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# SD-Separability in Topological Spaces

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**Abstract:** Our aim is to introduce a generalization of dense set in topological space, namely *SD*-dense set, when we used the notion of somewhere dense closure operator. We provide the characterization of this class of sets, and their implications with dense sets and with somewhere dense sets, and study their union and intersection properties, moreover we discuss their behavior as subspaces in some special cases, additionally, we investigate their properties in some particular spaces, and then we prove that *SD*-dense sets, dense sets, somewhere dense sets and open sets are equivalent in strongly hyperconnected space, after that we illustrate the image of *SD*-dense sets by particular maps; as *SD*-irresolute map and *SD*-continuous map. Finally, we define a new axiom of separability, namely *SD*-separable space using the notion of *SD*-dense sets, then we state that *SD*-separable space is stronger than separable space, and in submaximal space these notions become equivalent, moreover we study the subspaces and the images of *SD*-separable spaces.

**Keywords:** Separability; topological space and generalizations; continuous and generalization; somewhere dense set; strongly hyperconnected space.

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## 1 Introduction

Researchers have mentioned various forms of generalized open sets; for instance $\alpha$-open set, preopen set, semi open set, $b$-open set, $\beta$-open set and somewhere dense set. The notion of somewhere dense set was due to Pugh [19], then in 2017 [1] Al-shami provided the topological properties of this class of sets, and he studied some operators as; somewhere dense interior, somewhere dense closure and somewhere dense boundary and he used these notions to defined the axiom of $ST_1$space, and with Noiri [2,3] they investigated some particular maps as; *SD*-irresolute maps and *SD*-continuous maps, then they introduced the concepts of Lindelofness and compactness using somewhere dense sets, and recently, Arwini et al. [4] stated that somewhere dense sets and open sets are coinciding if and only if a space is strongly hyperconnected, moreover, Sarbout et al. [20] defined somewhere dense connected space, and they showed that this space is stronger than hyperconnected space but weaker than strongly hyperconneted space.
In 1906 [15] Frechet defined separable space, which is a space that contains a countable dense subset, since then different types of separability were defined, as $d$-separable space, $D$-separable space, weakly separable space, $b$-separable space, dense-separable space, $r$-separable space, pre-separable space etc. Kurepa [17] in 1936 introduced a generalized form of separable and metrizable spaces, namely d-separable space, and then in 1981 Arhangelskii [5] studied some properties of $d$-separable spaces and proved that any product of $d$-separable spaces is $d$-separable. $D$-Separable spaces were due to Bella et al. [8], and in 2012 Aurihi et al. [7] investigated the properties of $D$-separable space and showed their implication with $d$-separable spaces. Weakly separable spaces were defined by Beshimov in 1994 [9] when he proved that any weakly separable Hausdorff compact space is separable, moreover he studied its separable compactifications, see more in [10,11]. In 2013 Selvarani [21] defined $b$-dense sets and $b$-separable spaces using $b$-open sets, then in 2021 Arwini et al. [6,16] introduced two different types of separability, the first type is called dense separable space, and they showed that dense separable space, dense second countable space and separable space are equivalent, while in the second type they used the notion of regular open sets to defined $r$-separable space, then they illustrated that $r$-separable space is weaker than separable space, but they became equivalent in regular space. Recently, Elbhilil et al. [14] introduced pre-separable spaces

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using pre-dense set notion, and they showed that pre-separable space is placed between b-separable space and separable space, and in submaximal spaces, the axioms of separability and pre-separability became equivalent.
In this work we use the notion of somewhere dense set to introduce *SD*-dense set, which is a generalization of dense set, we provide its behavior with some operations as union and intersection, then we discuss the characterization of this class of sets in some particular spaces and study their images by some particular maps. After that, we define the axiom of *SD*-separable space using the notion of *SD*-dense sets, then we illustrate the implication between this space and separable space, and study their subspaces and images.
We divided this article into five sections; as follows: section two concludes the basic concepts concerning somewhere dense sets, section three presents the definition of *SD*-dense set, including some of its union and intersection properties and its behavior as a subspace, then we present its characterization in some spaces, after that we examine their images by some particular maps, and section four includes the basic studies of *SD*-separable space, and finally section five gives an overview of our results in the conclusion.
Throughout this paper, a topological space $(Z,\tau)$ denotes by $Z$ and $D(\tau)$ denotes the collection of all dense sets in $Z$, and if $E$ and $F$ are subsets of a space $Z$; $\overline{E}$, $E^{\circ}$, $P(E)$, $E^{c}$ and $E/F$ denote the closure of $E$, the interior of $E$, the power set of $E$, the complement of $E$ and the difference of $E$ and $F$; respectively. Additionally, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{K}$, $\mathbb{R}^{+}$, $\mathbb{R}^{-}$ are the sets of real numbers, integer numbers, rational numbers, irrational numbers, positive real numbers, negative real numbers; respectively.

