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Automorphisms of semidirect products fixing the non-normal subgroup

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Abstract: In this paper, we describe the automorphism group of semidirect product of two groups that fixes the non-normal subgroup of it. We have computed these automorphisms for the non-abelian metacyclic p-group and non-abelian p-groups $(p \geq 5)$ of order p^4, where p is a prime.

Keywords: Automorphism Group; Semidirect product; p-group.

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1 Introduction

Let H and K be two groups and $\phi : K \longrightarrow Aut(H)$ be a group homomorphism, where $Aut(H)$ is the group of automorphisms of the group H. Then $G = H \rtimes_\phi K$ is called the external semidirect product of groups H and K. On the other hand, let $G = HK$ be a group, where H and K are subgroups of G and K acts on H by conjugation defined as $h^k = khk^{-1}$ for all $h \in H$ and $k \in K$. Then $G = HK$ is called the internal semidirect product of subgroups H and K, where H is a normal subgroup of G and K is a non-normal subgroup of G.

Bidwell et. al. [1] studied the structure of automorphism group of direct product of two groups as the matrices of maps satisfying some certain conditions. The next interesting question was to study the structure of the automorphism group of semidirect product of two groups. The automorphism group of semidirect product of two groups was studied by Bidwell and Curran [2]. Later, M . J. Curran [5] and D. Jill [6] studied the automorphism group of the semidirect product of two groups that fixes the normal subgroup. In this paper, we study the structure of the automorphism group of the semidirect product of two groups that fixes the non-normal subgroup. We apply our main result to compute such automorphisms of non-abelian metacyclic p-groups and non-abelian p-groups $(p \geq 5)$ of order p^4, where p is a prime.

Let K be a non-normal subgroup of a group G. Let S be a right transversal to K in G with $1 \in S$. Then the group operation on G induces a binary operation on S with respect to it S becomes a right loop, a right action θ of K on S and two map $f : S \times S \longrightarrow K$ and $\sigma : S \times K \longrightarrow K$ (see [8] for details) . Let $Aut_K(G) = \{\Theta \in Aut(G) \mid \Theta(K) = K\}$. In [8, Theorem 2.6, p. 73], R. Lal obtained that $\Theta \in Aut_K(G)$ can be identified with the triple (α, γ, δ), where $\alpha \in Map(S,S)$, $\gamma \in Map(S,K)$ and $\delta \in Aut(K)$ satisfying the conditions in [8, Definition 2.5, p. 73] given below,

$(i)\alpha(xy) = (\alpha(x)\theta\gamma(y))\alpha(y)$
$(ii)\delta(f(x,y))\gamma(xy) = \gamma(x)\sigma_{\alpha(x)}(\gamma(y))f(\alpha(x)\theta\gamma(y),\alpha(y))$
$(iii)\alpha(x\theta k) = \alpha(x)\theta\delta(k)$
$(iv)\delta(\sigma_x(k))\gamma(x\theta k) = \gamma(x)\sigma_{\alpha(x)}(\delta(k))$

for all $x, y \in S$ and $k \in K$. In the case when there is a right transversal H to K in G which is a normal subgroup of G, the group G is the semidirect product of K and H. In this case, the conditions on α, γ and δ agree with the conditions given in [7, Lemma 1.1, p. 1000]. These conditions are given as follows.

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$(C1)\alpha(hh') = \alpha(h)\alpha(h')^{\gamma(h)}$,
$(C2)\gamma(h^k) = \gamma(h)^{\delta(k)}$,
$(C3)\alpha(h^k) = \alpha(h)^{\delta(k)}$,
$(C4)$For any $h'k' \in G$, there exists a unique $hk \in G$ such that $\alpha(h) = h'$ and $\gamma(h)\delta(k) = k'$.

*Remark.*In [8], the author put the non-normal subgroup in the left in the factorization of G. To match the terminology with that in [5], we put the non-normal subgroup K in the right, that is $G = HK$. Through out the paper, we will use the terminology used in [5]. We will identify the internal semidirect product $G = HK$ with the external semidirect product $H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is the corresponding homomorphism.

2 Structure of the automorphism group $Aut_K(G)$

In this section, we will give the structure of the group $Aut_K(G)$.Consider a set

$$\hat{\mathscr{M}}_K = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \mid \begin{matrix} \alpha \in Map(H,H), \ \gamma \in Hom(H,K), \\ \text{and} \qquad \delta \in Aut(K) \end{matrix} \right\},$$

where the maps α, γ and δ satisfy the conditions $(C1)-(C4)$. Let us define a binary operation on the set $\hat{\mathscr{M}}_K$ as,

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix}, \tag{1}$$

where $\alpha\alpha', \delta\delta'$ are the usual composition of maps and $(\gamma\alpha' + \delta\gamma')(h) = \gamma(\alpha'(h))\delta(\gamma'(h))$, for all $h \in H$. Then using $(C1)-(C4)$, for all $h,\ h' \in H$, we have $(\gamma\alpha' + \delta\gamma')(hh') = \gamma(\alpha'(hh'))\delta(\gamma'(hh')) = \gamma(\alpha'(h)\alpha'(h')^{\gamma'(h)})\delta(\gamma'(h)\gamma'(h')) = \gamma(\alpha'(h))\gamma(\alpha'(h')^{\gamma'(h)})\delta(\gamma'(h))\ \delta(\gamma'(h')) = \gamma\alpha'(h)\gamma(\alpha'(h'))^{\delta(\gamma'(h))}\delta(\gamma'(h))\delta(\gamma'(h')) = \gamma\alpha'(h)\delta(\gamma'(h))\gamma\alpha'(h')\delta(\gamma'(h')) = (\gamma\alpha' + \delta\gamma')(h)(\gamma\alpha' + \delta\gamma')(h')$. Thus the map $\gamma\alpha' + \delta\gamma' \in Hom(H,K)$. Since $\alpha,\ \alpha' \in Map(H,H)$ and $\delta,\ \delta' \in Aut(K)$, $\alpha\alpha' \in Map(H,H)$ and $\delta\delta' \in Aut(K)$.

