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Optimal Conjugate Gradient with Spline Scheme for Solving Bagley-Torvik Fractional Differential Problems

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Abstract: In this work, a non-polynomial spline function is constructed to solve the Bagley-Torvik Fractional Differential Problems involving derivatives in the Caputo sense. This method transforms the fractional differential equation into a system of linear equations using a spline scheme. The conjugate gradient method is employed for the iterative solution of the linear system. To validate the accuracy of the method, numerical examples with known analytical solutions are tested. The numerical experiments demonstrate satisfactory agreement with the exact solution.

Keywords: Fractional calculus; Bagley-Torvik Fractional Differential equation; Caputo fractional derivatives; Non-polynomial Spline; Conjugate gradient method.

2010 Mathematics Subject Classification. 26A25; 26A35.

1 Introduction

Fractional differential equations have received significant attention nowadays in several fields of science and engineering due to its applications such as : electrical engineering[1],economic[2],Modelling of Viscoelastic Systems[3],diffusion processes[4],medicine[5]. It is difficult to find an exact analytical solution of all fractional differential equations therefore several methods and techniques have been invented to solve fractional differential equation for instance: fractional finite difference method[6], Adomain decomposition method[7],spectral method[8],Bessel collocation method[9].

Spline technique has been investigated by many researchers for solving fractional differential equations due to its accurate and efficiency for example: W. K. ZAHRA and et al proposed cubic spline solution of fractional Bagley-Torvik equation[10], semiorthogonal B-spline collection is applied for solving the fractional differential equations[11], NonPolynomial Spline discussed by Faraidun K. Hamasalh and et al to solve FDE[12], Faraidun K. Hamasalh and Karzan A. Hamza, used Quintic B-spline polynomial for Solving Bagely-Torvik Fractional Differential Problems[13], fourth order homogeneous parabolic partial differential equations solved using non-polynomial cubic spline technique[14].

Conjugate gradient method is an appropriate and efficient method for solving a system of equations. The linear conjugate gradient method was proposed in the 1950s by Hestenes and Stiefel to solve a linear system of equations with positive definite matrices as an alternative to Gauss elimination[15], Fletcher and Reeves were discussed the nonlinear conjugate gradient method in 1964[16]. Presently, conjugate gradient (CG) techniques are considered as a popular and efficient approach to solve engineering optimization problems. As recent examples, shape optimization with nonlinear conjugate gradient method proposed in[17], application in signal processing of decent hybrid nonlinear conjugate gradient method discussed by Zohre Aminifard and etal[18] Abubakar and et al investigated a modified a three-term

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conjugate gradient projection with application in signal recovery[19].

The rest of this paper is organized as follows: in section 2, we briefly review the main definitions of fractional calculus, some definitions and properties of the matrix. Mathematical formulation of the nonpolynomial spline function discussed in section 3. In section 4 numerical results are illustrated to present applicability of the method. Finally, the conclusion is presented in section 5.

2 Some basic definitions

Definition 1.[20] *The Riemann-Liouville fractional derivative of order* $\lambda > 0$ *is defined by* $D^{\lambda} f(t) = \frac{1}{\Gamma(m-\lambda)} \frac{d^m}{d\tau^m} \int_a^t (t-\tau)^{m-\lambda-1} f(\tau) d\tau, \quad m-1 < \lambda < m \in N$

Definition 2.[21] *The Caputo fractional derivative of order* $\lambda > 0$ *is defined by* $D^{\lambda} f(t) = \frac{1}{\Gamma(m-\lambda)} \int_a^t (t-\tau)^{m-\lambda-1} \frac{d^m}{d\tau^m} f(\tau) d\tau, \quad m-1 < \lambda < m \in N$

Definition 3.[20] *The Riemann-Liouville fractional integral of order* $\lambda > 0$ *is defined by* $I^{\lambda} f(t) = \frac{1}{\Gamma\lambda} \int_a^t (t-\tau)^{\lambda-1} f(\tau) d\tau, \quad m-1 < \lambda < m \in N$

Definition 4.[20] *The Caputo derivative of order* λ *of a polynomial function* x^d *is defined by* $D^{\lambda} x^d = \frac{\Gamma(d+1)}{\Gamma(d-\lambda+1)} x^{d-\lambda}$

Definition 5.[22] *The Spectral radius* $\mu(M)$ *where* M *is an* $n \times n$ *matrix is given by* $\mu(M) = max(|\lambda|)$ *where* λ *is an eigenvalue of* M.