## 2 Preliminaries

In the present section, we provide the basic properties and characterizations of somewhere dense set, and their behavior in subspaces and in strongly hyperconnected spaces.

**Definition 1.**[1] *In a topological space $(Z,\tau)$, a subset $A$ is called somewhere dense (namely s-dense) if $\overline{A}^{\circ} \neq \phi$. The complement of s-dense set is called closed somewhere dense (namely cs-dense) set, and the collection of all s-dense sets in $Z$ is denoted by $S(\tau)$. Clearly any non-empty open (dense) set is s-dense.*

**Theorem 1.**[1] *If $A$ is s-dense subset in a space $Z$ and $A \subseteq B$, then $B$ is s-dense.*

**Theorem 2.**[1] *Any subset of a space $Z$ is s-dense or cs-dense.*

**Definition 2.**[20] *A subset $B$ of a space $Z$ is called SD-clopen if $B$ is s-dense and cs-dense. Clearly any clopen set is SD-clopen.*

**Definition 3.**[20] *In a space $Z$ if any open set is closed, then $Z$ is called partition space. Clearly any non-empty subset of partition space is s-dense.*

**Definition 4.**[14] *A space $(Z,\tau)$ is called S-space if any subset of $Z$ that contains a non-empty open set is also open.*

**Definition 5.**[1, 13, 18] *A space $Z$ is called:*

*i. submaximal if any dense set is open.*
*ii. hyperconnected if any non-empty open set is dense.*
*iii. strongly hyperconnected if $Z$ is submaximal and hyperconnected.*

**Theorem 3.**[4] *In a space $Z$, the following conditions are equivalent:*
*1. $Z$ is strongly hyperconnected space.*
*2. dense sets are equivalent with non-empty open sets.*
*3 s-dense sets are equivalent with non-empty open sets; where $Z$ is a non-discrete space.*

**Definition 6.**[10] *If $Z$ is a space, then a subset $B$ is called regular closed (namely r-closed) if $B = \overline{B^{\circ}}$, while the complement of regular closed set is called regular open (r-open). Clearly any r-closed set is closed.*

**Theorem 4.**[20] *Any proper non-empty r-closed set is SD-clopen.*

**Theorem 5.**[4, 12] *Let $Z$ be a space, $W$ be a subspace of $Z$ and $B \subseteq W$ then:*

*1) $\overline{B}^{W} = \overline{B} \cap W$, where $\overline{B}^{W}$ is the closure of $B$ with respect to the relative topology on $W$.*
*2) where $W$ is r-closed in $Z$, then $B$ is s-dense in $W$ if and only if $B$ is s-dense in $Z$.*


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**Definition 7.**[1] *Let Z be a space and B be a subset of Z, then:*

*i. the intersection of all cs-dense sets in Z containing B is denoted by* $\overline{B}^S$. *Clearly* $\overline{B}^S$ *is cs-dense set.*
*ii. the union of all s-dense sets contained in B is denoted by* $B^{\circ S}$. *Clearly* $B^{\circ S}$ *is s-dense set.*

**Theorem 6.**[1] *Let A and B be two subsets of a space Z, then:*

*1)* $A \subseteq \overline{A}^S \subseteq \overline{A}$ *and* $A^\circ \subseteq A^{\circ S} \subseteq A$.
*2) If* $A \subseteq B$, *then* $\overline{A}^S \subseteq \overline{B}^S$ *and* $A^{\circ S} \subseteq B^{\circ S}$.
*3) A is cs-dense if and only if* $\overline{A}^S = A$, *while A is s-dense if and only if* $A^{\circ S} = A$.

**Definition 8.**[1] *In a topological space* $(Z,\tau)$, *a subset A of Z is called:*

*i.* $\alpha$*-open if* $A \subseteq \overline{A^\circ}^\circ$.
*iii. pre-open if* $A \subseteq \overline{A}^\circ$.
*iv. b-open if* $A \subseteq \overline{A^\circ} \cup \overline{A}^\circ$.
*v.* $\beta$*-open if* $A \subseteq \overline{\overline{A}^\circ}$.

**Theorem 7.**[1] *The implications between the class of generalization open sets are given in the following diagram:*

open set $\Rightarrow$ $\alpha$-open set $\Rightarrow$ pre-open set $\Rightarrow$ *b*-open set $\Rightarrow$ $\beta$-open set $\Rightarrow$ *s*-dense set

**Definition 9.**[2] *A map* $f:(Z,\tau)\to(X,\sigma)$ *is called SD-irresolute if the inverse image of any s-dense in X is empty or s-dense in Z, while f is called SD-continuous if the inverse image of any open set in X is empty or s-dense in Z.*

**Definition 10.**[12] *A space is called separable if it contains a countable dense set.*

## 3 SD-Density

This section consists the definition of a new generalization of dense set using the concept of somewhere dense set, namely *SD*-dense set. Union and intersection properties of *SD*-dense sets and their implication with dense sets are given, additionally, the behavior of *SD*-dense sets as subspaces in particular conditions are shown, after that, we investigate the characterization of *SD*-dense sets in some spaces. Finally, we study the images of *SD*-dense sets by some particular maps; such as *SD*-continuous map and *SD*-irresolute map.

