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Relations between $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q}$ and $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$ and their applications

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Abstract: Let $\mathscr{B}^{+}(\mathscr{H})$ represent the cone comprising all positive invertible operators on a complex separable Hilbert space $\mathscr{H}$. When T and S belong to $\mathscr{B}^{+}(\mathscr{H})$, it holds true that for any $\gamma \geq 0$, $\delta \geq 0$, and $0 < q \leq 1$, the following two inequalities are equivalent:

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q} \quad \text{and} \quad T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$$

In this article, we will explore the connections between these inequalities and provide some applications of this discovery to operator class theory. Furthermore, we will provide a positive response to the question posed in [16].

Keywords: class p-$wA(\alpha,\beta)$; Löwner-Heinz theorem; Normal operator; Aluthge transformation.

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1 Introduction

Let $\mathscr{B}(\mathscr{H})$ denote the C^{*}-algebra encompassing all bounded linear operators acting on a complex, separable Hilbert space referred to as $\mathscr{H}$. Within this context, we use the symbol I to represent the identity operator. An operator, denoted as T, is characterized as positive, denoted as $T \geq 0$, if it satisfies the condition $\langle Tx, x\rangle \geq 0$ for every vector x in the Hilbert space $\mathscr{H}$. Additionally, an operator T is regarded as strictly positive, symbolized as $T > 0$, if it fulfills two criteria: firstly, it must be positive, and secondly, it must be invertible, meaning that $\langle Tx, x\rangle > 0$ for all nonzero vectors x within $\mathscr{H}$. To clarify further, when we express $T \geq S \geq 0$, it indicates that the operator $T - S$ is positive, or in other words, $\langle (T-S)x, x\rangle \geq 0$ for all vectors x within the Hilbert space $\mathscr{H}$.

The following result, which is crucial to understanding non-normal operators, is the first in this section.

Theorem 1(Furuta's inequality[10]). *If $T \geq S \geq 0$, then for each $t \geq 0$,*

(i) $(S^{\frac{t}{2}}T^{p}S^{\frac{t}{2}})^{\frac{1}{q}} \geq S^{\frac{t+p}{q}}$ *and*
(ii) $T^{\frac{t+p}{q}} \geq (T^{\frac{p}{2}}S^{t}T^{\frac{p}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+t)q \geq p+t$.

It's worth mentioning that if we substitute $t = 0$ into either condition (i) or (ii) from the previously mentioned theorems, we obtain the well-known Löwener-Heinz theorem, which asserts that "$T \geq S \geq 0$ guarantees $T^{\alpha} \geq S^{\alpha}$ for any $\alpha \in [0,1]$." The subsequent results were established as applications of Theorem 1 in the references [7] and [11]. For positive invertible operators T and S, the order relation $\log T \geq \log S$ (referred to as chaotic order) holds if and only if $(S^{\frac{r}{2}}T^{p}S^{\frac{r}{2}})^{\frac{r}{p+r}} \geq S^{r}$

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for all $p \geq 0$ and $r \geq 0$, and this equivalence also extends to $T^p \geq (T^{\frac{p}{2}}S^rT^{\frac{p}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$. It's worth noting that when $p = r$, this conclusion serves as an extension of the results presented in [2]. The following assertions are well-established concerning these operator inequalities: Let T and S be strictly positive operators. Then, we have

(a)$T \geq S \Rightarrow \log T \geq \log S$.

(b)$\log T \geq \log S \Rightarrow (S^{\frac{\alpha}{2}}T^{\beta}S^{\frac{\alpha}{2}})^{\frac{\alpha}{\beta+\alpha}} \geq S^{\alpha}$ and $T^{\beta} \geq (T^{\frac{\beta}{2}}S^{\alpha}T^{\frac{\beta}{2}})^{\frac{\beta}{\beta+\alpha}}$ for all $\beta \geq 0$ and $\alpha \geq 0$.

(c)For each $\beta \geq 0$ and $\alpha \geq 0$, $(S^{\frac{\alpha}{2}}T^{\beta}S^{\frac{\alpha}{2}})^{\frac{\alpha}{\beta+\alpha}} \geq S^{\alpha} \Leftrightarrow T^{\beta} \geq (T^{\frac{\beta}{2}}S^{\alpha}T^{\frac{\beta}{2}})^{\frac{\beta}{\beta+\alpha}}$ [11].

Regarding these findings, the requirement for invertibility in conditions (a) and (b) can be substituted with the condition $\ker(T) = \ker(S) = 0$. This condition implies that (a) and (b) remain valid even for specific non-invertible operators T and S, as established in [24]. The authors of [15] delved into the relationships between the following inequalities:

$$(S^{\frac{\alpha}{2}}T^{\beta}S^{\frac{\alpha}{2}})^{\frac{\alpha}{\beta+\alpha}} \geq S^{\alpha} \quad \text{and} \quad T^{\beta} \geq (T^{\frac{\beta}{2}}S^{\alpha}T^{\frac{\beta}{2}})^{\frac{\beta}{\beta+\alpha}}$$

when it is not possible to invert operators T and S.

