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Existence results for a coupled system of multi-term Katugampola fractional differential equations with integral conditions

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Abstract: This paper investigates a coupled system of nonlinear multi-term Katugampola fractional differential equations. Under sufficient conditions, it establishes the existence and uniqueness results of the solution by using standard fixed point theorems. Additionally, the paper includes some illustrative examples to strengthen the presented main results.

Keywords: Coupled system; Katugampola fractional derivative; Existence and uniqueness; Integral conditions; Fixed point theorems.

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1 Introduction

We consider the following coupled system of nonlinear multi-term fractional differential equations:

$$\begin{cases} {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t) = f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right), \\ {}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t) = f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right), \end{cases} \qquad t\in[0,\ell], \tag{1}$$

with the integral conditions

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi\right)(0^+) = \left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi\right)(0^+) = 0, \tag{2}$$

where $\rho,\ell>0,\ 0<\beta_{ij}<\alpha_i<1$ and $f_i:[0,\ell]\times\mathbb{R}^4\to\mathbb{R}$ are continuous functions for every $i,j\in\{1,2\}$. The operator ${}^{\rho}\mathscr{D}_{0^+}^{\alpha}$ and ${}^{\rho}\mathscr{I}_{0^+}^{1-\alpha}$ represents the Katugampola fractional derivative and integral of order $\alpha>0$, respectively.

The initial value problems are a vast and significant area of research, as these problems have applications in various scientific fields. Recently, so-called fractional initial value problems have appeared and become widespread, allowing the modeling of many real-world phenomena, as well as giving an understanding of some mathematical problems such as the Abel equation [22],

$$\int_a^t y(s)(t-s)^{\alpha-1}ds = f(t), \qquad 0<\alpha<1.$$

Recently, the resolvability of fractional differential equations with different kinds of initial or boundary conditions has witnessed a remarkable trend, which has led to the publication of many works in this regard, for example, but not limited to, see [2,4,5,6,7,8,9,10,11,12,13,14,16,21,23] and references cited therein.

The existence and uniqueness result of the coupled system of fractional differential equations (1) with integral boundary condition has been investigated in [3], but the functions f_1 dependent on time t, unknown functions φ and

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$\mathscr{D}_{0^+}^{\beta_{12}}\psi$ while f_2 dependent on time t, unknown functions ψ and $\mathscr{D}_{0^+}^{\beta_{21}}\varphi$. The authors in [20], studied the existence and uniqueness of the solution for system (1) with integral conditions where the functions f_1 and f_2 dependent only on time t and unknown functions φ and ψ. A similar result was found in [23], where the function f_1 dependent only on time t and unknown function φ and f_2 dependent only on time t and unknown function ψ.

The main contribution of this paper can be summarized in obtaining the existence and uniqueness result of a coupled system, with some conditions on the functions of second member f_1 and f_2.

The organization of this paper is as follows: In Section 2, we describe some preliminary concepts related to the proposed study; in Section 3, we give some existence and uniqueness results for the problem (1)–(2). The results are based on Schauder's and contraction mapping principle fixed point theorems in a special Banach space. In Section 4, two examples are presented to explain the application of our main results. Finally, we present some conclusions in Section 5.

2 Preliminaries

Here, as in [19], we will look at the Katugampola's fractional integral, derivative and some of their properties. Let $r\in\mathbb{R}$, $p\in[1,\infty]$ and

$$X_r^p([0,\ell],\mathbb{R})=\left\{\varphi:[0,\ell]\longrightarrow\mathbb{R}\text{ Lebesgue measurable and }\|\varphi\|_{X_r^p}<\infty\right\},$$

with the norm

$$\|\varphi\|_{X_r^p}=\begin{cases}\left(\int_0^\ell\frac{|t^r\varphi(t)|^p}{t}dt\right)^{1/p}, & \text{for } 1\le p<\infty,\\ \operatorname{ess}\sup\limits_{0\le t\le\ell}\{t^r|\varphi(t)|\}, & \text{for } p=\infty.\end{cases}$$

Let $C([0,\ell],\mathbb{R})$ be the collection of continuous functions from $[0,\ell]$ into $\mathbb{R}$ with the norm

$$\|\varphi\|_\infty=\sup_{0\le t\le\ell}|\varphi(t)|.$$

Then $C([0,\ell],\mathbb{R})$ is Banach space.

Definition 1([17]). *The Katugampola's fractional integral of order $\alpha\in\mathbb{R}_+$ of a function $g\in X_r^p([0,\ell],\mathbb{R})$ is defined as*

$$^{\rho}\mathscr{I}_{0^+}^{\alpha}g(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}g(s)\,ds,\qquad t\in[0,\ell],\tag{3}$$

for $\rho>0$. This is a left-sided integral.