### *3.1 SD-Dense Sets*

Here we provide some basic properties of the class of *SD*-dense sets.

**Definition 11.***A subset F of a space* $(Z,\tau)$ *is called SD-dense if* $\overline{F}^S = Z$. *The collection of all SD-dense sets in Z is denoted by* $SD(\tau)$.

*Example 1.*In a space $(Z,\tau)$ where $Z=\{1,2,3\}$ and $\tau=\{Z,\phi,\{1,2\}\}$, we have $S(\tau)=\{Z,\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\}\}$. Therefore $Z$ and $\{1,2\}$ are the only *SD*-dense sets in $Z$.

**Corollary 1.***Any SD-dense set is dense.*

*Proof.*Obvious, since $\overline{E}^S \subseteq \overline{E}$ for any subset $E$ of a space $(Z,\tau)$, i.e., $SD(\tau)\subseteq D(\tau)$; where $D(\tau)$ is the collection of all dense sets in $Z$.

*SD*-dense $\Rightarrow$ dense $\Rightarrow$ pre-open set $\Rightarrow$ *b*-open set $\Rightarrow$ $\beta$-open set $\Rightarrow$ *s*-dense

*Remark.*No general relations between *SD*-dense set and open set, for instance:

*1.* In the usual topology, $(0,1)$ is open (s-dense) set but not *SD*-dense nor dense, since $(0,1)^c$ is s-dense but disjoint from $(0,1)$. Moreover, the set $\mathbb{Q}$ is dense but not *SD*-dense, since the set $\mathbb{K}$ is also *s*-dense but disjoint from $\mathbb{Q}$.
2. In the space $(\mathbb{R},\tau)$, where $\tau=P(\mathbb{K})\cup\mathbb{R}$ we have $S(\tau)=P(\mathbb{K})\cup\{A\subseteq\mathbb{R}: A\cap\mathbb{Q}, A\cap\mathbb{K}\neq\phi\}$. Then the set $\mathbb{K}\cup B$ where $B\subseteq\mathbb{Q}$ is *SD*-dense, so if $B\neq\phi$ the set $\mathbb{K}\cup B$ is *SD*-dense but not open.

**Theorem 8.** *If $E$ is a subset of a space $Z$, then these statements are equivalent:*

*1) $E$ is SD-dense.*
*2) $E$ intersects all s-dense sets in $Z$.*
*3) $(E^c)^{oS} = \phi$.*
*4) $E^c$ is not s-dense.*
*5) $E^o$ is dense.*
*6) $E$ contains an open dense subset $F$ in $Z$.*

*Proof.* 1) $\Rightarrow$ 2) Let $F$ be a $SD$-dense in $Z$, and suppose that $A$ is a $s$-dense set which is disjoint from $F$, then $A^c$ is $cs$-dense that contained $F$, hence $\overline{F}^S \neq \mathbb{Z}$, which is a contradiction.
2) $\Rightarrow$ 3) Suppose that $(E^c)^{oS}$ is a non-empty set, then there exists a $s$-dense set $A$ containing in $E^c$, hence a $s$-dense set $A$ does not intersect $E$, which is a contradiction.
3) $\Rightarrow$ 4) Suppose that $(E^c)^{oS} = \phi$, then from theorem (6) the set $E^c$ is not $s$-dense.
4) $\Rightarrow$ 5) Suppose $E^c$ is not $s$-dense, then $\overline{E^c}^{\circ} = \phi$, i.e., $\overline{E^c}$ does not contains any non-empty open set, therefore $(\overline{E^c})^c = E^o$ is dense.
5) $\Rightarrow$ 6) Direct since $E^o$ is open dense set.
6) $\Rightarrow$ 1) Let $F$ be an open dense subset in $Z$ and $F \subseteq E$, and suppose that $\overline{E}^S \neq Z$, then there exists a $s$-dense set $A$ which does not intersect $E$, and since $F^c$ is closed set, then $A \subseteq \overline{A} \subseteq \overline{E^c} \subseteq F^c$, but $A$ is a $s$-dense, then $\overline{A}$ contains a non-empty open set which is disjoint from the dense set $F$, which is a contradiction, therefore $\overline{E}^S = Z$, thus $E$ is $SD$-dense.

**Corollary 2.** *In a space $(Z,\tau)$ we have $\tau \cap D(\tau) \subseteq SD(\tau)$, moreover $SD(\tau) = \{A \subseteq Z : B \subseteq A$ for some open dense set $B\}$.*

*Proof.* According to the previous theorem number (5) we obtain $\tau \cap D(\tau) \subseteq SD(\tau)$ , and by using number (6) clearly any subset that contains an open dense set is $SD$-dense.