An operator $T \in \mathscr{B}(\mathscr{H})$ is referred to as hyponormal when it satisfies the inequality $T^*T \geq TT^*$. The Aluthge transformation, denoted as $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, was introduced by Aluthge in [1]. It is a key component of the polar decomposition of $T \in \mathscr{B}(\mathscr{H})$, which can be represented as $T = U|T|$. Furthermore, the formula $\tilde{T}_{s,t} = |T|^sU|T|^t$ describes the generalized Aluthge transformation $\tilde{T}_{s,t}$ with $0 < s,t$. It's important to note that an operator $T \in \mathscr{B}(\mathscr{H})$ is defined as p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. Additionally, it falls into class $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([14]). The class $A(k)$, which encompasses p-hyponormal and log-hyponormal operators, was introduced by Furuta et al. in their study [9], where $A(1)$ corresponds to the class A operator. Furthermore, if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, we assert that an operator T belongs to class $A(k)$, where $k > 0$. In this paper, we aim to establish the relationships between the following inequalities:

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q} \quad \text{and} \quad T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}} \tag{1}$$

These relationships will be explored in cases where operators T and S are not invertible. We will also demonstrate the normality of the class p-$A(\alpha,\beta)$ for $\alpha > 0, \beta > 0$, and $0 < p \leq 1$. Furthermore, we will prove that if either T or T belongs to class p-$A(\alpha,\beta)$ for some $\alpha > 0, \beta > 0$, with $0 < p \leq 1$, and S is an operator such that $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is a self-adjoint operator.

2 Relations between $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q}$ and $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$

In this section, we will present the following outcome:

Theorem 2. *Let $T,S \in \mathscr{B}^+(\mathscr{H})$. Then for each $\gamma \geq 0$, $\delta \geq 0$ and $0 < q \leq 1$, the following assertions hold:*

(i)If $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q}$, then $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$.

(ii)If $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$ and $\ker(T) \subset \ker(S)$, then $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{q\delta}$.

We would like to note that Theorem 2 serves as an extension of Theorem 1 in [15]. The following results are organized to provide a proof and illustration of Theorem 2.

Lemma 1. *[13, Löwner-Heinz inequality] Let $T,S \in \mathscr{B}^+(\mathscr{H})$. If $T \geq S \geq 0$, then $T^{\gamma} \geq S^{\gamma}$ for every $\gamma \in [0,1]$.*

Lemma 2. *[8] Let $T,S \in \mathscr{B}(\mathscr{H})$. Assume that T is positive ($T > 0$), and that S is an invertible operator. Under these conditions, the following holds for any real number λ:*

$$(STS^*)^{\lambda} = ST^{\frac{1}{2}}(T^{\frac{1}{2}}S^*ST^{\frac{1}{2}})^{\lambda-1}T^{\frac{1}{2}}S^*.$$

Proof. For the sake of convenience, we provide a proof of this self-evident result. Let's start with the polar decomposition of the invertible operator $ST^{\frac{1}{2}}$ as $ST^{\frac{1}{2}} = UQ$, where U is a unitary operator and $Q = |ST^{\frac{1}{2}}|$. Then,

$$\begin{aligned}(STS^*)^{\lambda} &= (UQ^2U)^{\lambda} = UQ^{2\lambda}U^* \\ &= ST^{\frac{1}{2}}Q^{-1}Q^{2\lambda}Q^{-1}T^{\frac{1}{2}}S^* = ST^{\frac{1}{2}}(Q^2)^{\lambda-1}T^{\frac{1}{2}}S^* \\ &= ST^{\frac{1}{2}}(T^{\frac{1}{2}}S^*ST^{\frac{1}{2}})^{\lambda-1}T^{\frac{1}{2}}S^*.\end{aligned}$$

Proposition 1.*[21] Let $T,S \in \mathscr{B}^+(\mathscr{H})$. Consequently, the following statements are true:*

(i)If $(S^{\frac{\delta_0}{2}}T^{\gamma_0}S^{\frac{\delta_0}{2}})^{\frac{\delta_0 p}{\gamma_0+\delta_0}} \geq S^{\delta_0 p}$ maintains for fixed $\gamma_0 > 0$, $\delta_0 > 0$ and $0 < p \leq 1$, then

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\delta p_1}{\gamma_0+\delta}} \geq S^{\delta p_1} \tag{2}$$

holds for any $\delta \geq \delta_0$ and $0 < p_1 \leq p \leq 1$. Moreover, for each fixed $\gamma \geq -\gamma_0$,

$$f_{\gamma_0,\gamma}(\delta) = (T^{\frac{\gamma_0}{2}}S^{\delta}T^{\frac{\gamma_0}{2}})^{\frac{(\gamma_0+\gamma)p_1}{\gamma_0+\delta}}$$

is a decreasing function for $\delta \geq \max\{\delta_0, \gamma\}$. Hence the inequality

$$(T^{\frac{\gamma_0}{2}}S^{\delta_1}T^{\frac{\gamma_0}{2}})^{p_1} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_2}T^{\frac{\gamma_0}{2}})^{\frac{p_1(\gamma_0+\delta_1)}{\gamma_0+\delta_2}} \tag{3}$$

holds for any δ_1 and δ_2 such that $\delta_2 \geq \delta_1 \geq \delta_0$ and $0 < p_1 \leq p$.