Now, $\alpha\alpha'(hh') = \alpha(\alpha'(hh')) = \alpha(\alpha'(h)\alpha'(h')^{\gamma'(h)}) = \alpha(\alpha'(h))\alpha(\alpha'(h')^{\gamma'(h)})^{\gamma(\alpha'(h))} = \alpha\alpha'(h)(\alpha(\alpha'(h'))^{\delta(\gamma'(h))})^{\gamma\alpha'(h)} = \alpha\alpha'(h)\alpha\alpha'(h')^{(\gamma\alpha'+\delta\gamma')(h)}$. Also, $(\gamma\alpha' + \delta\gamma')(h^k) = \gamma(\alpha'(h^k))\delta(\gamma'(h^k)) = \gamma(\alpha'(h)^{\delta'(k)})\delta(\gamma'(h)^{\delta'(k)}) = \gamma(\alpha'(h))^{\delta(\delta'(k))}\delta(\gamma'(h))^{\delta(\delta'(k))} = (\gamma(\alpha'(h))\delta(\gamma'(h)))^{\delta(\delta'(k))} = (\gamma\alpha' + \delta\gamma')(h)^{\delta\delta'(k)}$. Clearly, $\alpha\alpha'(h^k) = \alpha(\alpha'(h)^{\delta'(k)}) = \alpha(\alpha'(h))^{\delta(\delta'(k))}$. Hence, $\begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix}$ satisfies $(C1)-(C4)$. The inverse of an arbitrary element $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in \hat{\mathscr{M}}_K$ is given as

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} & 0 \\ -\delta^{-1}\gamma\alpha^{-1} & \delta^{-1} \end{pmatrix}$$

and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element, where 1 denotes the identity group homomorphism and 0 denotes the trivial group homomorphism. Hence $\hat{\mathscr{M}}_K$ is a group with the binary operation as given in the Equation (1).

Proposition 1.*$Aut_K(G)$ is a subgroup of $Aut(G)$.*

*Proof.*Let $\Theta_1, \Theta_2 \in Aut_K(G)$. Then $\Theta_1(K) = K$ and $\Theta_2(K) = K$. Then $\Theta_1\Theta_2(K) = \Theta_1(\Theta_2(K)) = \Theta_1(K) = K$. Also, since $\Theta_1, \Theta_2 \in Aut(G)$, $\Theta_1\Theta_2 \in Aut(G)$. Hence, $\Theta_1\Theta_2 \in Aut_K(G)$. Further, for all $\Theta \in Aut_K(G)$, $\Theta^{-1}(K) = K$. Thus, $\Theta^{-1} \in Aut_K(G)$. Hence, $Aut_K(G)$ is a subgroup of $Aut(G)$.

Proposition 2.*Let $G = H \rtimes K$ be the semidirect product of groups H and K. Let $\hat{\mathscr{M}}_K$ be defined as above. Then the group $Aut_K(G)$ is isomorphic to the group $\hat{\mathscr{M}}_K$.*

*Proof.*Let $\Theta \in Aut_K(G)$. Then define the maps α, γ and δ by means of $\Theta(h) = \alpha(h)\gamma(h)$ and $\Theta(k) = \delta(k)$ for all $h \in H$ and $k \in K$. Now, for all $h, h' \in H$, $\alpha(hh')\gamma(hh') = \Theta(hh') = \Theta(h)\Theta(h') = \alpha(h)\gamma(h)\alpha(h')\gamma(h') = \alpha(h)\alpha(h')^{\gamma(h)}\gamma(h)\gamma(h')$. Therefore, by the uniqueness of representation, $\gamma \in Hom(H,K)$ and $(C1)$ holds. Using a similar argument, we get $\delta \in Hom(K)$. Now, $\alpha(h^k)\gamma(h^k)\delta(k) = \Theta(h^k k) = \Theta(khk^{-1}k) = \Theta(kh) = \Theta(k)\Theta(h) = \delta(k)\alpha(h)\gamma(h) = \alpha(h)^{\delta(k)}\delta(k)\gamma(h)$. Then, by the

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uniqueness of representation, $(C2)$ and $(C3)$ hold. Since Θ is a bijection, $(C4)$ holds. As a result, we can assign to every $\Theta \in Aut_K(G)$ a unique element $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in \hat{\mathcal{M}}_K$. This defines a map $\psi : Aut_K(G) \longrightarrow \hat{\mathcal{M}}_K$ given by $\Theta \mapsto \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$.