Definition 6.[23] *A square matrix M is called diagonally dominate if* $|m_{ij}| < \Sigma_{i \neq j} |m_{ij}|$

Definition 7.[22] *An* $n \times n$ *matrix* M *is converges if* $\mu(M) < 1$.

3 Mathematical Formulation

In this study we consider the fractional differential equation of the form

$$y^{(\frac{3}{2})} + \phi(x) y'' + \psi(x) y = \tau(x), \qquad x \in [a,b] \tag{1}$$

with the boundary conditions

$$y(a) = B_1, \quad y(b) = B_2 \tag{2}$$

Where $\phi(x), \psi(x) and \tau(x)$ are functions of x, B_1 and B_2 are constants.Then the interval $[a,b]$ can be uniformly divide into j subintervals the length of uniform subintervals can be define as:$\Delta x = h = \frac{b-a}{j}, \quad n = j-1$. In this existing literature we can modify the model of nonpolynomial spline and the factional continuity by using Caputo type as follows:

$$S(x) = S_i(x), x \in [x_i, x_{i+1}], i = 0, 1, 2, ..., n \tag{3}$$

Here the nonpolynomial spline function with fractional order defined by

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i sin(k(x-x_i)) + f_i cos(k(x-x_i)) \tag{4}$$

where $a_i, b_i, c_i, d_i, e_i, f_i$ are constants for $i = 0, 1, 2, ..., n$ and k is a free parameter .The function $S_i(x)$ interpolates $y(x)$ at the points x_i by depending on k. To find the value of constants in equation (4) we supposed the following conditions:

$$\begin{gathered} S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1}, S_i''(x_i) = y_i'', \\ S_{i+1}''(x_{i+1}) = y_{i+1}'', S_i^{(\frac{3}{2})}(x_i) = p_i, S_{i+1}^{(\frac{3}{2})}(x_{i+1}) = p_{i+1}. \end{gathered} \tag{5}$$

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Applying the conditions in equation (5) the value of all constants in equation (4) obtained as follows:

$$
\begin{gathered}
a_i=(1-\frac{2}{3A_1}\sqrt{\frac{h}{\pi}})y_i''+\frac{4}{3}\sqrt{\frac{h}{\pi}}y_{i+1}''-\frac{1}{A_1}p_{i+1}+\frac{A_2}{A_1}p_i,\\
b_i=\frac{y_{i+1}-y_i}{h}-\frac{1}{h}(\frac{\theta^2}{2A_1}+\frac{h^3\beta}{A_1}+\frac{sin\theta+cos\theta-1}{A_1})p_{i+1}-\frac{1}{h}((1-\theta^2)\frac{A_2}{A_1}+\frac{h^3}{A_5}\\
+sin\theta(\frac{1}{k}\sqrt{\frac{2}{k}}-\frac{A_2}{A_1})-\frac{A_2cos\theta}{A_1})P_i-\frac{1}{h}(\frac{4}{3}\sqrt{\frac{h}{\pi}}-\frac{2}{3}\theta^2\sqrt{\frac{h}{\pi}}+h^3A_4-\frac{4}{3}\sqrt{\frac{h}{\pi}}sin\theta\\
-\frac{\frac{4}{3}\sqrt{\frac{h}{\pi}}cos\theta}{A_1})y_{i+1}''+\frac{1}{h}(\frac{h^2}{2}+\frac{\theta^2A_3}{2A_1}-h^3(\frac{\beta A_3}{A_1}-\frac{1}{6h})+\frac{(sin\theta+cos\theta)A_3}{A_1})y_i'',\\
c_i=\frac{k^2}{2A_1}P_{i+1}-\frac{k^2A_2}{2A_1}p_i-\frac{2}{3}k^2\sqrt{\frac{h}{\pi}}y_{i+1}+(\frac{1}{2}-\frac{\frac{1}{3}k^2\sqrt{\frac{h}{\pi}}}{A_1})y_i''\\
d_i=(\frac{1}{6h}-\frac{4\beta}{3A_1}\sqrt{\frac{h}{\pi}})y_{i+1}''-(\frac{2\beta}{3A_1}\sqrt{\frac{h}{\pi}}+\frac{1}{6h})y_i''+\frac{\beta}{A_1}p_{i+1}+\\
(\frac{\sqrt{2}k^2sin\theta}{6hk^{\frac{3}{2}}}-\frac{\beta A_2}{A_1})p_i,\\
e_i=(\frac{\sqrt{2}}{k^{\frac{3}{2}}}-\frac{A_2}{A_1})p_i+\frac{1}{A_1}p_{i+1}-\frac{4}{3}\sqrt{\frac{h}{\pi}}y_{i+1}''+\frac{4}{3A_1}\sqrt{\frac{h}{\pi}}y_i'',\\
f_i=\frac{1}{A_1}p_{i+1}-\frac{A_2}{A_1}p_i-\frac{4}{3}\sqrt{\frac{h}{\pi}}y_{i+1}''+\frac{4}{3A_1}\sqrt{\frac{h}{\pi}}y_i''.
\end{gathered}
\tag{6}
$$

where $\beta=\frac{k^2(sin\theta+cos\theta-1)}{6h}, A_1=2k^2\sqrt{\frac{h}{\pi}}+8\beta h\sqrt{\frac{h}{\pi}}+k^{\frac{3}{2}}(cos(\theta+\frac{3\pi}{4})+sin(\theta+\frac{3\pi}{4}))$, $A_2=\frac{4\sqrt{2\theta sin\theta}}{3\sqrt{\pi}}-\sqrt{2}sin(\theta+\frac{3\pi}{4}), A_3=\frac{-2}{3}\sqrt{\frac{h}{\pi}}, A_4=\frac{1}{6h}-\frac{4}{3}\sqrt{\frac{h}{\pi}}, A_5=\frac{\sqrt{2k}sin\theta}{6h}-\frac{\beta A_2}{A_1}, \theta=kh, i=0,1,...,n$. Therefore we obtain the nonpolynomial spline function, it can be easily verified that the spline scheme approximation $S(x)$, is successfully uniquely determined using the equation (6) recurrence formula for all h which in the interval, see[24]. Substitute these values in equation (4) we obtain