*Remark.* Let $(Z,\tau)$ be a space, then $Z$ is the only $SD$-dense set if and only if $Z$ is the only open dense set in $Z$.

*Example 2.* In the usual topology, the set $\mathbb{Z}$ of integer numbers is not $s$-dense set, hence $\mathbb{R}/\mathbb{Z}$ is $SD$-dense. Note that $\mathbb{R}/\mathbb{Z}$ is open dense set.

### *3.2 Operation on SD-Dense Sets*

Union and intersection operations of the class of SD-dense sets are investigated.

**Corollary 3.** *1) Any set contains SD-dense set is SD-dense.*
*2) Union of SD-dense sets is SD-dense.*

*Proof.* Obvious using theorem (8).

**Theorem 9.** *Any two SD-dense sets have non-empty intersection.*

*Proof.* Suppose $E$ and $F$ are two $SD$-dense sets in a space $Z$, then from corollary (1) we get $E$ is $s$-dense and $F$ is $SD$-dense, and according to theorem (8) we obtain $E \cap F \neq \phi$.

*Remark.* The infinite intersection of $SD$-dense sets can be empty set; for example in the usual topology, $\{\{x\}^c\}_{x\in\mathbb{R}}$ is a collection of $SD$-dense sets in $\mathbb{R}$, but $\cap_{x\in\mathbb{R}}\{x\}^c = \phi$.

**Lemma 1.** *If $A$ and $B$ are not s-dense sets in a space $Z$, then $A \cup B$ is also not s-dense.*

*Proof.* If at least one of the sets $A$ and $B$ are empty, then the prove is obvious. Now let $A$ and $B$ be non-empty not $s$-dense sets, and suppose that $A \cup B$ is $s$-dense, so there exists a non-empty open set $V$ such that $V \subseteq \overline{A \cup B} = \overline{A} \cup \overline{B}$, since $A$ is not dense we have $\overline{A}^c$ is a non-empty open set, and $V \cap \overline{A}^c \subseteq (\overline{A} \cup \overline{B}) \cap \overline{A}^c \subseteq \overline{B}$, therefore $V \cap \overline{A}^c$ is a non-empty open set contained in $\overline{B}$, so $\overline{B}^{\circ}$ is a non-empty set, hence $B$ is $s$-dense, which contradicts the assumption. Therefore $A \cup B$ is not $s$-dense.

*Remark.* Infinite union of non $s$-dense sets can be $s$-dense; for example in the usual topology any singleton is not $s$-dense set, but $\cup_{x\in\mathbb{R}}\{x\} = \mathbb{R}$ is $s$-dense.

**Theorem 10.** *The intersection of SD-dense sets is SD-dense.*


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*Proof.*Suppose $E$ and $F$ are two $SD$-dense sets in a space $Z$, then from theorem (9) we have $E\cap F\neq\phi$, and by theorem (8) the sets $E^c$ and $F^c$ are not $s$-dense, and by the previous lemma we obtain $E^c\cup F^c=(E\cap F)^c$ is not $s$-dense, hence $E\cap F$ is $SD$-dense.

**Corollary 4.***Any finite intersection of SD-dense sets is SD-dense.*

*Proof.*Direct from the mathematical induction.

### *3.3 SD-Dense Sets as Subspaces*

Here we study the characterizations of $SD$-dense sets as subspaces, and states some conditions that make subspaces $SD$-dense.

**Theorem 11.***In a space $Z$, if $W$ is a subspace of $Z$ and $E\subseteq W$ where $E$ is SD-dense in $Z$, then $E$ is SD-dense in $W$.*

*Proof.*Since $E$ is $SD$-dense $Z$, then $E$ contains an open dense set $F$, then $F$ is open dense set in $W$, therefore $E$ is $SD$-dense in $W$.

*Example 3.*Let $(Z,\tau)$ is a space, where $Z=\mathbb{R}$ and $\tau=\{\phi\}\cup\{U\subseteq Z:0\in U\}$, then $S(Z)=\tau/\{\phi\}$. If $W=\{0\}^c$, then $W$ is $SD$-dense in $W$, but $W$ is not $SD$-dense in $Z$.

**Lemma 2.***In a space $Z$, if $W$ is an open (dense) subspace of $Z$ and $D\subseteq Z$ where $D$ is an open dense in $Z$, then $D\cap W$ is an open dense in $W$.*

**Theorem 12.***In a space $Z$, if $W$ is an open (dense) subspace of $Z$ and $E$ is SD-dense in $Z$, then $E\cap W$ is SD-dense in $W$.*

*Proof.*Suppose $E$ is $SD$-dense in $Z$, then $E$ contains an open dense subset $F$, and by the previous lemma we obtain $F\cap W$ is open dense subset in $W$, which contained in $E\cap W$, thus $E\cap W$ is $SD$-dense in $W$.