On the other hand, let $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in \hat{\mathcal{M}}_K$. Then we define a map $\Theta : G \longrightarrow G$ by $\Theta(hk) = \alpha(h)\gamma(h)\delta(k)$. Now, for all $h, h' \in H$ and $k, k' \in K$, using $(C1) - (C4)$, we get

$$\begin{aligned}
\Theta(hkh'k') &= \Theta(h(h')^k kk') \\
&= \alpha(h(h')^k)\gamma(h(h')^k)\delta(kk') \\
&= \alpha(h)\alpha((h')^k)^{\gamma(h)}\gamma(h)\gamma((h')^k)\delta(k)\delta(k') \\
&= \alpha(h)\gamma(h)\alpha(h')^{\delta(k)}\gamma(h')^{\delta(k)}\delta(k)\delta(k') \\
&= \alpha(h)\gamma(h)\delta(k)\alpha(h')\gamma(h')\delta(k') \\
&= \Theta(hk)\Theta(h'k').
\end{aligned}$$

Thus Θ is a group homomorphism. Using $(C4)$, it is clear that Θ is a bijection. Thus $\Theta \in Aut(G)$. Since $\Theta(K) = \delta(K)$ and $\delta \in Aut(K)$, $\Theta(K) = K$. Hence, $\Theta \in Aut_K(G)$. This shows that the map ψ is a bijection. Now, let $\Theta'(hk) = \alpha'(h)\gamma'(h)\delta'(k)$. Then we have

$$\begin{aligned}
\Theta\Theta'(hk) &= \Theta(\Theta'(hk)) \\
&= \Theta(\alpha'(h)\gamma'(h)\delta'(k)) \\
&= \alpha(\alpha'(h))\gamma(\alpha'(h))\delta(\gamma'(h)\delta'(k)) \\
&= \alpha\alpha'(h)\gamma(\alpha'(h))\delta(\gamma'(h))\delta(\delta'(k)) \\
&= \alpha\alpha'(h)(\gamma\alpha' + \delta\gamma')(h)\delta\delta'(k).
\end{aligned}$$

Write $\begin{pmatrix} h \\ k \end{pmatrix}$ for hk, then

$$\begin{pmatrix} \alpha' & 0 \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \alpha(h) \\ \gamma(h)\delta(k) \end{pmatrix}$$

and $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha(h) \\ \gamma(h)\delta(k) \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$

for all $h \in H$ and $k \in K$. Therefore, $\psi(\Theta\Theta') = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix} = \psi(\Theta)\psi(\Theta')$. Hence, ψ is an isomorphism of groups.

From now on we will identify automorphisms in $Aut_K(G)$ with the matrices in $\hat{\mathcal{M}}_K$. Now, we have the following remarks.

Remark. $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in Aut_K(G)$ if and only if $\alpha \in Aut(H)$ and $\alpha(h^k) = \alpha(h)^k$ for all $h \in H$ and $k \in K$.

Remark. $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in Aut_K(G)$ if and only if $\gamma(H) \subseteq C_K(H)$ and $\gamma(h^k) = \gamma(h)^k$, for all $h \in H$ and $k \in K$, where $C_K(H) = \{k \in K \mid h^k = h, \forall\, h \in H\}$ is the centralizer of H in K.

Remark. $\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \in Aut_K(G)$ if and only if $k^{-1}\delta(k) \in C_K(H)$ for all $k \in K$.

Now, let us consider the following subsets of $Aut(H)$, $Aut(K)$ and $Aut(H) \times Aut(K)$,

$$\begin{aligned}
U &= \{\alpha \in Aut(H) \mid \alpha(h^k) = \alpha(h)^k, \forall h \in H, k \in K\}, \\
V &= \{\delta \in Aut(K) \mid k^{-1}\delta(k) \in C_K(H), \forall k \in K\}, \\
W &= \{(\alpha, \delta) \in Aut(H) \times Aut(K) \mid \alpha(h^k) = \alpha(h)^{\delta(k)}, \forall h \in H, k \in K\}.
\end{aligned}$$

Clearly, U, V and W are the subgroups of $Aut(H)$, $Aut(K)$, and $Aut(H) \times Aut(K)$, respectively. The corresponding subgroups of the group $Aut_K(G)$ are

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in U \right\}, D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \mid \delta \in V \right\} \text{ and } E = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid (\alpha, \delta) \in W \right\}.$$

Note that, if $\alpha \in U$ and $\delta \in V$, then $(\alpha, \delta) \in W$. Therefore, $U \times V \leq W$.

Clearly, E is a subgroup of $Aut_K(G)$. However, one can check that E need not be a normal subgroup of $Aut_K(G)$. Let

$$C = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in Aut_K(G) \mid \gamma(H) \subseteq C_K(H) \text{ and } \gamma(h^k) = \gamma(h)^k, \forall h \in H \text{ and } k \in K \right\}.$$

Then, for all $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in C$ and $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in E$, we have

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \delta\gamma\alpha^{-1} & 1 \end{pmatrix} \tag{2}$$

Now, for all $h, h' \in H$ and $k \in K$, we have $h^{\delta\gamma\alpha^{-1}(h')} = h^{\delta(\gamma\alpha^{-1}(h'))} = h^{\gamma(\alpha^{-1}(h'))} = h$. This implies that $\delta\gamma\alpha^{-1}(h') \in C_K(H)$. Also, $\delta\gamma\alpha^{-1}(h^k) = \delta\gamma(\alpha^{-1}(h^k)) = \delta(\gamma(\alpha^{-1}(h)^{\delta^{-1}(k)})) = \delta(\gamma(\alpha^{-1}(h))^{\delta^{-1}(k)}) = \delta(\delta^{-1}(k)\gamma\alpha^{-1}(h)\delta^{-1}(k)^{-1}) = k\delta\gamma\alpha^{-1}(h)k^{-1} = \delta\gamma\alpha^{-1}(h)^k$. Thus $\begin{pmatrix} 1 & 0 \\ \delta\gamma\alpha^{-1} & 1 \end{pmatrix} \in C$ and so, C is a normal subgroup of the group $Aut_K(G)$. Clearly, $E \cap C = \{1\}$. Now, let $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in Aut_K(G)$. Then,

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in CE.$$

Hence, $Aut_K(G) = CE$. Thus, we have proved the following theorem,

Theorem 1. *Let $G = H \rtimes K$ be the semidirect product. Then $Aut_K(G) \simeq C \rtimes E$.*

3 Computation of $Aut_K(G)$ for some groups

In this section, we will compute the automorphism group $Aut_K(G)$ for non-abelian metacyclic p-groups and non-abelian p-groups $(p \geq 5)$ of order p^4, where p is a prime. The notation $\mathbb{Z}_m$ will denote the cyclic group of order m.