(ii)If $T^{\gamma_0 p} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{\gamma_0 p}{\gamma_0+\delta_0}}$ holds for fixed $\gamma_0 > 0$, $\delta_0 > 0$ and $0 < p \leq 1$, then

$$T^{\gamma p_1} \geq (T^{\frac{\gamma}{2}}S^{\delta_0}T^{\frac{\gamma}{2}})^{\frac{\gamma p_1}{\gamma+\delta_0}} \tag{4}$$

holds for any $\gamma \geq \gamma_0$ and $0 < p_1 \leq p \leq 1$. Furthermore, for each fixed $\delta \geq -\delta_0$,

$$g_{\delta_0,\delta}(\gamma) = (S^{\frac{\delta_0}{2}}T^{\gamma}S^{\frac{\delta_0}{2}})^{\frac{(\delta+\delta_0)p_1}{\gamma+\delta_0}}$$

is an increasing function for $\gamma \geq \max\{\gamma_0, \delta\}$. Therefore the inequality

$$(S^{\frac{\delta_0}{2}}T^{\gamma_2}S^{\frac{\delta_0}{2}})^{\frac{p_1(\gamma_1+\delta_0)}{\gamma_2+\delta_0}} \geq (S^{\frac{\delta_0}{2}}T^{\gamma_1}S^{\frac{\delta_0}{2}})^{p_1} \tag{5}$$

holds for any γ_1 and γ_2 such that $\gamma_2 \geq \gamma_1 \geq \gamma_0$ and $0 < p_1 \leq p$.

By applying the Furuta inequality, we derive Theorem 2. Our approach relies on the utilization of the subsequent expression, which constitutes a pivotal element of the Furuta inequality presented in Theorem 1.

Lemma 3.*Let $T,S \in \mathscr{B}(\mathscr{H})$. If $T \geq S \geq 0$, then*

(i)$(S^{x/2}T^{y}S^{x/2})^{\frac{1+x}{x+y}} \geq S^{1+x}$ and
(ii)$T^{1+x} \geq (T^{x/2}S^{y}T^{x/2})^{\frac{1+x}{x+y}}$

hold for $x \geq 0$ and $y \geq 1$.

Proof(Proof of Theorem 2). (i) Suppose that the following relation

$$(S^{\delta_0/2}T^{\gamma_0}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma_0+\delta_0}} \geq S^{q\delta_0} \tag{6}$$

holds for fixed $\gamma_0 > 0$ and $\delta_0 > 0$ and $0 < q \leq 1$. Applying Lemma 3 to (6), we have

$$\{S^{\frac{q\delta_0 r_1}{2}}(S^{\delta_0/2}T^{\gamma_0}S^{\delta_0/2})^{\frac{p_1 q\delta_0}{\gamma_0+\delta_0}}S^{\frac{q\delta_0 r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \geq S^{q\delta_0(1+r_1)} \tag{7}$$

for any $p_1 \geq 1$ and $r_1 \geq 0$. Putting $p_1 = \frac{\gamma_0+\delta_0}{q\delta_0}$ in (7), we have

$$(S^{\frac{\delta_0(1+qr_1)}{2}}T^{\gamma_0}S^{\frac{\delta_0(1+qr_1)}{2}})^{\frac{q\delta_0(1+r_1)}{\gamma_0+\delta_0+r_1 q\delta_0}} \geq S^{q\delta_0(1+r_1)} \tag{8}$$

for any $r_1 \geq 0$. Put $\delta = \delta_0(1+qr_1) \geq \delta_0$ in (8). Then we have

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\delta-(1-q)\delta_0}{\gamma_0+\delta}} \geq S^{\delta-(1-q)\delta_0}. \tag{9}$$

Hence we have

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\mu}{\gamma_0+\delta}} \geq S^{\mu} \text{ for } 0 < \mu \leq \delta-(1-q)\delta_0. \tag{10}$$

Next, we demonstrate $f(\delta) = (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}}$ is decreasing for $\delta \geq \delta_0$. By Löwner-Heinz theorem, (10) ensures the following (11)

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\mu}{\gamma_0+\delta}} \geq S^{\mu} \text{ for } 0 < \mu \leq \delta-(1-q)\delta_0. \tag{11}$$

Next, we have

$$\begin{aligned} f(\delta) &= (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}} \\ &= \{(T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{\gamma_0+\delta+\mu}{\gamma_0+\delta}}\}^{\frac{q\gamma_0}{\gamma_0+\delta+\mu}} \\ &= \{T^{\gamma_0/2}S^{\delta/2}(S^{\delta/2}T^{\gamma_0}S^{\delta/2})^{\frac{\mu}{\gamma_0+\delta}}S^{\delta/2}T^{\gamma_0/2}\}^{\frac{q\gamma_0}{\gamma_0+\delta+\mu}} \text{ (by Lemma 2)} \\ &\geq (T^{\gamma_0/2}S^{\delta+\mu}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta+\mu}} \\ &= f(\delta+\mu). \end{aligned}$$