$$
\begin{gathered}
S(x)=(1-\frac{2}{3A_1}\sqrt{\frac{h}{\pi}})y_i''+\frac{4}{3}\sqrt{\frac{h}{\pi}}y_{i+1}''-\frac{1}{A_1}p_{i+1}+\frac{A_2}{A_1}p_i+(\frac{y_{i+1}-y_i}{h}\\
-\frac{1}{h}(\frac{\theta^2}{2A_1}+\frac{h^3\beta}{A_1}+\frac{sin\theta+cos\theta-1}{A_1})p_{i+1}-\frac{1}{h}((1-\theta^2)\frac{A_2}{A_1}+\frac{h^3}{A_5}+\\
sin\theta(\frac{1}{k}\sqrt{\frac{2}{k}}-\frac{A_2}{A_1})-\frac{A_2cos\theta}{A_1})P_i-\frac{1}{h}(\frac{4}{3}\sqrt{\frac{h}{\pi}}-\frac{2}{3}\theta^2\sqrt{\frac{h}{\pi}}\\
+h^3A_4-\frac{4}{3}\sqrt{\frac{h}{\pi}}sin\theta-\frac{\frac{4}{3}\sqrt{\frac{h}{\pi}}cos\theta}{A_1})y_{i+1}''+\frac{1}{h}(\frac{h^2}{2}+\frac{\theta^2A_3}{2A_1}-h^3(\frac{\beta A_3}{A_1}\\
-\frac{1}{6h})+\frac{(sin\theta+cos\theta)A_3}{A_1})y_i'')(x-x_i)(\frac{k^2}{2A_1}P_{i+1}-\frac{k^2A_2}{2A_1}p_i-\frac{2}{3}k^2\sqrt{\frac{h}{\pi}}y_{i+1}+\\
(\frac{1}{2}-\frac{\frac{1}{3}k^2\sqrt{\frac{h}{\pi}}}{A_1})y_i'')(x-x_i)^2+((\frac{1}{6h}-\frac{4\beta}{3A_1}\sqrt{\frac{h}{\pi}})y_{i+1}''-(\frac{2\beta}{3A_1}\sqrt{\frac{h}{\pi}}+\frac{1}{6h})y_i''+\\
\frac{\beta}{A_1}p_{i+1}+(\frac{\sqrt{2}k^2sin\theta}{6hk^{\frac{3}{2}}}-\frac{\beta A_2}{A_1})p_i)(x-x_i)^3+\\
((\frac{\sqrt{2}}{k^{\frac{3}{2}}}-\frac{A_2}{A_1})p_i+\frac{1}{A_1}p_{i+1}-\frac{4}{3}\sqrt{\frac{h}{\pi}}y_{i+1}''+\frac{4}{3A_1}\sqrt{\frac{h}{\pi}}y_i'')sin(k(x-x_i))+\\
(\frac{1}{A_1}p_{i+1}-\frac{A_2}{A_1}p_i-\frac{4}{3}\sqrt{\frac{h}{\pi}}y_{i+1}''+\frac{4}{3A_1}\sqrt{\frac{h}{\pi}}y_i'')cos(k(x-x_i)).
\end{gathered}
\tag{7}
$$

Now apply the fractional continuity conditions of the spline function $S_i(x)$ where the splines $S_{i-1}^m(x)=S_i^m(x), m=\frac{1}{2},1$ joined, we obtained the following equations:

$$
S_i^{(\frac{1}{2})}(x_i)=\frac{\sqrt{2k}}{A_1}p_{i+1}+\sqrt{\frac{k}{2}}(\frac{\sqrt{2}}{k^{\frac{3}{2}}}-\frac{2A_2}{A_1})p_i-\frac{4\sqrt{\theta}}{3\sqrt{2\pi}}(1+\frac{1}{A_1})y_{i+1}''-\frac{4\sqrt{\theta}}{3\sqrt{2\pi}A_1}y_i'' \tag{8}
$$

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, And

$$S_{i-1}^{(\frac{1}{2})}(x_i) = \frac{2}{\sqrt{h\pi}}(y_i - y_{i-1}) + (\frac{\sqrt{k}(cos(\theta+\frac{\pi}{4})+sin(\theta+\frac{\pi}{4}))}{A_1} + \frac{42\beta h^{\frac{5}{2}}}{15A_1\sqrt{\pi}} + \frac{4k^2h^{\frac{3}{2}}}{3A_1\sqrt{\pi}} - \frac{2L_1}{\sqrt{\pi h}})p_i + (\sqrt{k}sin(\theta+\frac{\pi}{4})(\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{A_2}{A_1}) - \frac{\sqrt{k}cos(\theta+\frac{\pi}{4})A_2}{A_1} + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{\sqrt{2k}sin\theta}{6h} - \frac{A_2\beta}{A_1}) - \frac{4k^2h^{\frac{3}{2}}}{3\sqrt{\pi}A_1} - \frac{2L_2}{\sqrt{h\pi}})p_{i-1} + (\frac{42\beta h^{\frac{5}{2}}}{15A_1\sqrt{\pi}}(\frac{1}{6h} - \frac{4\beta\sqrt{h}}{3A_1\sqrt{\pi}}) - \frac{4\sqrt{\theta}sin(\theta+\frac{\pi}{4})}{3\sqrt{\pi}} - \frac{4\sqrt{\theta}cos(\theta+\frac{\pi}{4})}{3A_1\sqrt{\pi}} - \frac{16\theta^2}{9\pi} - \frac{2L_3}{\sqrt{h\pi}})y_i'' + (\frac{8h^{\frac{3}{2}}}{3\sqrt{\pi}}(\frac{1}{2} - \frac{k^2\sqrt{h}}{3A_1\sqrt{\pi}}) - \frac{2L_4}{\sqrt{\pi h}} - \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{2\beta\sqrt{h}}{3A_1\sqrt{\pi}} + \frac{1}{6h}) - \frac{2\sqrt{\theta}(sin(\theta+\frac{\pi}{4})+cos(\theta+\frac{\pi}{4}))}{3A_1\sqrt{\pi}})y_{i-1}'' \tag{9}$$