Metacyclic p-groups

First, assume that p is odd. A non-abelian split metacyclic p-group G is of the form $G = \langle a, b \mid a^{p^m} = 1 = b^{p^n}, a^b = a^{1+p^{m-r}} \rangle$, where $m \geq 2, n \geq 1$, and $1 \leq r \leq \min\{m-1, n\}$. Let $H = \langle a \rangle$, $K = \langle b \rangle$ and $\phi : K \longrightarrow Aut(H)$ be defined by $\phi(b)(a) = a^{1+p^{m-r}}$. Then $G = H \rtimes_\phi K$.

Note that $[H, K] = \langle a^{p^{m-r}} \rangle \simeq \mathbb{Z}_{p^r}$. Since K is abelian, by [2, Corollary 2.2, p. 490], $\gamma(h^k) = \gamma(h)$ is equivalent to $\gamma \in Hom(H/[H,K], K)$. Define $\gamma_i : H \to K$ by $\gamma_i(a) = b^i, 1 \leq i \leq p^n$ when $m - r \geq n$ and by $\gamma_i(a) = b^{ip^{n-m+r}}, 1 \leq i \leq p^{m-r}$ when $m - r < n$. Since $[H,K] \subseteq Ker\gamma_i$, it will induce a homomorphism from $H/[H,K]$ to K. Let $\hat{\gamma}_1 = \begin{pmatrix} 1 & 0 \\ \gamma_1 & 1 \end{pmatrix}$. Then, one can easily observe that $\gamma_1(H) \subseteq C_K(H)$. Therefore, $Hom(H/[H,K], K) \simeq C = \langle \hat{\gamma}_1 \rangle \simeq \mathbb{Z}_{p^{\min\{m-r,n\}}}$. Also, $C_K(H) = \langle b^{p^r} \rangle \simeq \mathbb{Z}_{p^{n-r}}$ and for $b \in K$, $b^{-1}\delta(b) \in C_K(H)$. Therefore, there are p^{n-r} choices for $\delta(b)$. If $\delta_1(b) = b^{1+p^r}$, then $V = \langle \delta_1 \rangle \simeq \mathbb{Z}_{p^{n-r}}$ and so, $D \simeq \mathbb{Z}_{p^{n-r}}$. Now, for all $\alpha \in Aut(H)$, $\alpha(a^b) = \alpha(a^{1+p^{m-r}}) = \alpha(a)^{1+p^{m-r}} = \alpha(a)^b$. Therefore, $U = Aut(H) \simeq \mathbb{Z}_{p^{m-1}(p-1)}$ and so, $A \simeq \mathbb{Z}_{p^{m-1}(p-1)}$. Then, by Theorem [5, Theorem 2, p. 207], $E = A \times D$. Now, by Theorem 1, $Aut_K(G) \simeq \mathbb{Z}_{p^{\min\{m-r,n\}}} \rtimes (\mathbb{Z}_{p^{m-1}(p-1)} \times \mathbb{Z}_{p^{n-r}})$. Hence, $Aut_K(G)$ is a subgroup of index $p^{\min\{m,n\}}$ in the group $Aut(G)$.

Now, assume $p = 2$. Then, as given in [4], the non-abelian split metacyclic 2-group is one of the following three forms,

$(i) G = \langle a, b \mid a^{2^m} = 1 = b^{2^n}, a^b = a^{1+2^{m-r}} \rangle, 1 \leq r \leq \min\{m-2, n\}, m \geq 3, n \geq 1.$

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

$(ii) G = \langle a, b \mid a^{2^m} = 1 = b^{2^n}, a^b = a^{-1+2^{m-r}} \rangle, 1 \le r \le \min\{m-2, n\}, m \ge 3, n \ge 1.$
$(iii) G = \langle a, b \mid a^{2^m} = 1 = b^{2^n}, a^b = a^{-1} \rangle, m \ge 2, n \ge 1.$

Let $H = \langle a \rangle \simeq \mathbb{Z}_{2^m}$ and $K = \langle b \rangle \simeq \mathbb{Z}_{2^n}$. We will compute the automorphism group, $Aut_K(G)$ in the above three cases $(i)-(iii)$. Using the similar argument as for odd prime p above, in the $case(i)$, $[H,K] = \langle a^{2^{m-r}} \rangle \simeq \mathbb{Z}_{2^r}$ and $C_K(H) = \langle b^{2^r} \rangle \simeq \mathbb{Z}_{2^{n-r}}$. Then $Hom(H/[H,K],K) \simeq \mathbb{Z}_{2^{\min\{m-r,n\}}}$. Thus $C \simeq \mathbb{Z}_{2^{\min\{m-r,n\}}}$, $A \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}}$. Hence, $Aut_K(G) \simeq \mathbb{Z}_{2^{\min\{m-r,n\}}} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}})$.