Similarly, for the right-sided integrals definition. From Definition 1 we can infer

$$\left(t^{1-\rho}\frac{d}{dt}\right)\,^{\rho}\mathscr{I}_{0^+}^{\alpha+1}g(t)=\,^{\rho}\mathscr{I}_{0^+}^{\alpha}g(t).\tag{4}$$

Definition 2([18]). *The generalized fractional derivative of order $\alpha\in\mathbb{R}_+$, corresponding to the Katugampola's fractional integral (3) is defined for any $t\in[0,\ell]$ as*

$$\begin{aligned}^{\rho}\mathscr{D}_{0^+}^{\alpha}g(t)&=\left(t^{1-\rho}\frac{d}{dt}\right)^n\left(^{\rho}\mathscr{I}_{0^+}^{n-\alpha}g\right)(t)\\ &\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{n-\alpha-1}g(s)\,ds,\end{aligned}\tag{5}$$

if the integral exists. Here $\rho>0$ and $n=[\alpha]+1$, with $[\cdot]$ denotes the integer part.

Lemma 1([7]). *Let $\alpha,\rho>0$ and $g\in C([0,\ell],\mathbb{R})$. Then:*

1. The equation $^{\rho}\mathscr{D}_{0^+}^{\alpha}g(t)=0$ has a unique solution

$$g(t)=\sum_{i=1}^n c_i t^{\rho(\alpha-n)},\qquad n=[\alpha]+1,\ c_i\in\mathbb{R}_+.$$

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2. If ${}^{\rho}\mathscr{D}_{0^+}^{\alpha} g(t) \in C([0,\ell],\mathbb{R})$ *and* $0<\alpha\leq 1$, *then*

$$ {}^{\rho}\mathscr{I}_{0^+}^{\alpha}\, {}^{\rho}\mathscr{D}_{0^+}^{\alpha} g(t) = g(t) + ct^{\rho(\alpha-1)}, \tag{6}$$

for some constant $c\in\mathbb{R}_+$.

3. Let $0<\beta<\alpha\leq 1$ *be such that* ${}^{\rho}\mathscr{D}_{0^+}^{\alpha} g(t) \in C([0,\ell],\mathbb{R})$ *then*

$$ {}^{\rho}\mathscr{I}_{0^+}^{\alpha-\beta}\, {}^{\rho}\mathscr{D}_{0^+}^{\alpha} g(t) = {}^{\rho}\mathscr{D}_{0^+}^{\beta} g(t) - \frac{\rho^{1-\alpha+\beta}\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha} g\right)(0^+)}{\Gamma(\alpha-\beta)} t^{\rho(\alpha-\beta-1)}. \tag{7}$$

Moreover, if $\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha} g\right)(0^+)=0$, *we have*

$$ \left|{}^{\rho}\mathscr{D}_{0^+}^{\beta} g(t)\right| \leq \lambda_{\alpha-\beta}^{\rho} \left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha} g(t)\right\|_{\infty}, \tag{8}$$

where $\lambda_{\alpha-\beta}^{\rho} = \frac{\ell^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}$.

3 Main results

Below, we prepare some important lemmas to illustrate our main results.

Lemma 2. *Let* $(\varphi,\psi), \left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi, {}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi\right) \in C([0,\ell],\mathbb{R})\times C([0,\ell],\mathbb{R})$. *Then the problem* (1)–(2) *is equivalent to the fractional integral equations:*

$$ \begin{cases} \varphi(t) = \int_0^t G_{\alpha_1}(t,s) f_1\left(s,\varphi(s),\psi(s), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\right) ds, \\ \psi(t) = \int_0^t G_{\alpha_2}(t,s) f_2\left(s,\varphi(s),\psi(s), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(s), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(s)\right) ds, \end{cases} \tag{9}$$

where $G_{\alpha_i}(t,s) = \frac{\rho^{1-\alpha_i} s^{\rho-1}}{\Gamma(\alpha_i)} (t^{\rho}-s^{\rho})^{\alpha_i-1}$.

Proof. Applying ${}^{\rho}\mathscr{I}_{0^+}^{\alpha_1}$ and ${}^{\rho}\mathscr{I}_{0^+}^{\alpha_2}$ to the first and second equations in (1), respectively, we get

$$ \begin{cases} {}^{\rho}\mathscr{I}_{0^+}^{\alpha_1}\, {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t) = {}^{\rho}\mathscr{I}_{0^+}^{\alpha_1} f_1\left(t,\varphi(t),\psi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right), \\ {}^{\rho}\mathscr{I}_{0^+}^{\alpha_2}\, {}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t) = {}^{\rho}\mathscr{I}_{0^+}^{\alpha_2} f_2\left(t,\varphi(t),\psi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right). \end{cases} \tag{10}$$