Such that,

$L_1 = \frac{\theta^2}{2A_1} + \frac{h^3\beta+cos\theta+sin\theta-1}{A_1}$,

$L_2 = \frac{A_2}{A_1}(1-\theta^2 - sin\theta) + h^3A_5 + (\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{A_2}{A_1})$

$L_3 = h^3A_4 + (\frac{4}{3} - \frac{2\theta^2}{3} - \frac{4sin\theta}{3} - \frac{4cos\theta}{3A_1})\sqrt{\frac{h}{\pi}}$,

$L_4 = \frac{h^2}{2} + \frac{\theta^2A_3}{2A_1} + h^3(\frac{\beta A_3}{A_1} - \frac{1}{6h}) + \frac{A_3}{A_1}(sin\theta + cos\theta)$

Here by equating equation (8) and equation (9) we obtain

$$\frac{\sqrt{2}}{A_1}p_{i+1} + C_1p_i - \frac{4\sqrt{\theta}}{3\sqrt{2\pi}}(1+\frac{1}{A_1})y_{i+1}'' - C_2y_i'' - \frac{2}{\sqrt{\pi h}}(y_i - y_{i-1}) + C_3p_{i-1} + C_4y_{i-1}'' = 0 \tag{10}$$

Where, $C_1 = \frac{1}{k} - \frac{\sqrt{2k}A_2}{A_1} - \frac{4k^2h^{\frac{3}{2}}}{3A_1\sqrt{\pi}} - \frac{42h^{\frac{5}{2}}\beta}{15\sqrt{\pi}A_1} - \frac{\sqrt{k}(sin(\theta+\frac{\pi}{4})+cos(\theta+\frac{\pi}{4}))}{A_1}$

$C_2 = \frac{4\sqrt{\theta}}{3\sqrt{2\pi}A_1} - \frac{2L_3}{\sqrt{h\pi}} - \frac{16\theta^2}{9\pi} + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{1}{6h} - \frac{4\beta\sqrt{h}}{3A_1\sqrt{\pi}}) - \frac{4\sqrt{\theta}sin(\theta+\frac{\pi}{4})}{3\sqrt{\pi}} - \frac{4\sqrt{\theta}cos(\theta+\frac{\pi}{4})}{3\sqrt{\pi}A_1}$,

$C_3 = \frac{2L_2}{\sqrt{h\pi}} + \frac{4k^2h^{\frac{3}{2}}}{3A_1\sqrt{\pi}} + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{\sqrt{2k}sin\theta}{6h} - \frac{A_2\beta}{A_1}) - \sqrt{k}sin(\theta+\frac{\pi}{4})(\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{A_2}{A_1}) + \sqrt{k}cos(\theta+\frac{\pi}{4})\frac{A_2}{A_1}$,