Hence $f(\delta)$ is decreasing for $\delta \geq \delta_0$. Consequently,

$$T^{q\gamma_0} \geq (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}} \text{ for } \delta \geq \delta_0 \tag{12}$$

holds since

$$T^{q\gamma_0} \geq (T^{\gamma_0/2}S^{\delta_0}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} = f(\delta_0) \geq f(\delta) = (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}}.$$

Again applying Theorem 1 to (12), we have

$$T^{q\gamma_0(1+r_2)} \geq (T^{\frac{qr_2\gamma_0}{2}}(T^{qr_2\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{p_2q\gamma_0}{\gamma_0+\delta}}T^{\frac{qr_2\gamma_0}{2}})^{\frac{1+r_2}{p_2+r_2}} \tag{13}$$

for any $p_2 \geq 1$ and $r_2 \geq 0$. Putting $p_2 = \frac{\gamma_0+\delta}{q\gamma_0} \geq 1$ in (13), we have

$$T^{q\gamma_0(1+r_2)} \geq (T^{\frac{\gamma_0(1+qr_2)}{2}}S^{\delta}T^{\frac{\gamma_0(1+qr_2)}{2}})^{\frac{q\gamma_0(1+r_2)}{\gamma_0+\delta+qr_2\gamma_0}} \tag{14}$$

for any $r_2 \geq 0$. Put $\gamma = \gamma_0(1+qr_2) \geq \gamma_0$ in (14). Then we have

$$T^{\gamma+\gamma_0(q-1)} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{\gamma+\gamma_0(q-1)}{\delta+\gamma}} \tag{15}$$

for all $\gamma \geq \gamma_0$ and $\delta \geq \delta_0$. Now, since $0 < \frac{q_1\gamma}{\gamma+\gamma_0(q-1)} \leq 1$, making use of Löwner-Heinz theorem to (15), we have

$$T^{q_1\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q_1\gamma}{\delta+\gamma}}$$

for all $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$.
(ii) Suppose that $\ker(T) \subset \ker(S)$ and

$$T^{q\gamma_0} \geq (T^{\gamma_0/2}S^{\delta_0}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} \tag{16}$$

holds for fixed $\gamma_0 > 0$ and $\delta_0 > 0$ and $0 < q \leq 1$. Applying Lemma 3 to (16), we have

$$T^{q\gamma_0(1+r_3)} \geq (T^{\frac{qr_3\gamma_0}{2}}(T^{\gamma_0/2}S^{\delta_0}T^{\gamma_0/2})^{\frac{p_3q\gamma_0}{\gamma_0+\delta_0}}T^{\frac{qr_3\gamma_0}{2}})^{\frac{1+r_3}{p_3+r_3}} \tag{17}$$

for any $p_3 \geq 1$ and $r_3 \geq 0$. Putting $p_3 = \frac{\gamma_0+\delta_0}{q\gamma_0} \geq 1$ in (17), we have

$$T^{q\gamma_0(1+r_3)} \geq (T^{\frac{\gamma_0(1+qr_3)}{2}}S^{\delta_0}T^{\frac{\gamma_0(1+qr_3)}{2}})^{\frac{q\gamma_0(1+r_3)}{\gamma_0+\delta_0+qr_3\gamma_0}} \tag{18}$$

for any $r_3 \geq 0$. Put $\gamma = \gamma_0(1+qr_3) \geq \gamma_0$ in (18). Then we have

$$T^{\gamma+\gamma_0(q-1)} \geq (T^{\frac{\gamma}{2}}S^{\delta_0}T^{\frac{\gamma}{2}})^{\frac{\gamma+\gamma_0(q-1)}{\delta_0+\gamma}} \text{ for } \gamma \geq \gamma_0. \tag{19}$$

Next we show that $g(\gamma) = (S^{\delta_0/2} A^\gamma S^{\delta_0/2})^{\frac{q\delta_0}{\gamma_0+\delta_0}}$ is increasing for $\gamma \geq \gamma_0$. Löwner-Heinz theorem , when applied to (19), guarantees the following:

$$T^u \geq (T^{\frac{\gamma}{2}} S^{\delta_0} T^{\frac{\gamma}{2}})^{\frac{u}{\delta_0+\gamma}} \text{ for } 0 \leq u \leq \gamma + \gamma_0(q-1). \tag{20}$$