By using the relation (6), we obtain

$$ \begin{cases} \varphi(t) = {}^{\rho}\mathscr{I}_{0^+}^{\alpha_1} f_1\left(t,\varphi(t),\psi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right) - c_1 t^{\rho(\alpha_1-1)}, \\ \psi(t) = {}^{\rho}\mathscr{I}_{0^+}^{\alpha_2} f_2\left(t,\varphi(t),\psi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t), {}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right) - c_2 t^{\rho(\alpha_2-1)}, \end{cases} \tag{11}$$

for some $c_1,c_2\in\mathbb{R}$. Taking into account the condition (2) and the fact that

$$ {}^{\rho}\mathscr{I}_{0^+}^{\alpha} t^{\rho(\alpha-1)} = \rho^{\alpha-1}\Gamma(\alpha), $$

we find

$$ 0 = \left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi\right)(0^+) = -c_1\rho^{\alpha_1-1}\Gamma(\alpha_1) \quad \Longrightarrow c_1 = 0 \tag{12}$$

and

$$ 0 = \left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi\right)(0^+) = -c_2\rho^{\alpha_2-1}\Gamma(\alpha_2) \quad \Longrightarrow c_2 = 0. \tag{13}$$

Combining the results (11), (12) and (13), we obtain (9).

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Let us define the following Banach spaces [7],

$$E=\left\{\varphi\in C\left([0,\ell],\mathbb{R}\right)/\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi\right)\left(0^+\right)=0\right\},$$

with the norm

$$\|\varphi\|_E=\sup_{0\le t\le\ell}|\varphi(t)|$$

and

$$F=\left\{\psi\in C\left([0,\ell],\mathbb{R}\right)/\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi\right)\left(0^+\right)=0\right\},$$

with the norm

$$\|\psi\|_F=\sup_{0\le t\le\ell}|\psi(t)|.$$

Again the product space $(\Omega,\|\cdot\|_\Omega)$ is a Banach space with norm $\|(\varphi,\psi)\|_\Omega=\|\varphi\|_E+\|\psi\|_F$ for any $(\varphi,\psi)\in\Omega=E\times F$.

Now, we define an operator $\mathscr{T}:\Omega\to C([0,\ell],\mathbb{R})\times C([0,\ell],\mathbb{R})$ by

$$\mathscr{T}(\varphi,\psi)(t)=\left(\mathscr{T}_\varphi(\varphi,\psi)(t),\mathscr{T}_\psi(\varphi,\psi)(t)\right), \tag{14}$$

where

$$\mathscr{T}_\varphi(\varphi,\psi)(t)=\int_0^t G_{\alpha_1}(t,s)f_1\left(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\right)ds,$$
$$\mathscr{T}_\psi(\varphi,\psi)(t)=\int_0^t G_{\alpha_2}(t,s)f_2\left(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(s)\right)ds,$$

and $G_{\alpha_i}(t,s)=\frac{\rho^{1-\alpha_i}s^{\rho-1}}{\Gamma(\alpha_i)}\left(t^\rho-s^\rho\right)^{\alpha_i-1}$.

Lemma 3. *Let the integral operator $\mathscr{T}:\Omega\to C([0,\ell],\mathbb{R})\times C([0,\ell],\mathbb{R})$ given in (14), equipped with the norm*

$$\|\mathscr{T}(\varphi,\psi)\|_\infty=\sup_{0\le t\le\ell}\left|\mathscr{T}_\varphi(\varphi,\psi)\right|+\sup_{0\le t\le\ell}\left|\mathscr{T}_\psi(\varphi,\psi)\right|.$$

Then $\mathscr{T}(\Omega)\subset\Omega$.

Proof. Let $(\varphi,\psi)\in\Omega$. From (14), we have

$$\begin{aligned}\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\mathscr{T}_\varphi(\varphi,\psi)\right)(t)&={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\,{}^{\rho}\mathscr{I}_{0^+}^{\alpha_1}f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right)\\&={}^{\rho}\mathscr{I}_{0^+}^{1}\,f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right)\end{aligned}$$

and

$$\begin{aligned}\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\mathscr{T}_\psi(\varphi,\psi)\right)(t)&={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\,{}^{\rho}\mathscr{I}_{0^+}^{\alpha_2}f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right)\\&={}^{\rho}\mathscr{I}_{0^+}^{1}f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right).\end{aligned}$$

Using Definition 2 and relation (4), we get

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\mathscr{T}_\varphi(\varphi,\psi)\right)(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi(t)={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi(t)$$

and

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\mathscr{T}_\psi(\varphi,\psi)\right)(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi(t)={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi(t).$$

Thus

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\mathscr{T}_\varphi(\varphi,\psi)\right)\left(0^+\right)=\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\mathscr{T}_\psi(\varphi,\psi)\right)\left(0^+\right)=0.$$

As a result $\mathscr{T}(\Omega)\subset\Omega$.