*Example 4.*In the previous example if $W=\{0\}^c$, then the singleton $\{0\}$ is $SD$-dense in $Z$ but does not intersect $W$, additionally, the set $\{0,1\}$ is also $SD$-dense in $Z$, but $\{0,1\}\cap W=\{1\}$ is not $SD$-dense in $W$. Note that the subspace $W$ is closed but not open nor dense in $Z$.

**Theorem 13.***If $Z$ is a space, $W$ is an r-closed subspace of $Z$ and $E$ is SD-dense subset in $Z$, then $E\cap W$ is SD-dense in $W$.*

*Proof.*Suppose $E$ is $SD$-dense in $Z$, then by theorem (4) the set $W$ is $s$-dense in $Z$, hence we have $E\cap W$ is non-empty set. Now suppose $A$ is $s$-dense subset in $W$, and by theorem (5) since $W$ is $r$-closed we obtain $A$ is $s$-dense in $Z$, so we have $A\cap E\neq\phi$, therefore $A\cap(E\cap W)=A\cap E\neq\phi$, thus $E\cap W$ is $SD$-dense in $W$.

### *3.4 SD-Dense Sets in Some Special Spaces*

In the present subsection, we study the characterizations of $SD$-dense sets in some spaces, as partition space, $S$-space, submaximal space, hyperconnected space and strongly hyperconnected space

**Theorem 14.***A space $(Z,\tau)$ is partition if and only if the only SD-dense set is $Z$.*

*Proof.*If a space $Z$ is partition, then we have $S(\tau)=P(Z)/\{\phi\}$, therefore $Z$ is the only $SD$-dense set. Conversely, suppose $Z$ is not partition space, then there is an open set $V$ which is not closed, so $V^c$ is not open, hence $V^{c\circ}=\phi$ or $V^{c\circ}\neq\phi$. In the case where $V^{c\circ}=\phi$ we obtain $V$ is open dense set, hence it is $SD$-dense, while in the second case, we obtain $V\cup V^{c\circ}$ is open dense set, so it is $SD$-dense. Thus complete the prove.

**Corollary 5.***In S-space $(Z,\tau)$, we have $SD(\tau)=\tau\cap D(\tau)$.*

*Proof.*Direct since any subset of $Z$ that contains an open dense set is also open dense set, so it is $SD$-dense.

*Example 5.*A space $(Z,\tau)$ that satisfies $SD(\tau)=\tau\cap D(\tau)$ can be not $S$-space; for example the usual topological space $(\mathbb{R},\tau)$ is not $S$-space but $SD(\tau)=\tau\cap D(\tau)$.

**Theorem 15.***If $F$ is a subset of a submaximal space $Z$, then these statements are equivalents:*

*1) $F$ is dense in $Z$.*
*2) $F$ is SD-dense in $Z$.*

*Proof.*1) $\Rightarrow$ 2) Let $F$ be a dense set in $Z$, then $F$ is open dense set, and by using theorem (8) the set $F$ is $SD$-dense.
2) $\Rightarrow$ 1) Obvious using corollary (1).

*Remark.*In submaximal space $(Z,\tau)$, we have $SD(\tau)=D(\tau)\subseteq\tau$.

$SD$-dense $\equiv$ dense $\Rightarrow$ open set $\equiv$ pre-open set $\Rightarrow$ $b$-open set $\Rightarrow$ $\beta$-open set $\Rightarrow$ $s$-dense

*Example 6.*A space $(Z,\tau)$ that satisfies $SD(\tau)=D(\tau)$ may not be submaximal space; for example in $(\mathbb{R},\tau)$ given in remark (3.1), we have $SD(\tau)=D(\tau)=\{\mathbb{K}\cup A\subseteq\mathbb{R}: A\cap\mathbb{Q}\neq\phi\}\cup\{\mathbb{K}\}$, but $\mathbb{R}$ is not submaximal, since $\mathbb{K}\cup\{0\}$ is dense but not open.

**Theorem 16.***A space $(Z,\tau)$ is hyperconnected if and only if $\tau/\{\phi\}\subseteq SD(\tau)$.*

*Proof.*If $V$ is a non-empty open set, then it is open dense, hence it is $SD$-dense. Conversely, suppose that $V$ is a non-empty open set, then $V$ is $SD$-dense, i.e., $V$ contains an open dense set, therefore it is dense, thus complete the prove.

Non-empty open set $\Rightarrow$ SD-dense $\Rightarrow$ dense open set $\Rightarrow$ pre-open set $\Rightarrow$ $b$-open set $\Rightarrow$ $\alpha$-open set $\Rightarrow$ $s$-dense

*Remark.*In hyperconnected space $(Z,\tau)$, we have: $SD(\tau)=\{A\subseteq Z: V\subseteq$ A for some non-empty open set $V\}$.

*Proof.*Direct since any non-empty open set is dense, so it is $SD$-dense. Moreover, any set that contains a non-empty open set is $SD$-dense.