Then we have

$$\begin{aligned} g(\gamma) &= (S^{\delta_0/2} T^\gamma S^{\delta_0/2})^{\frac{q\delta_0}{\gamma+\delta_0}} \\ &= \{(S^{\delta_0/2} T^\gamma S^{\delta_0/2})^{\frac{\gamma+\delta_0+u}{\gamma+\delta_0}}\}^{\frac{q\delta_0}{u+\delta_0+\gamma}} \\ &= \{S^{\delta_0/2} T^{\gamma/2} (T^{\gamma/2} S^{\delta_0} T^{\gamma/2})^{\frac{u}{\gamma+\delta_0}} T^{\gamma/2} S^{\delta_0/2}\}^{\frac{q\delta_0}{u+\delta_0+\gamma}} \\ &\leq (S^{\delta_0/2} T^{\gamma+u} S^{\delta_0/2})^{\frac{q\delta_0}{u+\delta_0+\gamma}} \\ &= g(\gamma+u). \end{aligned}$$

Hence $g(\gamma)$ is increasing for $\gamma \geq \gamma_0$. Therefore

$$(S^{\delta_0/2} T^\gamma S^{\delta_0/2})^{\frac{q\delta_0}{\gamma+\delta_0}} \geq S^{q\delta_0} \text{ for } \gamma \geq \gamma_0 \tag{21}$$

holds since

$$(S^{\delta_0/2} T^\gamma S^{\delta_0/2})^{\frac{q\delta_0}{\gamma+\delta_0}} = g(\gamma) \geq g(\gamma_0) = (S^{\delta_0/2} T^{\gamma_0} S^{\delta_0/2})^{\frac{q\delta_0}{\gamma_0+\delta_0}} \geq S^{q\delta_0}.$$

Again applying Theorem 1 to (21), we have

$$\{S^{\frac{qr_4\delta_0}{2}} (S^{\delta_0/2} T^\gamma S^{\delta_0/2})^{\frac{p_4 q\delta_0}{\gamma+\delta_0}} S^{\frac{qr_4\delta_0}{2}}\}^{\frac{1+r_4}{p_4+r_4}} \geq S^{q\delta_0(1+r_4)} \tag{22}$$

for any $p_4 \geq 1$ and $r_4 \geq 0$. Putting $p_4 = \frac{\gamma+\delta_0}{q\delta_0} \geq 1$ in (22), we have

$$(S^{\frac{\delta_0(1+qr_4)}{2}} T^\gamma S^{\frac{\delta_0(1+qr_4)}{2}})^{\frac{q\delta_0(1+r_4)}{\gamma+\delta_0+q\delta_0 r_4}} \geq S^{q\delta_0(1+r_4)} \tag{23}$$

for any $r_4 \geq 0$. Put $\delta = \delta_0(1+qr_4) \geq \delta_0$ in (23). Then we have

$$(S^{\frac{\delta}{2}} T^\gamma S^{\frac{\delta}{2}})^{\frac{\delta+\delta_0(q-1)}{\gamma+\delta}} \geq S^{\delta+\delta_0(q-1)} \text{ for } \gamma \geq \gamma_0 \text{ and } \delta \geq \delta_0. \tag{24}$$

Applying the Löwner-Heinz theorem to (24), we now obtain since $0 < \frac{q_1\delta}{\delta+\delta_0(q-1)} \leq 1$,

$$(S^{\frac{\delta}{2}} T^\gamma S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}} \geq S^{q_1\delta}$$

for all $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$, consequently, the proof is conclusive.

Proposition 2.*Let $T, S \in \mathscr{B}^+(\mathscr{H})$ and let $\gamma_0 > 0$, $\delta_0 > 0$ and $0 < q \leq 1$. Suppose that*

$$(S^{\frac{\delta_0}{2}} T^{\gamma_0} S^{\frac{\delta_0}{2}})^{\frac{q\delta_0}{\gamma_0+\delta_0}} \geq S^{q\delta_0} \tag{25}$$

and

$$T^{q\gamma_0} \geq (T^{\frac{\gamma_0}{2}} S^{\delta_0} T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta_0}}. \tag{26}$$

Consequently, the following statements are true:

(i)For every $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$

$$(S^{\frac{\delta}{2}} T^\gamma S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}} \geq S^{q_1\delta}.$$

Moreover, for each fixed $\gamma \geq -\gamma_0$,

$$f_{\gamma_0,\gamma}(\delta) = (T^{\frac{\gamma_0}{2}} S^\delta T^{\frac{\gamma_0}{2}})^{\frac{(\gamma_0+\gamma)p_1}{\gamma_0+\delta}}$$

is a decreasing function for $\delta \geq \max\{\delta_0, \gamma\}$. Hence the inequality

$$(T^{\frac{\gamma_0}{2}} S^{\delta_1} T^{\frac{\gamma_0}{2}})^{p_1} \geq (T^{\frac{\gamma_0}{2}} S^{\delta_2} T^{\frac{\gamma_0}{2}})^{\frac{p_1(\gamma_0+\delta_1)}{\gamma_0+\delta_2}} \tag{27}$$

holds for any δ_1 and δ_2 such that $\delta_2 \geq \delta_1 \geq \delta_0$ and $0 < p_1 \leq p$.