In the case (ii), $[H,K] = \langle a^2 \rangle \simeq \mathbb{Z}_{2^{m-1}}$ and $C_K(H) = \langle b^{2^r} \rangle \simeq \mathbb{Z}_{2^{n-r}}$. Thus, $C \simeq \mathbb{Z}_2, A \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}}$. Hence, $Aut_K(G) \simeq \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}})$. Similarly, in the $case(iii)$, $[H,K] = \langle a^2 \rangle \simeq \mathbb{Z}_{2^{m-r}}$, and $C_K(H) = \langle b^2 \rangle \simeq \mathbb{Z}_{2^{n-1}}$. Thus, $C \simeq \mathbb{Z}_2$, $A \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}}$. Hence, $Aut_K(G) \simeq \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})$.

Non-abelian p-groups of order p^4 ($p \ge 5$)

Burnside in [3] classified p-groups of order p^4, where p is a prime. Below, we list 10 non-abelian p-groups ($p \ge 5$) of order p^4 up to isomorphism.

(i)$G_1 = \langle a, b \mid a^{p^3} = 1 = b^p, a^b = a^{1+p^2} \rangle$,
(ii)$G_2 = \langle a, b \mid a^{p^2} = 1 = b^{p^2}, a^b = a^{1+p} \rangle$,
(iii)$G_3 = \langle a, b, c \mid a^{p^2} = 1 = b^p = c^p, cb = a^p bc, ab = ba, ac = ca \rangle$,
(iv)$G_4 = \langle a, b, c \mid a^{p^2} = 1 = b^p = c^p, ca = a^{1+p} c, ab = ba, cb = bc \rangle$,
(v)$G_5 = \langle a, b, c \mid a^{p^2} = 1 = b^p = c^p, ca = abc, ab = ba, bc = cb \rangle$,
(vi)$G_6 = \langle a, b, c \mid a^{p^2} = 1 = b^p = c^p, ba = a^{1+p} b, ca = abc, bc = cb \rangle$,
(vii)$G_7 = \langle a, b, c \mid a^{p^2} = 1 = b^p = c^p, ba = a^{1+p} b, ca = a^{1+p} bc, cb = a^p bc \rangle$,
(viii)$G_8 = \langle a, b, c \mid a^{p^2} = 1 = b^p = c^p, ba = a^{1+p} b, ca = a^{1+dp} bc, cb = a^{dp} bc, d \not\equiv 0, 1 \pmod p \rangle$,
(ix)$G_9 = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca, ab = ba \rangle$,
(x)$G_{10} = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1, dc = bcd, db = abd, ad = da, bc = cb, ac = ca, ab = ba \rangle$.

Observe that G_1 and G_2 are metacyclic p-groups. $Aut_K(G_1)$ and $Aut_K(G_2)$ (for the corresponding K) can be calculated as in the previous case.

The group G_3. Let $H = \langle a, b \mid a^{p^2} = b^p = 1, ab = ba \rangle$ and $K = \langle c \mid c^p = 1 \rangle$. Then $G_3 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = a$ and $\phi(c)(b) = a^p b$. Note that $[a^u b^v, c] = (a^u b^v) c (a^u b^v)^{-1} c^{-1} = a^u b^v (a^{u+pv} b^v)^{-1} = a^{-pv}$. Therefore, $[H,K] = \langle a^p \rangle \simeq \mathbb{Z}_p$. Also, if $c^s \in C_K(H)$, then $a^i b^j = c^s a^i b^j c^{-s} = a^{i+pjs} b^j$. Therefore, $js \equiv 0 \pmod p$ for all j and hence, $C_K(H) = \{1\}$. This implies that $Hom(H/[H,K],K)$ is the trivial group. Since K is abelian, by [2, Corollary 2.2, p. 490] C is the trivial group. Note that, each $\alpha \in Aut(H)$ defined by $\alpha(a) = a^i b^j$ and $\alpha(b) = a^{pm} b^l$ can be expressed as a matrix $\begin{pmatrix} i & j \\ m & l \end{pmatrix}$, where $0 \le i \le p^2-1, \gcd(p,i) = 1$, $0 \le m, j \le p-1$ and $1 \le l \le p-1$. Also, let $\delta \in Aut(K) \simeq \mathbb{Z}_{p-1}$ be defined by $\delta(c) = c^r$, where $1 \le r \le p-1$. Now, if $(\alpha, \delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and (ii) $\alpha(b^c) = \alpha(b)^{\delta(c)}$. By (i), $a^i b^j = \alpha(a) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^i b^j)^{c^r} = a^i a^{prj} b^j = a^{i+prj} b^j$. Thus, $prj \equiv 0 \pmod{p^2}$ which implies that $j = 0$. Now, by (ii), $a^{pi+pm} b^l = \alpha(a^p b) = \alpha(b^c) = \alpha(b)^{\delta(c)} = (a^{pm} b^l)^{c^r} = a^{pm} (b^l)^{c^r} = a^{pm} a^{prl} b^l = a^{pm+prl} b^l$. Thus, $i \equiv rl \pmod p$. Let t be a primitive root of 1 $\pmod p$ and $x = \left(\begin{pmatrix} t+p & 0 \\ 0 & t \end{pmatrix}, \delta_1 \right)$, $y = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \delta_1 \right)$ and $z = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t \right)$, where $\delta_\rho(c) = c^\rho$. Then $W \simeq \langle x, y, z \mid x^{p(p-1)} = 1 = y^p = z^{p(p-1)}, xz = zx, xy = yx, zyz^{-1} = y^{t^{-1}} \rangle$. Therefore, $W \simeq (\mathbb{Z}_p \times \mathbb{Z}_{p(p-1)}) \rtimes \mathbb{Z}_{p(p-1)}$ and so, $E \simeq (\mathbb{Z}_p \times \mathbb{Z}_{p(p-1)}) \rtimes \mathbb{Z}_{p(p-1)}$. Hence, by Theorem 1, $Aut_K(G_3) \simeq (\mathbb{Z}_p \times \mathbb{Z}_{p(p-1)}) \rtimes \mathbb{Z}_{p(p-1)}$.