*Example 7.*Hyperconnected space can contains a dense subset which is not $SD$-dense, for instance: If $(Z,\tau)$ is the trivial space where $Z$ has more than one element, then $Z$ is hyperconnected space and $S(\tau)=P(Z)/\{\phi\}$, so any singleton $\{a\}$ is dense in $Z$ but not $SD$-dense. Clearly, the only $SD$-dense in $Z$ is $Z$.

**Corollary 6.***In hyperconnected S-space $(Z,\tau)$, non-empty open sets and SD-dense sets are equivalent, i.e., $SD(\tau)=\tau/\{\phi\}$.*

*Proof.*Direct using the previous remark and definition (4).

**Theorem 17.***If $F$ is a non-empty subset of a strongly hyperconnected space $Z$, then these statements are equivalents:*

*1) $F$ is SD-dense in $Z$.*
*2) $F$ is dense in $Z$.*
*3) $F$ is s-dense set in $Z$.*
*4) $F$ is open set in $Z$.*

*Proof.*1) $\Rightarrow$ 2) Obvious.
2) $\Rightarrow$ 3) Obvious.
3) $\Rightarrow$ 4) Obvious using theorem (3).
4) $\Rightarrow$ 1) Obvious using the submaximality in theorem (15).

*Remark.1.* A space that satisfy any dense set is $SD$-dense may not by strongly hyperconnected space, for example, In a space $\mathbb{R}$ with $\tau=\{U\subseteq\mathbb{R}: 0\notin U\}\cup\{\mathbb{R}\}$, we have $S(\tau)=P(\mathbb{R})/\{\phi,\{0\}\}$, so $SD(\tau)=\{\mathbb{R},\{0\}^c\}=D(\tau)$. Therefore, $SD$-dense set and dense set are coinciding, but the space $(\mathbb{R},\tau)$ is not strongly hyperconnected space since the singleton $\{1\}$ is open but not dense.

*2.* A space that satisfy any $s$-dense set is $SD$-dense may not by strongly hyperconnected space, for example, In a space $(Z,\tau)$, where $Z=\{1,2,3\}$, with $\tau=\{Z,\phi,\{1\},\{1,2\}\}$ we have $S(\tau)=\{Z,\{1\},\{1,2\},\{1,3\}\}=SD(\tau)$. Therefore, $SD$-dense set and dense set are coinciding, but the space $Z$ is not strongly hyperconnected space, since it is not submaximal, since $\{1,3\}$ is dense but not open.

*3.* A space that satisfy $SD(\tau)=\tau/\{\phi\}$ may not be strongly hyperconnected space, for example in the space $(\mathbb{R},\tau)$, where $\tau=\{\phi\}\cup\{U\subseteq\mathbb{R}:\{0\}\cup\mathbb{R}^+\subseteq U\}$, we obtain $SD(\tau)=\tau/\{\phi\}$, but the singleton $\{0\}$ is dense but not open, so $\mathbb{R}$ is not submaximal, therefore is not strongly hyperconnected.

**Corollary 7.***In a strongly hyperconnected space $Z$ if $E\subseteq Z$, these conditions are equivalents:*

*1) $E$ is a non-empty open set.*
*2) $E$ is a non-empty $\alpha$-open set.*


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*3) E is a non-empty pre-open set.*
*5) E is a non-empty b-open set.*
*6) E is a non-empty $\beta$-open set.*
*7) E is a s-dense set.*
*8) E is a dense set.*
*9) E is a SD-dense set.*

*Proof.* According to theorems (3,17).

### *3.5 Images of SD-Dense Sets*

Here we show that *SD*-irresolute map preserves *SD*-dense sets, while *SD*-continuous map dose not, moreover, the image of *SD*-dense set by *SD*-continuous map is dense.

**Theorem 18.** *If $f:(Z,\tau)\to(X,\sigma)$ is surjective SD-irresolute map, then the image of any SD-dense set in Z is SD-dense in $X$.*

*Proof.* Suppose $A$ is *SD*-dense in $Z$, but $f(A)$ is not *SD*-dense in $X$, then there is a $s$-dense set $B$ in $X$ such that $f(A)\cap B=\phi$. Since $f$ is surjective and *SD*-irresolute, then $A\cap f^{-1}(B)=\phi$, where $f^{-1}(B)$ is $s$-dense in $Z$, which contradict that $A$ is *SD*-dense.

*Example 8.* The image of *SD*-dense set in $Z$ by *SD*-irresolute map need not be *SD*-dense set in $X$, for example: Let $\tau=\{U\subseteq\mathbb{R}:0\in U\}\cup\{\phi\}$ and $\sigma=\{U\subseteq\mathbb{R}:0\notin U\}\cup\{\mathbb{R}\}$ be two topologies on $\mathbb{R}$, then $SD(\tau)=\tau/\{\phi\}$ while $SD(\sigma)=\{R,\{0\}^c\}=D(\sigma)$. Therefore the map $f:(\mathbb{R},\sigma)\to(\mathbb{R},\sigma)$ which defined by: $f(r)=\begin{cases}1, & r\in\mathbb{Q}\\ 0, & r\in\mathbb{K}\end{cases}$ is *SD*-irresolute map but not surjective, while the singleton $\{0\}$ is *SD*-dense in $\tau$ but $f(\{0\})=\{1\}$ is not *SD*-dense in $\sigma$.