$C_4 = \frac{2L_4}{\sqrt{h\pi}} - \frac{8h^{\frac{3}{2}}}{3\sqrt{\pi}}(\frac{1}{2} - \frac{k^2\sqrt{h}}{3A_1\sqrt{\pi}}) + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{1}{6h} + \frac{2\beta\sqrt{h}}{3A_1\sqrt{\pi}}) + \frac{\sqrt{2\theta}(cos(\theta+\frac{\pi}{4})+sin(\theta+\frac{\pi}{4}))}{3A_1\sqrt{\pi}}$.

from equation (1), and using backward, central, and forward difference formula for $y_{i+1}'', y_i'', and y_{i-1}''$ respectively we have

$$\begin{aligned} p_{i+1} &= -\phi_{i+1}(x)y_{i+1}'' - \psi_{i+1}(x)y_{i+1} + \tau_{i+1}(x) \\ p_i &= -\phi_i(x)y_i'' - \psi_i(x)y_i + \tau_i(x), \\ p_{i-1} &= -\phi_{i-1}(x)y_{i-1}'' - \psi_{i-1}(x)y_{i-1} + \tau_{i-1}(x) \\ y_{i+1}'' &= \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}, y_i'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}, y_{i-1}'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} \end{aligned} \tag{11}$$

substitute equation (11) in equation (10) we obtain:

$$a_iy_{i-1} + b_iy_i + c_iy_{i+1} = F_i \tag{12}$$

Then, a system of linear equation is formulated using equation(12) as follows :

$$Ay = F \tag{13}$$

such that

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

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$y=[y_1\ \ y_2\ \ y_3\ \ \cdots\ \ y_{n-1}\ \ y_n]^T$, and $F=[F_1-a_1y_0\ \ F_2\ \ \ \ \cdots\ \ F_{n-1}\ \ F_n-c_ny_{n+1}]$
Such that,
$a_i=\frac{-\sqrt{2k}\phi_{i+1}}{h^2A_1}-\frac{4\sqrt{\theta}}{3\sqrt{2\pi}h^2}(1+\frac{1}{A_1})-\frac{C_1\phi_i}{h^2}-\frac{1}{h^2}(C_4-C_2)+\frac{2}{\sqrt{h\pi}}-\frac{C_3\phi_{i-1}}{h^2}-C_3\psi_{i-1},$

$b_i=\frac{2\sqrt{2k}\phi_{i+1}}{h^2A_1}-\frac{8\sqrt{\theta}}{3\sqrt{2\pi}h^2}(1+\frac{1}{A_1})+\frac{2C_1\phi_i}{h^2}-\frac{2C_3\phi_{i-1}}{h^2}-\frac{2}{h^2}(C_4-C_2)-C_1\psi_i-\frac{2}{\sqrt{h\pi}},$
$c_i=\frac{-\sqrt{2k}\phi_{i+1}}{h^2A_1}-\frac{\sqrt{2k}\psi_{i+1}}{A_1}-\frac{4\sqrt{\theta}}{3\sqrt{2\pi}h^2}(1+\frac{1}{A_1})-\frac{C_1\phi_i}{h^2}-\frac{C_3\phi_{i-1}}{h^2}+\frac{1}{h^2}(C_4-C_2)$
$F_i=\frac{\sqrt{2k}}{A_1}\tau_{i+1}-C_1\tau_i-C_3\tau_{i-1}, i=1,2,\cdots,n.$

4 Numerical experiments

In this section the method applied to several numerical examples of boundary fractional differential equations, the result compared with the exact analytical solution to show the methods efficiency. The computational programs were written in MatLab. Here the algorithms of the conjugate gradient method is presented.