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Getting ready to present our results, we propose the following hypotheses:

Hyp.1. Let $f_1, f_2 : [0,\ell] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions and there are two strictly positive constants k_1 and k_2 such that

$$|f_i(t,\varphi_1,\varphi_2,\varphi_3,\varphi_4) - f_i(t,\psi_1,\psi_2,\psi_3,\psi_4)| \leq k_i \sum_{j=1}^{4} |\varphi_j - \psi_j|, \qquad i = 1,2,$$

for all $t \in [0,\ell]$ and $\varphi_i, \psi_i \in \mathbb{R},\ i = 1,2,3,4$.

Hyp.2. There exist a positive functions $a_i, b_i \in C([0,\ell],\mathbb{R}),\ i = 1,2,\ldots,5$ such that

$$|f_1(t,\varphi_1,\varphi_2,\varphi_3,\varphi_4)| \leq a_1(t) + \sum_{i=2}^{5} a_i(t)|\varphi_i|$$

and

$$|f_2(t,\varphi_1,\varphi_2,\varphi_3,\varphi_4)| \leq b_1(t) + \sum_{i=2}^{5} b_i(t)|\varphi_i|,$$

for any $\varphi_i \in \mathbb{R},\ i = 1,2,3,4$ and $t \in [0,\ell]$.

To simplify the computation, we adopt the notation:

$$\begin{aligned}
\lambda_{ij}^{\rho} &= \lambda_{\alpha_i-\beta_{ji}}^{\rho} = \frac{\ell^{\rho(\alpha_i-\beta_{ji})}}{\rho^{\alpha-\beta}\Gamma(1+\alpha_i-\beta_{ji})}, \quad i,j = 1,2,\\
\bar{a}_i &= \max_{0\leq t\leq \ell}|a_i(t)|, \quad \bar{b}_i = \max_{0\leq t\leq \ell}|b_i(t)|, \quad i = 1,2,\ldots,5,\\
\bar{G}_{\alpha} &= \frac{\rho^{-\alpha}\ell^{\rho\alpha}}{\Gamma(\alpha+1)}, \quad \bar{G} = \max\left\{\bar{G}_{\alpha_1},\bar{G}_{\alpha_2}\right\},\\
d_1 &= \frac{\bar{a}_1+\bar{b}_1}{\min\left\{1-\bar{a}_4\lambda_{11}^{\rho}-\bar{b}_4\lambda_{12}^{\rho}, 1-\bar{a}_5\lambda_{21}^{\rho}-\bar{b}_5\lambda_{22}^{\rho}\right\}},\\
d_2 &= \frac{\max\left\{\bar{a}_2+\bar{b}_2,\bar{a}_3+\bar{b}_3\right\}}{\min\left\{1-\bar{a}_4\lambda_{11}^{\rho}-\bar{b}_4\lambda_{12}^{\rho}, 1-\bar{a}_5\lambda_{21}^{\rho}-\bar{b}_5\lambda_{22}^{\rho}\right\}},
\end{aligned}$$

with

$$\max_{i\in\{1,2\}}\left\{\bar{a}_{3+i}\lambda_{i1}^{\rho}+\bar{b}_{3+i}\lambda_{i2}^{\rho}, k_1\lambda_{i1}^{\rho}+k_2\lambda_{i2}^{\rho}, \frac{\bar{G}_{\alpha_1}k_1\lambda_{i1}^{\rho}+\bar{G}_{\alpha_2}k_2\lambda_{i2}^{\rho}}{\bar{G}_{\alpha_i}}\right\} < 1. \tag{15}$$

Now, we present the principal theorems

Theorem 1. *Assume (Hyp.1) holds. If*

$$k_G = \frac{\left(k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}\right)\bar{G}}{\min\limits_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}} < 1, \tag{16}$$

then the problem (1)–(2) *has a unique solution on* $[0,\ell]$.

Proof. First, we define the fixed point problem, which is equivalent to the one problem (1)–(2) by

$$\mathscr{T}(\varphi,\psi)(t) = (\varphi,\psi)(t). \tag{17}$$

Let $(\varphi,\psi), (\bar{\varphi},\bar{\psi}) \in \Omega$, then we have

$$\begin{aligned}
&\left|\mathscr{T}_{\varphi}(\varphi,\psi)(t) - \mathscr{T}_{\varphi}(\bar{\varphi},\bar{\psi})(t)\right|\\
=&\left|\int_0^t G_{\alpha_1}(t,s)\left[f_1\left(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\right)\right.\right.\\
&\left.\left.-f_1\left(s,\bar{\varphi}(s),\bar{\psi}(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\bar{\varphi}(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\bar{\psi}(s)\right)\right]ds\right|\\
=&\left|\int_0^t G_{\alpha_1}(t,s)\left[{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s) - {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(s)\right]ds\right|\\
\leq&\int_0^t G_{\alpha_1}(t,s)\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s) - {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(s)\right|ds
\end{aligned}$$