The group G_4. Let $H = \langle a, b \mid a^{p^2} = b^p = 1, ab = ba \rangle$ and $K = \langle c \mid c^p = 1 \rangle$. Then $G_4 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = a^{1+p}$ and $\phi(c)(b) = b$. Note that $[H,K] = \langle a^p \rangle \simeq \mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H) = \{1\}$. Since K is abelian, by [2, Corollary 2.2, p. 490] C is the trivial group. Note that, any $\alpha \in Aut(H)$ defined by, $\alpha(a) = a^i b^j$ and $\alpha(b) = a^{pm} b^l$ can be expressed as a matrix $\begin{pmatrix} i & j \\ m & l \end{pmatrix}$, where $0 \le i \le p^2-1, \gcd(p,i) = 1, 0 \le m, j \le p-1$ and $1 \le l \le p-1$. Also, let $\delta \in Aut(K) \simeq \mathbb{Z}_{p-1}$ be defined by $\delta(c) = c^r$, where $1 \le r \le p-1$. Now, if $(\alpha, \delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and (ii) $\alpha(b^c) = \alpha(b)^{\delta(c)}$. Note that $\alpha(b^c) = \alpha(b) = a^{pm} b^l$ and $\alpha(b)^{\delta(c)} = (a^{pm} b^l)^{c^r} = (a^{pm})^{c^r} b^l = a^{pm(1+p)^r} b^l = a^{pm} b^l$. Therefore, each $\alpha \in Aut(H)$ satisfies (ii). Now, by (i), $(a^i b^j)^{1+p} = \alpha(a^{1+p}) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^i b^j)^{c^r} = (a^i)^{c^r} b^j = a^{i(1+p)^r} b^j$. Thus, $i(p+1) \equiv i(1+p)^r \pmod{p^2}$ which implies that $r = 1$. Therefore, $W \simeq Aut(H) \simeq$

$\mathbb{Z}_{p-1} \times (((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$. Hence, $E \simeq \mathbb{Z}_{p-1} \times (((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$. Thus, $Aut_K(G_4) \simeq \mathbb{Z}_{p-1} \times (((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$.

The group G_5. Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb\rangle$ and $K = \langle a \mid a^{p^2} = 1\rangle$. Then $G_5 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(a)(b) = b$ and $\phi(a)(c) = b^{-1}c$. Note that $[H,K] = \langle b\rangle \simeq \mathbb{Z}_p$. Also, if $a^s \in C_K(H)$, then $b^ic^j = a^sb^ic^ja^{-s} = b^{i-js}c^j$. Therefore, $s \equiv 0 \pmod p$ and $C_K(H) = \langle a^p\rangle$. Since K is abelian, by [2, Corollary 2.2, p. 490], $\gamma(h^k) = \gamma(h)$ is equivalent to $\gamma \in Hom(H/[H,K],K)$. Define $\gamma_k \in Hom(H/[H,K],K)$ by $\gamma_k(b) = 1$ and $\gamma_k(c) = a^{pk}$ for all $0 \le k \le p-1$. Since $[H,K] \subseteq Ker\gamma_k$, it will induce a homomorphism from $H/[H,K]$ to K. Let $\hat{\gamma}_1 = \begin{pmatrix} 1 & 0 \\ \gamma_1 & 1 \end{pmatrix}$. Then, one can easily observe that $\gamma_1(H) \subseteq C_K(H)$. Therefore, $Hom(H/[H,K],K) \simeq C = \langle \hat{\gamma}_1\rangle \simeq \mathbb{Z}_p$. Note that, any $\alpha \in Aut(H) \simeq GL(2,p)$ defined as, $\alpha(b) = b^ic^j$ and $\alpha(c) = b^lc^m$ can be represented as a matrix, $\begin{pmatrix} i & j \\ l & m \end{pmatrix}$, where $0 \le l, j \le p-1$ and $1 \le i, m \le p-1$. Also, let $\delta \in Aut(K) \simeq \mathbb{Z}_{p(p-1)}$ be defined by $\delta(a) = a^r$, where $r \in \mathbb{Z}_{p^2}, \gcd(p,r) = 1$. Now, if $(\alpha,\delta) \in W$, then (i) $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and (ii) $\alpha(c^a) = \alpha(c)^{\delta(a)}$. By (i), $b^ic^j = \alpha(b) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^ic^j)^{a^r} = b^ib^{-rj}c^j = b^{i-rj}c^j$. Thus, $rj \equiv 0 \pmod p$ which implies that $j = 0$. Now, by (ii), $b^{-i+l}c^m = \alpha(b^{-1}c) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^lc^m)^{a^r} = b^l(c^m)^{a^r} = b^lb^{-rm}c^m = b^{l-rm}c^m$. Thus, $i \equiv rm \pmod p$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} t & 0 \\ 1 & t \end{pmatrix}, \delta_1\right)$, and $y = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t\right)$, where $\delta_\rho(a) = a^\rho$. Then $W \simeq \langle x, y \mid x^{p(p-1)} = 1, y^{p(p-1)} = 1, yxy^{-1} = x^\lambda\rangle$, where $x^\lambda = \begin{pmatrix} t & 0 \\ (t+p)^{-1} & t \end{pmatrix}$. Then $W \simeq \mathbb{Z}_{p(p-1)} \rtimes \mathbb{Z}_{p(p-1)}$ and so, $E \simeq \mathbb{Z}_{p(p-1)} \rtimes \mathbb{Z}_{p(p-1)}$. Hence, $Aut_K(G_5) \simeq \mathbb{Z}_p \rtimes (\mathbb{Z}_{p(p-1)} \rtimes \mathbb{Z}_{p(p-1)})$.