Algorithm 1*suppose that we have the linear system (13) where A is symmetric positive definite matrix The conjugate gradient algorithm expressed as:*

–*chose* $y_0\in R^n$, *and put* $d_0=r_0=F-Ay_0$ *for* $k=0,1,2,\cdots$
–*If* $d_k=0$, *stop and* y_k *is a solution of* $Ay=F$.
otherwise compute

–$\alpha_k=\frac{r_k^Tr_k}{d_k^TAd_k}, y_{k+1}=y_k+\alpha_kd_k,$

–$r_{k+1}=r_k-\alpha_kAd_k, \beta_k=\frac{r_{k+1}^Tr_{k+1}}{r_k^Tr_k}$

–$d_{k+1}=r_{k+1}+\beta_kd_k.$

Example 1.*[20] Consider the fractional differential equation*

$$D^2y(x)+D^{(\frac{3}{2})}y(x)+y(x)=1+x, x\in[0,1]. \tag{14}$$

with the boundary conditions $y(0)=1, y(1)=2$
,The exact solution of (14) is given by $y(x)=1+x$.

The numerical results using conjugate gradient method with, $h=\frac{1}{32}$, and 31 iterations tabulated in Table1

x	Exact solution	proposed method	Absolute error
0.125	1.125	1.125927	9.27×10^{-4}
0.25	1.25	1.251416	1.41×10^{-3}
0.375	1.375	1.376678	1.67×10^{-3}
0.5	1.5	1.501712	1.71×10^{-3}
0.625	1.625	1.626516	1.51×10^{-3}
0.75	1.75	1.751092	1.09×10^{-3}
0.875	1.875	1.875442	4.42×10^{-4}

Table 1: Exact, approximation solution, absolute error of example 1

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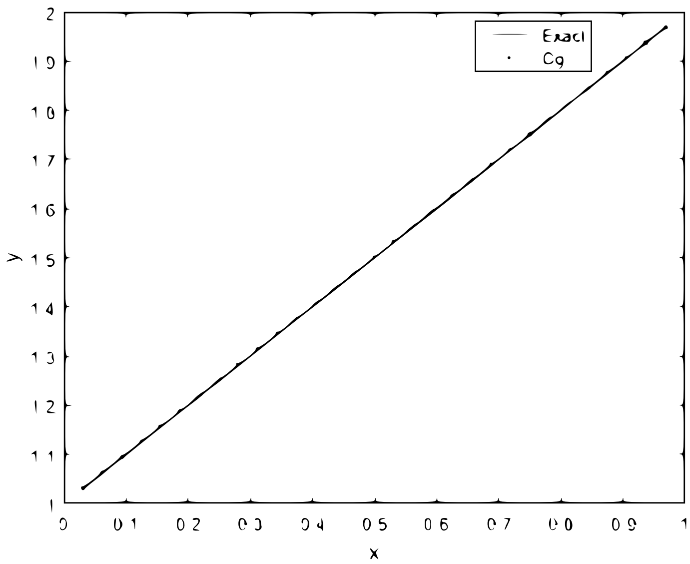

Fig. 1: Exact and approximate solution of example 1 with $h = \frac{1}{32}$

Example 2.*[25] Consider the fractional differential equation*

$$D^{(\frac{3}{2})}y(x) = cos(x+\frac{\pi}{4}), x \in [0,1]. \tag{15}$$

with the boundary conditions $y(0) = 1, y(1) = 1.84147$
,The exact solution of (15) is given by $y(x) = sin(x) + 1$.

The numerical results using conjugate gradient method with, $h = 0.01$, and 99 iterations tabulated in Table 2 with comparison to reference [25].

x	Exact solution	proposed method	Absolute error	Absolute error [25]
0.1	1.09983	1.09051	9.32×10^{-3}	2.29×10^{-3}
0.2	1.19866	1.17982	1.884×10^{-2}	9.97×10^{-2}
0.3	1.29552	1.26783	2.768×10^{-2}	1.03×10^{-1}
0.4	1.38941	1.35445	3.496×10^{-2}	8.901×10^{-2}
0.5	1.47942	1.43959	3.982×10^{-2}	1.995×10^{-2}
0.6	1.56464	1.52320	4.144×10^{-2}	9.144×10^{-2}
0.7	1.64421	1.60521	3.900×10^{-2}	8.577×10^{-2}
0.8	1.71735	1.68560	3.174×10^{-2}	9.177×10^{-2}
0.9	1.7833	1.76435	1.896×10^{-2}	7.467×10^{-2}

