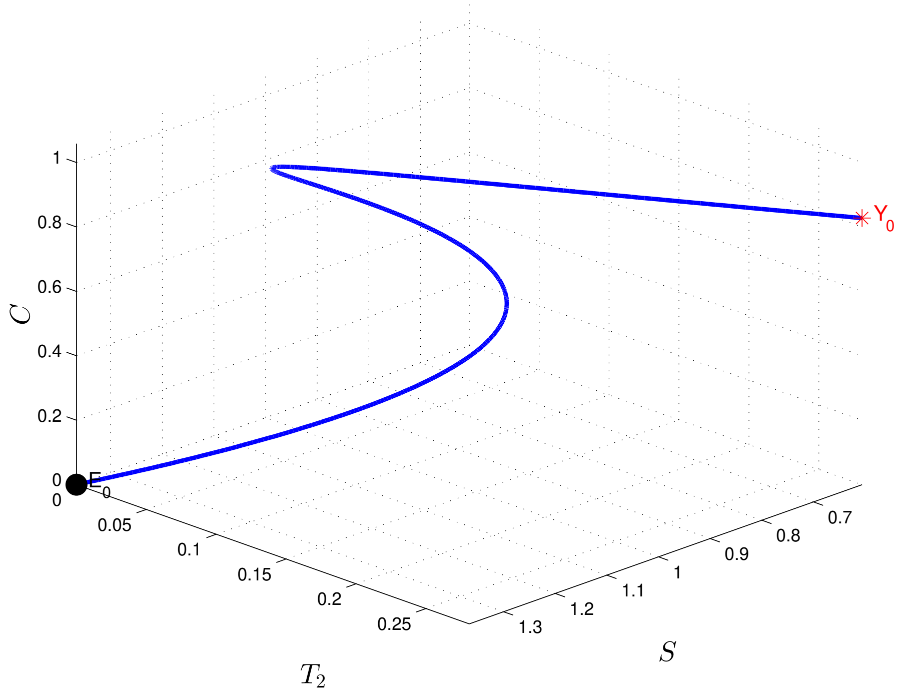

Fig. 5: Graph of numerical convergence to E_0.

a stable DFE at

$$E_0 = (1.3663, 0, 0, 0, 0)$$

in Ω. The existence and stability conditions from Theorem 3 are also satisfied, with $R_0 = 1.4121$, which is greater than 1. This condition leads to the existence of a unique endemic equilibrium point in Ω, represented by

$$E_1 = (1.6708, 0.2265, 0.0569, 0.0357, 0.0132)$$

as shown in Figure (6). The conditions $R_0 < 1$ for stability of the DFE and $R_0 > 1$ for the endemic equilibrium demonstrate the dynamic behavior of the system, where the disease-free equilibrium is stable when $R_0 < 1$, and an endemic equilibrium emerges when $R_0 > 1$. These results provide strong numerical evidence supporting the theoretical analysis of the fractional model. The stability conditions for both the disease-free and endemic equilibria are satisfied and clearly reflected in the system's dynamics. This validates the model's ability to describe real-world epidemiological transitions depending on the value of R_0, reinforcing the relevance of the fractional approach in capturing complex biological behaviors.

Conclusion

In this study, we proposed a finite element method tailored to numerically solve a fractional-order mathematical model describing the prevalence of infertility. The model accounts for five key compartments representing various stages of infertility and treatment among couples. To assess the efficiency of the proposed numerical scheme, we compared approximate solutions for different values of the fractional order α, including the classical case when $\alpha = 1$, in the absence of an exact analytical solution. Our findings suggest that the use of the Caputo fractional derivative operator provides an effective framework for capturing memory effects inherent in infertility dynamics. By employing the Lax-Milgram lemma and conducting a stability analysis, we verified the existence and uniqueness of stable solutions under appropriate conditions. Numerical simulations confirmed the convergence of the system toward the equilibrium points, and visualizations using the finite element method demonstrated the reliability and applicability of our approach. This fractional model offers a powerful and flexible tool not only for studying infertility but also for generalizing to other epidemiological models involving memory-dependent dynamics. When accurate transmission parameters and real-time data are available, the proposed framework can significantly enhance the predictive power and reliability of such models.

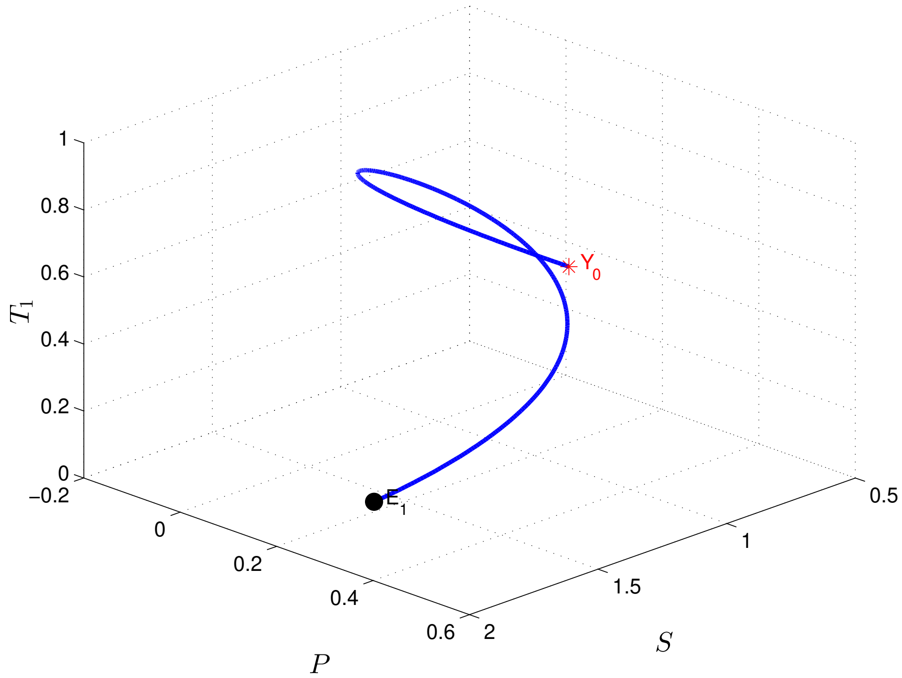

Fig. 6: Graph of numerical convergence to E_1.

Future Research Directions

- Extending the model to incorporate stochastic effects or delays in treatment.
- Integrating demographic or behavioral factors to improve biological realism.
- Applying the method to other health conditions or epidemic diseases with similar compartmental structures.
- Comparing the finite element method with other numerical schemes such as spectral or mesh-free methods in fractional contexts.

This work thus bridges the gap between mathematical modeling and real-world medical concerns, offering a solid foundation for further developments in the mathematical analysis of infertility and related health challenges.

Declarations

Competing interests: The authors declare no known financial or personal conflicts of interest that could have appeared to influence the work reported in this paper.
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