The group G_6. Let $H = \langle a, b \mid a^{p^2} = b^p = 1, ba = a^{1+p}b\rangle$ and $K = \langle c \mid c^p = 1\rangle$. Then $G_6 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = ab$ and $\phi(c)(b) = b$. Note that $[H,K] = \langle b^{-1}, a^p\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H) = \{1\}$ and hence C is the trivial group. Now, $\alpha \in Aut(H)$ as given in [2] can be expressed as a matrix $\begin{pmatrix} \eta & \beta \\ \xi & 1 \end{pmatrix}$, where $\eta(a) = a^i, 0 \le i \le p^2-1, \gcd(p,i) = 1$, $\beta(b) = a^{pj}, 0 \le j \le p-1$, $\xi(a) = b^k$, $0 \le k \le p-1$, and $1(b) = b$. Also, $\delta \in Aut(K)$ is given by $\delta(c) = c^r, 1 \le r \le p-1$. Now, if $(\alpha,\delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and (ii) $\alpha(b^c) = \alpha(b)^{\delta(c)}$. Note that $\alpha(b^c) = \alpha(b) = a^{pj}b$ and $\alpha(b)^{\delta(c)} = (a^{pj}b)^{c^r} = (c^rac^{-r})^{pj}b = (ab^r)^{pj}b = a^{pj+rp\frac{pj(pj-1)}{2}}b^{pjr+1} = a^{pj}b$. Therefore, each $\alpha \in Aut(H)$ satisfies (ii). Now, by (i), $a^{i+pj}b^{k+1} = \alpha(ab) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^ib^k)^{c^r} = (c^rac^{-r})^ib^k = (ab^r)^ib^k = a^{i+rp\frac{i(i-1)}{2}}b^{ri+k}$. Thus, $ri \equiv 1 \pmod p$ which gives that $i \equiv 2j+1 \pmod p$. Therefore, $i \in \{(2j+1)+\lambda p \mid \lambda \in \mathbb{Z}_p\}$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t\right)$, $y = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \delta_1\right)$, and $z = \left(\begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_1\right)$, where $\delta_\rho(c) = c^\rho$. Then $W \simeq \langle x, y, z \mid x^{p(p-1)} = 1 = y^p = z^p, xyx^{-1} = y^e, xz = zx, yz = zy\rangle$, where $y^e = \begin{pmatrix} 1 & 0 \\ (t+p)^{-1} & 1 \end{pmatrix}$. Hence, $W \simeq \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$ and so, $E \simeq \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$. Thus, $Aut_K(G_6) \simeq \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$.

The group G_7. Let $H = \langle a, b \mid a^{p^2} = b^p = 1, ba = a^{1+p}b\rangle$ and $K = \langle c \mid c^p = 1\rangle$. Then, $G_7 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = a^{1+p}b$ and $\phi(c)(b) = a^pb$. Note that $[H,K] = \langle b, a^p\rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H) = \{1\}$ and hence C is the trivial group. Each $\alpha \in Aut(H)$ can be expressed as a matrix $\begin{pmatrix} \eta & \beta \\ \xi & 1 \end{pmatrix}$, where $\eta(a) = a^i, 0 \le i \le p^2-1, \gcd(i,p) = 1$, $\beta(b) = a^{pj}, \xi(a) = b^k, 0 \le j,k \le p-1$, and $1(b) = b$. Also, $\delta \in Aut(K)$ is given by $\delta(c) = c^r, 1 \le r \le p-1$. Now, if $(\alpha,\delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and $(ii)\alpha(b^c) = \alpha(b)^{\delta(c)}$.

By (ii), $a^{pi+pj}b = \alpha(a^pb) = \alpha(b^c) = \alpha(b)^{\delta(c)} = (a^{pj}b)^{c^r} = (c^rac^{-r})^{pj}(c^rbc^{-r}) = (a^{1+p\frac{r(r+1)}{2}}b^r)^{pj}(a^{rp}b) = a^{p^2j\frac{r(r+1)}{2}}(ab^r)^{pj}a^{rp}b = a^{pj+pr}b$. Thus $i \equiv r \pmod p$. Now, by (i), $a^{i(1+p)+pj}b^{k+1} = \alpha(a^{1+p}b) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^ib^k)^{c^r} = (c^rac^{-r})^i(c^rbc^{-r})^k = a^{i+pri\frac{r+i}{2}}b^{ri}(a^{rp}b)^k = a^{i+rpi(\frac{r+i}{2})+rpk}b^{ri+k}$. Thus, $ri \equiv 1 \pmod p$ and $ip+pj \equiv pri(\frac{r+i}{2})+rpk \pmod{p^2}$ implies that $i+j \equiv r+rk \pmod p$. So, $j \equiv rk \pmod p$. Using $r \equiv i \pmod p$ and $ri \equiv 1 \pmod p$, we get $i^2 \equiv 1 \pmod{p^2}$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \delta_1\right)$, $y = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \delta_1\right)$ and $z = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t\right)$, where $\delta_\rho(c) = c^\rho$. Then $W \simeq \langle x, y, z \mid x^p = 1 = y^2 = z^{p(p-1)}, xy = yx^{-1}, zxz^{-1} = x^t, yz = zy\rangle \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_p \rtimes \mathbb{Z}_2)$ and so, $E \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_p \rtimes \mathbb{Z}_2)$. Hence, $Aut_K(G_7) \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_p \rtimes \mathbb{Z}_2) \simeq D_{2p} \times \mathbb{Z}_{p(p-1)}$, where D_{2p} is the dihedral group of order $2p$.