**Theorem 19.** *If $f:(Z,\tau)\to(X,\sigma)$ is surjective SD-continuous map, then the image of any SD-dense set in Z is dense in $X$.*

*Proof.* Suppose $A$ is *SD*-dense in $Z$ but $f(A)$ is not dense in $X$, so there exists an open set $B$ in $X$ such that $f(A)\cap B=\phi$. Since $f$ is surjective and *SD*-continuous, then $A\cap f^{-1}(B)=\phi$, where $f^{-1}(B)$ is $s$-dense in $Z$, which contradict that $A$ is *SD*-dense.

*Example 9.* The image of *SD*-dense in $Z$ by *SD*-continuous (continuous) map need not be *SD*-dense in $X$, for instance: If $\tau=\{U\subseteq\mathbb{R}:0\in U\}\cup\{\phi\}$ while $\sigma$ is the trivial topology on $R$, hence the identity map $I:(R,\tau)\to(R,\sigma)$ is surjective *SD*-continuous (also is continuous) from; while the singleton $\{0\}$ is *SD*-dense in $\tau$ but $I\{0\}=\{0\}$ is not *SD*-dense in $\sigma$, since the only *SD*-dense in $\sigma$ is $\mathbb{R}$.

## 4 SD-Separability

In this section, we define *SD*-separable space, which is stronger than separable space and then we study its properties; as subspaces and images.

### *4.1 SD-Separable Spaces*

**Definition 12.** *A space that contains a countable SD-dense set is called SD-separable space.*

**Corollary 8.** *Every SD-separable space is separable space.*

*Proof.* Obvious since any *SD*-dense is dense.

*Example 10.*
*1)* If $(Z,\tau)$ is the trivial topological space where $Z$ is uncountable set, then $Z$ is separable space but not *SD*-separable, because the only *SD*-dense set is $Z$.
*2)* If $(\mathbb{R},\mu)$ is the usual space, then $\mathbb{R}$ is separable but not *SD*-separable, because if $F$ is a non-empty countable subset of $R$ then $F^c$ is also $s$-dense and $F\cap F^c=\phi$, so $F$ is not *SD*-dense, i.e., any *SD*-dense in $\mathbb{R}$ is uncountable.

*3)* If $(\mathbb{R},\tau)$ is a space where $\tau = P(Q)\cup\{\mathbb{R}\}$, then $S(\tau) = P(\mathbb{Q})\cup\{A\subseteq\mathbb{R} : A\cap\mathbb{Q}\neq\phi, A\cap\mathbb{K}\neq\phi\}$. Hence the set $\mathbb{Q}\cup B$ where $B\subseteq\mathbb{K}$ is *SD*-dense, so $\mathbb{Q}\cup\{\sqrt{2}\}$ is a countable *SD*-dense in $\mathbb{R}$, therefore $(\mathbb{R},\tau)$ is *SD*-separable space.

**Definition 13.***A space $(Z,\tau)$ is called SD-countable if the collection $S(\tau)$ is countable.*

**Corollary 9.***Any SD-countable space is countable.*

*Proof.*Obvious since $\tau/\{\phi\}\subseteq S(\tau)$.

*Example 11.*Countable space need not be *SD*-countable, for instance if $(Z,\tau)$ is the trivial space on infinite set $Z$, then $\tau$ is countable space but not *SD*-countable, since $S(\tau) = P(Z)/\{\phi\}$.

**Corollary 10.***Every SD-countable space is SD-separable space.*

*Proof.*Since the collection $S(\tau)$ is countable, now we can choose a point from each *s*-dense set from $S(\tau)$, the set $F$ of all such point is clearly countable and *SD*-dense, therefore $Z$ is *SD*-separable.

*Example 12.SD*-separable space need not be *SD*-countable, for instance the space $\tau = \{\phi\}\cup\{U\subseteq\mathbb{R} : 0\in U\}$ on $\mathbb{R}$, we obtain $S(\tau) = \tau/\{\phi\}$, then $\mathbb{R}$ is SD-separable space, since $\{0\}$ is a countable *SD*-dense in $\mathbb{R}$, but $S(\tau)$ is uncountable, therefore $\tau$ is not *SD*-countable.

**Corollary 11.***If Z is a SD-separable space where Z is uncountable, then there exists an uncountable set which is not s-dense.*

*Proof.*Since Z contains a countable *SD*-dense subset $E$, then $E^c$ is uncountable and not *s*-dense.

*Remark.*The inverse of the previous corollary is not true in general, for example: in the space $(\mathbb{R},\tau)$ given in remark (3.4) number (1), the set of negative real numbers $\mathbb{R}^-$ is uncountable and it is not *s*-dense, while $\mathbb{R}$ it is not *SD*-separable space, since all *SD*-dense sets are uncountable.