(ii)For each $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$

$$T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}.$$

Additionally, for every fixed $\delta \geq -\delta_0$,

$$g_{\delta_0,\delta}(\gamma) = (S^{\frac{\delta_0}{2}}T^{\gamma}S^{\frac{\delta_0}{2}})^{\frac{(\delta+\delta_0)p_1}{\gamma+\delta_0}}$$

is an increasing function for $\gamma \geq \max\{\gamma_0, \delta\}$. Hence the inequality

$$(S^{\frac{\delta_0}{2}}T^{\gamma_2}S^{\frac{\delta_0}{2}})^{\frac{p_1(\gamma_1+\delta_0)}{\gamma_2+\delta_0}} \geq (S^{\frac{\delta_0}{2}}T^{\gamma_1}S^{\frac{\delta_0}{2}})^{p_1} \tag{28}$$

holds for any γ_1 and γ_2 such that $\gamma_2 \geq \gamma_1 \geq \gamma_0$ and $0 < p_1 \leq p$.

*Proof.*We will provide the proof for part (ii), noting that the proof for part (i) follows a similar pattern. We begin by observing that inequality (2) implies inequality (4), as established in Proposition 1. Therefore, we have:

$$T^{q\gamma_0} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} \geq (T^{\frac{\gamma_0}{2}}S^{\delta}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta}}$$

This inequality holds for all $\beta \geq \beta_0$ based on inequality (4) and the Löwner-Heinz inequality. Consequently, we can conclude part (ii) by invoking Proposition 1 (ii).

In Proposition 2, when considering $\gamma > 0$, $\delta > 0$, and $0 < q \leq 1$, one might naturally anticipate that the inequality $T^{q\gamma} \geq \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}}$ is equivalent to $\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} \geq S^{q\delta}$, even in cases where T and S are not invertible. However, this assumption is disproven by the following example.

*Example 1.*There exists positive bounded linear operators T and S such that $T^{q\gamma} \geq \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}}$ and $\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} \ngeq S^{q\delta}$. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$T^{q\gamma} - \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

and

$$\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} - S^{q\delta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \ngeq 0$$

for $\gamma > 0$, $\delta > 0$, and $0 < q \leq 1$. Therefore $T^{q\gamma} \geq \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}}$ and $\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} \ngeq S^{q\delta}$ for $\gamma > 0$, $\delta > 0$, and $0 < q \leq 1$.

Corollary 1.*Let $T, S \in \mathscr{B}^+(\mathscr{H})$ and let $\gamma_0 > 0$, $\delta_0 > 0$. Then, the following claims are true:*

(i)If $0 < q \leq 1$, then

$$(S^{\frac{\delta_0}{2}}T^{\gamma_0}S^{\frac{\delta_0}{2}})^{\frac{q\delta_0}{\gamma_0+\delta_0}} \geq S^{q\delta_0} \Longrightarrow (S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}} \geq S^{q_1\delta} \tag{29}$$

holds for any $\gamma \geq \gamma_0$ and $\delta \geq \delta_0$, thus $T^{q_1\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q_1\gamma}{\gamma+\delta}}$ holds for any $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$.

(ii)If $0 < q \leq 1$ and $\ker(T) \subset \ker(S)$, then

$$T^{q\gamma_0} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} \Longrightarrow T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}} \tag{30}$$

holds for any $\gamma \geq \gamma_0$ and $\delta \geq \delta_0$, thus $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}} \geq S^{q_1\delta}$ holds for any $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$.

*Proof.*We present the proof for part (i), and it's worth noting that the proof for part (ii) follows a similar line of reasoning.

Based on the provided hypothesis, the Löwner-Heinz theorem, and Proposition 2, we can establish the following inequality for all $\delta \geq \delta_0$, $\gamma \geq \gamma_0$, and $0 < q \leq 1$:

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{q\delta}$$

This inequality, derived from the hypothesis and known theorems, validates Corollary 1 (i). The application of the Löwner-Heinz theorem and Theorem 2 further supports this conclusion.

*Remark.*We need to keep in mind the assumptions (i) and (ii) of Theorem 2. In the context of Theorem 2, we consider the scenario where $\gamma = \delta = 1$ and $0 < q \leq 1$. The following conditions are relevant:

(a)$(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \geq S^q$.
(b)$T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$ and $\ker(T) \subset \ker(S)$.

We have shown that in Theorem 2, condition (a) implies $T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$, and condition (b) ensures condition (a). Consequently, one might expect that conditions (a) and (b) are analogous. However, we have a counterexample to demonstrate otherwise.

Example 2.$(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \geq S^q$ and $T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$, but $\ker(T) \not\subseteq \ker(S)$.
Let $T = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $T^{\frac{1}{2}} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, S^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = S$. Hence

$$2^{\frac{q}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \geq S^q$$

and

$$T^q = \begin{pmatrix} 2^q \cdot 5^{q-1} & 2^{q+1} \cdot 5^{q-1} \\ 5^{q-1} \cdot 2^{q+1} & 5^{q-1} \cdot 2^{q+2} \end{pmatrix} \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}} = \begin{pmatrix} \frac{2^{\frac{q}{2}}}{5} & \frac{2^{\frac{q+2}{2}}}{5} \\ \frac{2^{\frac{q+2}{2}}}{5} & \frac{2^{\frac{q+4}{2}}}{5} \end{pmatrix}.$$

But $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in \ker(A)$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \notin \ker(S)$, so that $\ker(T) \not\subseteq \ker(S)$.