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Using Hölder inequality and the fact that

$$\sup_{0\leq t\leq \ell}\int_0^t G_{\alpha_1}(t,s)\,ds=\frac{\rho^{-\alpha_1}\ell^{\rho\alpha_1}}{\Gamma(\alpha_1+1)},$$

we get

$$\begin{aligned}\left\|\mathscr{T}_{\varphi}(\varphi,\psi)(t)-\mathscr{T}_{\varphi}(\bar{\varphi},\bar{\psi})(t)\right\|_{\infty}&\leq\int_0^t G_{\alpha_1}(t,s)\,ds\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}\\&\leq\frac{\rho^{-\alpha_1}\ell^{\rho\alpha_1}}{\Gamma(\alpha_1+1)}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}.\end{aligned}\tag{18}$$

And in the same way, we obtain

$$\left\|\mathscr{T}_{\psi}(\varphi,\psi)(t)-\mathscr{T}_{\psi}(\bar{\varphi},\bar{\psi})(t)\right\|_{\infty}\leq\frac{\rho^{-\alpha_2}\ell^{\rho\alpha_2}}{\Gamma(\alpha_2+1)}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}.\tag{19}$$

Also, we have

$$\begin{aligned}\left\|\mathscr{T}(\varphi,\psi)(t)-\mathscr{T}(\bar{\varphi},\bar{\psi})(t)\right\|_{\infty}&\leq\bar{G}_{\alpha_1}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}+\bar{G}_{\alpha_2}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}\\&\leq\bar{G}\left(\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}\right).\end{aligned}\tag{20}$$

By taking into account the hypothesis (Hyp.1), we obtain

$$\begin{aligned}\frac{1}{k_1}\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right|&\leq|\varphi(t)-\bar{\varphi}(t)|+|\psi(t)-\bar{\psi}(t)|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\bar{\varphi}(t)\right|\\&\quad+\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\bar{\psi}(t)\right|.\end{aligned}$$

Using the equality (8), we get

$$\begin{aligned}\frac{1}{k_1}\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right|&\leq|\varphi(t)-\bar{\varphi}(t)|+\lambda_{11}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}\\&\quad+|\psi(t)-\bar{\psi}(t)|+\lambda_{21}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty},\end{aligned}$$

Consequently

$$\begin{aligned}\frac{1}{k_1}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}&\leq\|\varphi(t)-\bar{\varphi}(t)\|_{\infty}+\lambda_{11}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}\\&\quad+\|\psi(t)-\bar{\psi}(t)\|_{\infty}+\lambda_{21}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}.\end{aligned}\tag{21}$$

In the same way, we can get

$$\begin{aligned}\frac{1}{k_2}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}&\leq\|\varphi(t)-\bar{\varphi}(t)\|_{\infty}+\lambda_{12}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}\\&\quad+\|\psi(t)-\bar{\psi}(t)\|_{\infty}+\lambda_{22}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}.\end{aligned}\tag{22}$$

Multiplying (21) by $k_1\bar{G}_{\alpha_1}$ and (22) by $k_2\bar{G}_{\alpha_2}$, then take the sum, we obtain

$$\begin{aligned}&\bar{G}_{\alpha_1}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}+\bar{G}_{\alpha_2}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}\\&\leq\left(k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}\right)\left\{\|\varphi(t)-\bar{\varphi}(t)\|_{\infty}+\|\psi(t)-\bar{\psi}(t)\|_{\infty}\right\}\\&+\left(k_1\bar{G}_{\alpha_1}\lambda_{11}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{12}^{\rho}\right)\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}\\&+\left(k_1\bar{G}_{\alpha_1}\lambda_{21}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{22}^{\rho}\right)\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty},\end{aligned}\tag{23}$$

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thus

$$\begin{aligned}
&\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}\left[\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}\right]\\
&\leq \bar{G}_{\alpha_1}-\left(k_1\bar{G}_{\alpha_1}\lambda_{11}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{12}^{\rho}\right)\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}\\
&+\bar{G}_{\alpha_2}-\left(k_1\bar{G}_{\alpha_1}\lambda_{21}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{22}^{\rho}\right)\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}\\
&\leq \left(k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}\right)\left\|(\varphi(t),\psi(t))-(\bar{\varphi}(t),\bar{\psi}(t))\right\|_{\Omega}, \qquad (24)
\end{aligned}$$

relation (15) guarantees that $\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}>0$, then

$$\begin{aligned}
&\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_{\infty}+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_{\infty}\\
&\leq \frac{k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}}{\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}}\left\|(\varphi(t),\psi(t))-(\bar{\varphi}(t),\bar{\psi}(t))\right\|_{\Omega}. \qquad (25)
\end{aligned}$$

Combining (20) and (25), we get

$$\left\|\mathscr{T}(\varphi,\psi)(t)-\mathscr{T}(\bar{\varphi},\bar{\psi})(t)\right\|_{\Omega}\leq k_G\left\|(\varphi(t),\psi(t))-(\bar{\varphi}(t),\bar{\psi}(t))\right\|_{\Omega},$$

where

$$k_G=\frac{\left(k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}\right)\bar{G}}{\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}}.$$

Since $k_G<1$ according to (16), then $\mathscr{T}$ is a contraction operator and has unique fixed point following the Banach's contraction principle [15]. Which means that the problem (1)–(2) has a unique solution on $[0,\ell]$.