Table 2: Exact, approximation solution, absolute error of example 2

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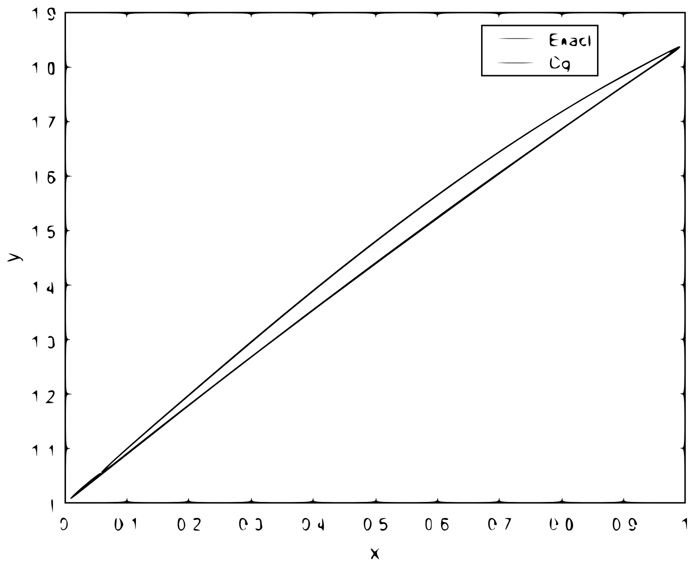

Fig. 2: Exact and approximate solution of example 2 with $h = 0.01$

Example 3.*[26] Consider the fractional differential equation*

$$D^2 y(x) + \sqrt{\pi} D^{(\frac{3}{2})} y(x) + y(x) = 0, x \in [0,1]. \quad (16)$$

with the boundary conditions $y(0) = 1, y(1) = 0.775989$.

The numerical results using conjugate gradient method with, $h = 0.125$, and 7 iterations tabulated in Table 3, with comparison to reference [26]

x	Exact solution	proposed method	Absolute error	Absolute error[26]
0.125	0.99437	0.98819	6.17×10^{-3}	1.24×10^{-3}
0.25	0.979919	0.971592	8.32×10^{-3}	5.11×10^{-3}
0.375	0.958424	0.95024	8.17×10^{-3}	1.387×10^{-2}
0.5	0.930957	0.92424	6.71×10^{-3}	2.614×10^{-2}
0.625	0.898335	0.89367	4.65×10^{-3}	4.039×10^{-2}
0.75	0.861241	0.85868	2.56×10^{-3}	5.579×10^{-2}
0.875	0.820277	0.81939	8.8×10^{-4}	7.148×10^{-2}

Table 3: Exact, approximation solution, absolute error of example 3

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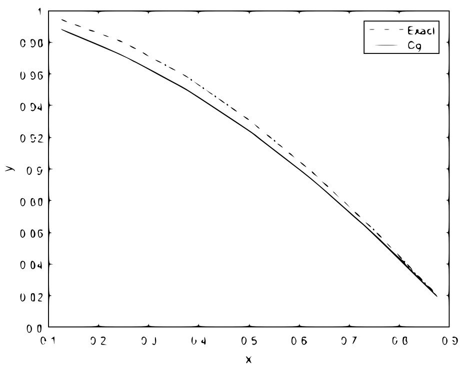

Fig. 3: Exact and approximate solution of example 3 with $h = 0.125$

5 Conclusion

This study constructs a non-polynomial spline function to approach the Bagley-Torvik Fractional Differential Problems with the conjugate gradient method. The numerical examples demonstrate that the non-polynomial spline and conjugate gradient techniques are more adaptable for approximating functions. The graphs of exact and approximate solutions for numerical examples show the superiority of our approach.

Declarations

Competing interests
The author declare that he has neither financial nor conflict interest.
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Ethical approval The author state that this research paper complies with ethical standards. This research paper does not involve either human participants or animals.
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