**Corollary 12.***A space Z is SD-separable if and only if there exists a set A which is not s-dense, where A, $A^c$ are uncountable and countable sets; respectively.*

**Corollary 13.***Submaximal separable space is SD-separable space.*

*Proof.*Obvious since dense sets and *SD*-dense sets are equivalent from theorem (15).

## *4.2 Subspaces and Images of SD-Separable Spaces*

*Example 13.*Subspace of *SD*-separable space need not *SD*-separable space in general, for instance: where $Z$ is uncountable set with topology $\tau = \{U\subseteq Z : a\in U\}\cup\{\phi\}$, where a is a fixed point in $Z$, so $S(\tau) = \tau/\{\phi\}$. The singleton $\{a\}$ is *SD*-dense, hence $Z$ is *SD*-separable space but the subspace $\{a\}^c$ is not *SD*-separable space, since it is the discrete space. Note that the subspace $\{a\}^c$ is not open subspace nor *r*-closed.

**Corollary 14.***Any open (dense or r-closed) subspace of SD-separable space is SD-separable space.*

*Proof.*Direct using theorem (12) (theorem (13)).

**Corollary 15.***If a map $f:(Z,\tau)\to(X,\sigma)$ is surjective and SD-irresolute from SD-separable space Z, then X is SD-separable.*

*Proof.*Direct using theorem (18).

*Example 14.*The image of *SD*-separable space by *SD*-continuous map need not be *SD*-separable, for instance if $(\mathbb{R},\tau)$ a space given in example (3) while $(\mathbb{R},\sigma)$ is the trivial topological space, hence the identity map $I : (\mathbb{R},\tau)\Rightarrow(\mathbb{R},\sigma)$ is *SD*-continuous (also is continuous) from *SD*-separable space $(\mathbb{R},\tau)$; since $\{0\}$ is countable *SD*-dense in $(\mathbb{R},\tau)$, while the space $(\mathbb{R},\sigma)$ is not *SD*-separable, since the only *SD*-dense in $(\mathbb{R},\sigma)$ is $\mathbb{R}$.

**Corollary 16.***If a map $f:(Z,\tau)\to(X,\sigma)$ is surjective SD-continuous from SD-separable space Z, then X is separable.*

*Proof.*Suppose $E$ is a countable *SD*-dense subset in $Z$, according to theorem (19) we obtain $f(E)$ is a countable dense in $X$, therefore $X$ is separable space.


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## 5 Conclusion

Using the concept of somewhere dense closure operator, we define a generalization of dense set; namely *SD*-dense set, then we introduce a new type of separability; namely *SD*-separable space. Here we outline the results that summarized the properties of *SD*-dense sets and *SD*-separable space:

**A**. *SD*-Dense Set $\Rightarrow$ Dense Set.
**B**. A subset $F$ of a space $Z$ is *SD*-dense if and only if $F$ intersect all $s$-dense sets; equivalently if $(F^c)^{oS} = \varphi$; equivalently if $F^c$ is not $s$-dense; equivalently if $F^o$ is open dense set; equivalently if $F$ contains an open dense set.
**C**. The union of two non $s$-dense sets is also non $s$-dense.
**D**. The intersection of two *SD*-dense sets is *SD*-dense.
**E**. If $W$ is open (dense or regular closed) subspace of a space $Z$, and $F$ is *SD*-dense subset in $Z$, then $F \cap W$ is *SD*-dense in $W$.
**F**. A space is partition if and only if it has no proper *SD*-dense set.
**G**. *SD*-Dense Set $\xRightarrow{S\text{-Space}}$ Open Dense Set.
**H**. *SD*-Dense Set $\xLeftrightarrow{\text{Submaximal Space}}$ Dense Set.
**I**. Non-empty Open Set $\xRightarrow{\text{Hyperconnected Space}}$ *SD*-Dense Set.
**J**. *SD*-Dense Set $\xLeftrightarrow{\text{Hyperconnected } S\text{-Space}}$ Non-empty Open Set
**K**. In strongly hyperconnected space, all these statements are equivalent: *SD*-dense set, dense set, $s$-dense set, $\beta$-open set, $b$-open set, preopen set and open set.
**L**. *SD*-Separable Space $\Rightarrow$ Separable Space.
**M**. *SD*-Separable Space $\xLeftrightarrow{\text{Submaximal Space}}$ Separable Space.
**N**. *SD*-separable space satisfy the open (dense or regular closed) hereditary property.
**O**. *SD*-irresolute map preserves *SD*-separable space (*SD*-dense set).
**P**. *SD*-continuous map does not preserve *SD*-separable space (*SD*-dense set), but the image of *SD*-separable space (*SD*-dense set) is separable space (dense set).

Note that some properties of *SD*-dense sets are different from dense sets, as in A, B (the third part), C and D.

## Declarations

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