Theorem 2. *Assume that hypotheses (Hyp.1) and (Hyp.2) hold. If we put*

$$\bar{G}d_2<1, \qquad (26)$$

then the problem (1)–(2) *has at least one solution on* $[0,\ell]$.

Proof. As in the previous proof, we will prove that the operator (17) has a fixed point using Schauder's theorem [15]. This is done through three steps:

Step 1: A is a continuous operator. Let $(\varphi_n,\psi_n)_{n\in\mathbb{N}}$ be real sequences such that $(\varphi_n,\psi_n)\to(\varphi,\psi)$ in Ω.
Using the same techniques used to prove theorem 1, then by replacing $(\bar{\varphi},\bar{\psi})$ by (φ_n,ψ_n), the relations (21) and (22) became

$$\begin{aligned}
\frac{1}{k_1}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_{\infty}&\leq\left\|\varphi_n(t)-\varphi(t)\right\|_{\infty}+\lambda_{11}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_{\infty}\\
&+\left\|\psi_n(t)-\psi(t)\right\|_{\infty}+\lambda_{21}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_{\infty} \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{k_2}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_{\infty}&\leq\left\|\varphi_n(t)-\varphi(t)\right\|_{\infty}+\lambda_{12}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_{\infty}\\
&+\left\|\psi_n(t)-\psi(t)\right\|_{\infty}+\lambda_{22}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_{\infty}. \qquad (28)
\end{aligned}$$

By combining (27) and (28), we obtain

$$\begin{aligned}
&\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_{\infty}+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_{\infty}\\
&\leq\frac{(k_1+k_2)}{\min_{i\in\{1,2\}}\left\{1-k_1\lambda_{i1}^{\rho}-k_2\lambda_{i2}^{\rho}\right\}}\left\|(\varphi_n(t),\psi_n(t))-(\varphi(t),\psi(t))\right\|_{\Omega},
\end{aligned}$$

and from (15), we answer that $\min_{i\in\{1,2\}}\left\{1-k_1\lambda_{i1}^{\rho}-k_2\lambda_{i2}^{\rho}\right\}>0$. As $(\varphi_n,\psi_n)\underset{n\to\infty}{\longrightarrow}(\varphi,\psi)$ in Ω, then $\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n\right)\underset{n\to\infty}{\longrightarrow}\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi\right)$, for all $t\in[0,\ell]$.

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Now, let $\delta > 0$ be such that for each $t \in [0,\ell]$, we have

$$\sup\left\{ \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)\right|, \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)\right|, \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right|, \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right| \right\} \leq \delta.$$

Then, we have

$$\begin{aligned}
&\Big|G_{\alpha_1}(t,s)\Big[f_1\Big(s,\varphi_n(s),\psi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi_n(s)\Big)\\
&-f_1\Big(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\Big)\Big]\Big|\\
=&\left|G_{\alpha_1}(t,s)\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(s)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s)\right)\right|\\
\leq&\, G_{\alpha_1}(t,s)\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(s)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s)\right|\\
\leq&\, G_{\alpha_1}(t,s)\left(\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(s)\right|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s)\right|\right)\\
\leq&\, 2\delta G_{\alpha_1}(t,s)
\end{aligned}$$

and in the same way we find

$$G_{\alpha_2}(t,s)\left(\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(s)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(s)\right|\right) \leq 2\delta G_{\alpha_2}(t,s)\,ds.$$

Which means that the functions $s \to \delta G_{\alpha_i}(t,s)$, $i=1,2$ are integrable for all $t \in [0,\ell]$.
Then Lebesgue dominated convergence theorem is applicable to the following

$$\begin{aligned}
&\Big|\int_0^t G_{\alpha_1}(t,s)\Big[f_1\Big(s,\varphi_n(s),\psi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi_n(s)\Big)\\
&-f_1\Big(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\Big)\Big]ds\Big|\\
=&\left|\mathscr{T}_{\varphi}(\varphi_n,\psi_n)(t)-\mathscr{T}_{\varphi}(\varphi,\psi)(t)\right| \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_0^t G_{\alpha_2}(t,s)\Big[f_2\Big(s,\varphi_n(s),\psi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi_n(s)\Big)\\
&-f_2\Big(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\Big)\Big]ds\Big|\\
=&\left|\mathscr{T}_{\psi}(\varphi_n,\psi_n)(t)-\mathscr{T}_{\psi}(\varphi,\psi)(t)\right| \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore

$$\left\|\mathscr{T}(\varphi,\psi)(t)-\mathscr{T}(\bar{\varphi},\bar{\psi})(t)\right\|_{\Omega} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence the continuity of the operator $\mathscr{T}$.