Moreover, we have the following example.

*Example 3.*We have $T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$, but $(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \ngeq S^q$ and $\ker(T) \not\subseteq \ker(S)$.
Set $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $T^q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ngeq S^q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\ker(T) \not\subseteq \ker(S)$.

3 Applications

In this section, we will illustrate the application of Theorem 2 to various operator classes.

Definition 1.*Consider the following operator classes defined in terms of $\alpha > 0$, $\beta > 0$, $0 < p \leq 1$, the polar decomposition of T as $T = U|T|$, and the generalized Aluthge transformation $\tilde{T}_{\alpha,\beta} = |T|^{\alpha}U|T|^{\beta}$:*

(i)T is classified as belonging to the p-$A(\alpha,\beta)$ class if it satisfies the inequality $(|T^*|^{\beta}|T|^{2\alpha}|T^*|^{\beta})^{\frac{p\beta}{\alpha+\beta}} \geq |T^*|^{2p\beta}$ [16].
(ii)T is categorized as part of the p-w$A(\alpha,\beta)$ class if it meets the criteria:

$$(|T^*|^{\beta}|T|^{2\alpha}|T^*|^{\beta})^{\frac{p\beta}{\alpha+\beta}} \geq |T^*|^{2p\beta} \quad \text{and} \quad |T|^{2p\alpha} \geq (|T|^{\alpha}|T^*|^{2\beta}|T|^{\alpha})^{\frac{p\alpha}{\alpha+\beta}}$$

or equivalently, $|\tilde{T}_{\alpha,\beta}|^{\frac{2p\beta}{\beta+\alpha}} \geq |T|^{2p\beta}$ and $|T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\beta})^*|^{\frac{2p\alpha}{\alpha+\beta}}$ as defined in [16].
(iii)T is classified as a member of the p-A class if $|T^2|^p \geq |T|^{2p}$, which is equivalent to T being part of the p-$A(1,1)$ class, as stated in [16].
(iv)T is considered p-w-hyponormal if and only if it satisfies the inequalities: $|\tilde{T}|^{\frac{p}{2}} \geq |T|^p \geq |(\tilde{T})^*|^{\frac{p}{2}}$. This classification corresponds to T belonging to the p-w$A(\frac{1}{2},\frac{1}{2})$ class, where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation, as outlined in [3].
(v)T is termed (α,p)-w-hyponormal if and only if it satisfies the following inequalities: $|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\alpha})^*|^{\frac{p}{2}}$. This characterization corresponds to T belonging to the p-w$A(\alpha,\alpha)$ class, where $\tilde{T}_{\alpha,\alpha} = |T|^{\alpha}U|T|^{\alpha}$ is the generalized Aluthge transformation, as discussed in [12] and [19].

Operators classified as p-$wA(\alpha,\beta)$ exhibit several significant properties typical of hyponormal operators. These properties encompass the Fuglede-Putnam type theorem, Weyl type theorem, subscalarity, and Putnam's inequality, as documented in [4], [5], [17], [18], and [23]. It's important to note that the Aluthge transformation has garnered considerable attention from various authors, including [1], [4], [6], and [25]. These classes are categorized as normaloid operators, denoted by $\|T\| = r(T)$, where $r(T)$ represents the spectral radius of T, as discussed in [17], [3], and [12]. For α, β, and $0 < p \leq 1$, it has been established that class p-$A(\alpha,\beta)$ includes class p-$A(\alpha,\beta)$ based on the definition in 1 (i) and (ii). Furthermore, as demonstrated in [16], both class p-$wA(\alpha,\beta)$ and class p-$wA(\alpha,\beta)$ are invertible for any $\alpha > 0$, $\beta > 0$, and $0 < p \leq 1$. Previous research has also provided more precise inclusion relations among class p-$wA(\alpha,\beta)$.

Lemma 4.*[4] If $T \in B(\mathscr{H})$ is class p-$wA(s,t)$ and $0 < s \leq \gamma, 0 < t \leq \delta, 0 < p_1 \leq p \leq 1$, then T is class p_1-$wA(\gamma,\delta)$.*

In their study [16], the authors posed the following question:
Question: Does the class p-$A(s,t)$ imply p-$wA(s,t)$ for $0 < p < 1$?
The subsequent theorem provides an affirmative answer to this question.

Theorem 3.*For each $\alpha > 0, \beta > 0$ and $0 < p \leq 1$, the following assertions hold:*

(i)class p-$A(\alpha,\beta)$ and class p-$wA(\alpha,\beta)$ are equivalent.
(ii)class p-A and class p-wA are equivalent.
(iii)class p-$A(\frac{1}{2},\frac{1}{2})$ and the class of p-w-hyponormal operators are equivalent, i.e., class p-$wA(\frac{1}{2},\frac{1}{2})$.
(iv)class p-$A(\alpha,\alpha)$ and class (α,p)-w-hyponormal operators are equivalent, i.e., class p-$wA(\alpha,\alpha)$.