Step 2: $A(B_\tau) \subset B_\tau$. Let B_τ be bounded, closed and convex subset of Ω, define by

$$B_\tau = \left\{(\varphi,\psi) \in \Omega \ / \ \|(\varphi,\psi)\|_{\Omega} \leq \tau\right\},$$

where $\tau \geq \frac{d_1}{(1/\bar{G}-d_2)}$.

Let $\mathscr{T}: B_\tau \to \Omega$ be the operator defined in (14). Then by applying the inequality (8) and hypothses (Hyp.2) for all $t \in [0,\ell]$, we have

$$\begin{aligned}
\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right| =&\left|f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right)\right| \\
\leq& a_1(t)+a_2(t)|\varphi(t)|+a_3(t)|\psi(t)|+a_4(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t)\right|+a_5(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right|\\
\leq& \bar{a}_1+\bar{a}_2\|\varphi(t)\|_\infty+\bar{a}_3\|\psi(t)\|_\infty+\bar{a}_4\lambda_{11}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty+\bar{a}_5\lambda_{21}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\varphi(t)\right| =&\left|f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right)\right| \\
\leq& b_1(t)+b_2(t)|\varphi(t)|+b_3(t)|\psi(t)|+b_4(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t)\right|+b_5(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right|\\
\leq& \bar{b}_1+\bar{b}_2\|\varphi(t)\|_\infty+\bar{b}_3\|\psi(t)\|_\infty+\bar{b}_4\lambda_{12}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty+\bar{b}_5\lambda_{22}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty,
\end{aligned} \tag{30}$$

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Combining the results (29) and (30). Similarly to (20), for all $(\varphi,\psi)\in B_\tau$ we get

$$\begin{aligned}\|\mathscr{T}(\varphi,\psi)(t)\|_\Omega &\le \bar{G}d_1+\bar{G}d_2\tau\\ &=\bar{G}\left[\left(1/\bar{G}-d_2\right)\frac{d_1}{\left(1/\bar{G}-d_2\right)}+d_2\tau\right]\\ &\le\tau.\end{aligned}$$

Then, we conclude that $\mathscr{T}(B_\tau)\subset B_\tau$.

Step 3: $A(B_\tau)$ is relatively compact. Let $t_1,t_2\in[0,\ell]$, $t_1<t_2$ and $(\varphi,\psi)\in B_\tau$. Then, we get

$$\begin{aligned}&\left|\mathscr{T}_\varphi(\varphi,\psi)(t_2)-\mathscr{T}_\varphi(\varphi,\psi)(t_1)\right|+\left|\mathscr{T}_\psi(\varphi,\psi)(t_2)-\mathscr{T}_\psi(\varphi,\psi)(t_1)\right|\\ &\le(d_1+d_2\tau)\left[\max_{i\in\{1,2\}}\int_0^{t_1}\left|G_{\alpha_i}(t_2,s)-G_{\alpha_i}(t_1,s)\right|ds+\max_{i\in\{1,2\}}\int_{t_1}^{t_2}G_{\alpha_i}(t_2,s)\,ds\right].\end{aligned}\tag{31}$$

On the other hand

$$\begin{aligned}\int_0^{t_1}\left|G_{\alpha_i}(t_2,s)-G_{\alpha_i}(t_1,s)\right|ds&=\frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}\int_0^{t_1}s^{\rho-1}\left|\left(t_2^\rho-s^\rho\right)^{\alpha_i-1}-\left(t_1^\rho-s^\rho\right)^{\alpha_i-1}\right|ds\\ &\le\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left[\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}+\left(t_2^{\rho\alpha_i}-t_1^{\rho\alpha_i}\right)\right]\end{aligned}\tag{32}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\left|G_{\alpha_i}(t_2,s)\right|ds&=\frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}s^{\rho-1}\left(t_2^\rho-s^\rho\right)^{\alpha_i-1}ds\\ &=\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}.\end{aligned}\tag{33}$$

Applying (32) and (33), then (31) becomes

$$\begin{aligned}&\left|\mathscr{T}_\varphi(\varphi,\psi)(t_2)-\mathscr{T}_\varphi(\varphi,\psi)(t_1)\right|+\left|\mathscr{T}_\psi(\varphi,\psi)(t_2)-\mathscr{T}_\psi(\varphi,\psi)(t_1)\right|\\ &\le(d_1+d_2\tau)\left[\max_{i\in\{1,2\}}\left\{\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left[\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}+\left(t_2^{\rho\alpha_i}-t_1^{\rho\alpha_i}\right)\right]\right\}\right.\\ &\left.\quad+\max_{i\in\{1,2\}}\left\{\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}\right\}\right].\end{aligned}$$

Hence, we conclude that for all $(\varphi,\psi)\in B_\tau$, $\|\mathscr{T}(\varphi,\psi)(t_2)-\mathscr{T}(\varphi,\psi)(t_1)\|_\Omega\underset{t_1\to t_2}{\longrightarrow}0$.

From step 1-3 and Ascoli-Arzelà Theorem [1], we show that $\mathscr{T}:B_\tau\to B_\tau$ is continuous, compact and so by Schauder's fixed point, the operator $\mathscr{T}$ has at least one fixed point which corresponds to the solution of the problem (1)–(2) on $[0,\ell]$.

4 Examples

Example 1. Consider the following problem

$$\begin{cases}{}^\rho\mathscr{D}_{0^+}^{\frac12}\varphi(t)=\dfrac{1/2}{\sqrt2\cos\left(\frac{\pi t}{4}\right)+|\varphi(t)|+|\psi(t)|}+\dfrac{1/11}{\cosh t+\left|{}^\rho\mathscr{D}_{0^+}^{\frac14}\varphi(t)\right|+\left|{}^\rho\mathscr{D}_{0^+}^{\frac18}\psi(t)\right|}, & t\in[0,1],\\ {}^\rho\mathscr{D}_{0^+}^{\frac23}\psi(t)=\dfrac{1/4}{1+t+|\varphi(t)|+|\psi(t)|+\left|{}^\rho\mathscr{D}_{0^+}^{\frac12}\varphi(t)\right|+\left|{}^\rho\mathscr{D}_{0^+}^{\frac13}\psi(t)\right|}, & t\in[0,1],\\ \left({}^\rho\mathscr{I}_{0^+}^{\frac12}\varphi\right)(0^+)=\left({}^\rho\mathscr{I}_{0^+}^{\frac13}\psi\right)(0^+)=0.\end{cases}\tag{34}$$

Obviously, the condition (Hyp.1) is satisfied with $k_1=1/11$ and $k_2=1/4$. Then;

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ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
k_G	10.76	3.917	2.050	1.458	1.148	0.963	0.838	0.745	0.674	0.628
	Theorem 1 is not applicable in this example.					from Theorem 1, the problem (34) has a unique solution.				

Example 2. Consider the following problem

$$\begin{cases} {}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{4}}\varphi(t) = \dfrac{10^{-2}\sin t}{1+|\varphi(t)|+|\psi(t)|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{9}}\varphi(t)\right|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{7}}\psi(t)\right|}, & t\in[0,2], \\ {}^{\rho}\mathscr{D}_{0^+}^{\frac{4}{5}}\psi(t) = \frac{3e^{t-2}}{5}\frac{|\varphi(t)|}{1+|\varphi(t)|} + \dfrac{10^{-2}\cos t}{1+t+|\psi(t)|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{5}}\varphi(t)\right|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{2}{9}}\psi(t)\right|}, & t\in[0,2], \\ \left({}^{\rho}\mathscr{I}_{0^+}^{\frac{3}{4}}\varphi\right)(0^+) = \left({}^{\rho}\mathscr{I}_{0^+}^{\frac{1}{5}}\psi\right)(0^+) = 0. \end{cases} \tag{35}$$

Obviously, the hypotheses (Hyp.1) and (Hyp.2) are satisfied with $k_1 = 10^{-2}$, $k_2 = 3/5$, $\bar{a}_1 = \bar{b}_1 = 1$, $\bar{a}_2 = 0$, $\bar{b}_2 = 3/5$ and $\bar{a}_i = \bar{b}_i = 0$ for $i = 2,3,4,5$. Then, $d_1 = 2,\ d_2 = 0.6$; Thus

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\bar{G}d_2$	4.065	2.334	1.688	1.341	1.122	0.969	0.857	0.770	0.701	0.662
	Theorem 2 is not applicable in this example.					from Theorem 2, the problem (35) has at least a solution.				

5 Conclusion

Using the Banach contraction principle and Schauder's fixed point theorem, this paper explores the existence and main properties of at least one solution and its uniqueness for a class of new coupled systems of nonlinear multi-term fractional differential equations with integral conditions. Katugampola's fractional derivative is used as the differential operator, which is crucial to generalizing Hadamard and Riemann-Liouville's fractional derivatives into a single form.

Declarations

Competing interests: The authors declare no competing interests.
Authors' contributions:
Billal Lekdim: Formal analysis; Investigation; Resources; Software; Visualization; Writing-original draft.
Bilal Basti: Methodology; Supervision; Validation; Writing-review and editing; Project administration.
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