*Proof.*We choose not to provide a proof here, as we can easily establish Theorem 3 by applying Theorem 2 to the definitions of these classes.

Notice that Theorem 3 in reference [15] corresponds to a specific case where $q = 1$, and therefore, Theorem 3 can be seen as an extension or generalization of it.

*Remark.*By (iv) of Theorem 3, we have

$$\begin{aligned}|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} &\Leftrightarrow (|T^*|^{\alpha}|T|^{2\alpha}|T^*|^{\alpha})^{\frac{p}{2}} \geq |T^*|^{2p\alpha} \Leftrightarrow T : \text{class } p-A(\alpha,\alpha) \\ &\Leftrightarrow T : (\alpha,p)-w-\text{hyponormal} \Leftrightarrow |\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\alpha})^*|^{\frac{p}{2}}.\end{aligned}$$

Hence

$$|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} \Rightarrow |T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\alpha})^*|^{\frac{p}{2}},$$

that is, we may as will define (α,p)-w-hyponormal by only $|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha}$.

Next, we shall show some properties of class p-$A(s,t)$.

Theorem 4.*If $T \in B(\mathscr{H})$ is class p-$A(s,t)$ and $0 < s \leq \gamma, 0 < t \leq \delta, 0 < p_1 \leq p \leq 1$, then T is class p_1-$A(\gamma,\delta)$.*

*Proof.*We skip the proof because it can be accomplished easily using (i) of Theorem 3 and Theorem 5.

We will show that certain non-normal operators can be proven to be normal. It is established that an operator T is normal if both T and T^* belong to the class A. However, the situation becomes less clear when T and T^* belong to classes weaker than class A. Thanks to the research efforts of various authors on this topic, the following results were previously unknown until now.

Lemma 5.*[21] Let $\alpha_i,\beta_i > 0$ and $0 < p_i \leq 1$, where $i = 1,2$. If T is a class p_1-$wA(\alpha_1,\beta_1)$ operator and T^* is a class p_2-$wA(\alpha_2,\beta_2)$ operator, then T is normal.*

Corollary 2.*Let $\alpha_i,\beta_i > 0$ and $0 < p_i \leq 1$, where $i = 1,2$. If T is a class p_1-$A(\alpha_1,\beta_1)$ operator and T^* is a class p_2-$A(\alpha_2,\beta_2)$ operator, then T is normal.*

*Proof.*Theorem 3 and Lemma 5 lead directly to the proof.

Lemma 6.*[21] Let $p,r > 0$, $0 < q \leq 1$, $s \geq p$ and $t \geq r$. If T is a class q-$wA(p,r)$ operator and $\tilde{T}_{s,t}$ is normal, then T is normal.*

Corollary 3.*Let $p,r > 0$, $0 < q \leq 1$, $s \geq p$ and $t \geq r$. If T is a class q-$A(p,r)$ operator and $\tilde{T}_{s,t}$ is normal, then T is normal.*

*Proof.*Theorem 3 and Lemma 6 are prerequisites for the proof.

*Remark.*Please take note that Corollaries 2 and 3, along with Lemmas 5 and 6, offer generalizations of several findings found in the existing literature. Notable examples include the extension of Theorem 6 in reference [15], as well as other results in papers such as [19] and [3].

The numerical range of an operator M, represented as $W(M)$, is defined as the set given by:

$$W(M) = \{\langle Mx, x\rangle : \|x\| = 1\}.$$

In a general context, it's important to note that neither the condition $N^{-1}MN = M^*$ nor the statement $0 \notin \overline{W(M)}$ guarantees that the operator M is normal. This is exemplified when considering the case of $M = NB$, where N is positive and invertible, B is self-adjoint, and N and B do not commute. In this scenario, $N^{-1}MN = M^*$ and $0 \notin \overline{W(N)}$, but the operator M is not normal. This naturally leads to the following question:
Question: Under what conditions does an operator M become normal when both $N^{-1}MN = M^*$ and $0 \notin \overline{W(N)}$ hold true?
In 1966, Sheth demonstrated in [22] that if M is a hyponormal operator and $N^{-1}MN = M^*$ for certain operators N, where $0 \notin \overline{W(N)}$, then M is self-adjoint. Rashid later extended Sheth's result to encompass the class $A(k)$ operators for $k > 0$ in [20]. This work further expands upon Sheth's result, demonstrating that it holds true for the class p-$A(\alpha,\beta)$ operators, as detailed below.

Corollary 4.*Let $M \in \mathscr{B}(\mathscr{H})$. If M or M^* belongs to class p-$A(\alpha,\beta)$ for every $\alpha > 0, \beta > 0$ and $0 < p \leq 1$ and N is an operator for which $0 \notin \overline{W(N)}$ and $NM = M^*N$, then M is self-adjoint.*

*Proof.*The conclusion drawn is a result of Theorem 3 and the findings presented in [21, Theorem 2.14].

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