The group G_8. Let $H=\langle a,b \mid a^{p^2}=b^p=1, ba=a^{1+p}b\rangle$ and $K=\langle c \mid c^p=1\rangle$. Then $G_8\simeq H\rtimes_\phi K$, where $\phi: K\longrightarrow Aut(H)$ is given by $\phi(c)(a)=a^{1+dp}b$ and $\phi(c)(b)=a^{dp}b$, $d\not\equiv 0,1 \pmod p$. By the similar argument as for the group G_7, we get, C is the trivial group and $E\simeq \mathbb{Z}_{p(p-1)}\times(\mathbb{Z}_p\rtimes\mathbb{Z}_2)$. Hence, $Aut_K(G_8)\simeq D_{2p}\times\mathbb{Z}_{p(p-1)}$.

The group G_9. Let $H=\langle a,b,c \mid a^p=b^p=c^p=1, ab=ba, bc=cb, ac=ca\rangle$, and $K=\langle d\mid d^p=1\rangle$. Then $G_9\simeq H\rtimes_\phi K$, where $\phi: K\longrightarrow Aut(H)$ is given by $\phi(d)(a)=a, \phi(d)(b)=b$, and $\phi(d)(c)=ac$.

Note that $[H,K]=\langle a\rangle\simeq\mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H)=\{1\}$ and hence C is the trivial group. Note that, $Aut(H)\simeq GL(3,p)$ and $Aut(K)\simeq\mathbb{Z}_{p-1}$. So, any automorphism $\alpha\in Aut(H)$ can be identified as an element $\begin{pmatrix} i & j & k\\ l & m & n\\ \lambda & \mu & \rho\end{pmatrix}$ in $GL(3,p)$. Let $\alpha\in Aut(H)$ and $\delta\in Aut(K)$ be defined as, $\alpha(a)=a^ib^jc^k, \alpha(b)=a^lb^mc^n, \alpha(c)=a^\lambda b^\mu c^\rho$, and $\delta(d)=d^r$, where $1\le i,m,\rho,r\le p-1$ and $0\le j,k,l,n,\lambda,\mu\le p-1$. Now, if $(\alpha,\delta)\in W$, then (i) $\alpha(a^d)=\alpha(a)^{\delta(d)}$, (ii) $\alpha(b^d)=\alpha(b)^{\delta(d)}$ and (iii) $\alpha(c^d)=\alpha(c)^{\delta(d)}$.

Note that, $d^rcd^{-r}=a^rc$. By (i), $a^ib^jc^k=\alpha(a)=\alpha(a^d)=\alpha(a)^{\delta(d)}=(a^ib^jc^k)^{d^r}=a^ib^j(d^rcd^{-r})^k=a^ib^j(a^rc)^k=a^{i+rk}b^jc^k$. Therefore, $rk\equiv 0(\mod p)$ which implies that $k=0$. Now, by (ii), $a^lb^mc^n=\alpha(b)=\alpha(b^d)=\alpha(b)^{\delta(d)}=(a^lb^mc^n)^{d^r}=a^{l+rn}b^mc^n$. Therefore, $rn\equiv 0(\mod p)$ which implies that $n=0$. By (iii), $a^{i+\lambda}b^{j+\mu}c^\rho=\alpha(ac)=\alpha(c^d)=\alpha(c)^{\delta(d)}=(a^\lambda b^\mu c^\rho)^{d^r}=a^{\lambda+r\rho}b^\mu c^\rho$. Thus, $i=r\rho$ and $j=0$. So, we have, $\alpha=\begin{pmatrix} r\rho & 0 & 0\\ l & m & 0\\ \lambda & \mu & \rho\end{pmatrix}$. Let t be a primitive root of $1 \pmod p$ and $u=\left(\begin{pmatrix} t&0&0\\0&1&0\\0&0&t\end{pmatrix},\delta_1\right)$, $v=\left(\begin{pmatrix} 1&0&0\\0&t&0\\0&0&1\end{pmatrix},\delta_1\right)$, $w=\left(\begin{pmatrix} t&0&0\\0&1&0\\0&0&1\end{pmatrix},\delta_t\right)$, $x=\left(\begin{pmatrix} 1&0&0\\1&1&0\\0&0&1\end{pmatrix},\delta_1\right)$, $y=\left(\begin{pmatrix} 1&0&0\\0&1&0\\1&0&1\end{pmatrix},\delta_1\right)$, $z=\left(\begin{pmatrix} 1&0&0\\0&1&0\\0&1&1\end{pmatrix},\delta_1\right)$, where $\delta_s(d)=d^s$. Then $W\simeq\langle u,v,w,x,y,z \mid u^{p-1}=1=v^{p-1}=w^{p-1}=x^p=y^p=z^p, uv=vu, uw=wu, uy=yu, vw=wv, vy=yv, wz=zw, xy=yx, yz=zy, uxu^{-1}=x^{t^{-1}}, uzu^{-1}=z^t, vxv^{-1}=x^t, vzv^{-1}=z^{t^{-1}}, wxw^{-1}=x^{t^{-1}}, wyw^{-1}=y^{t^{-1}}, zx=xyz\rangle\simeq(((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)\times\mathbb{Z}_{p-1})\rtimes(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})$ and so, $E\simeq(((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)\times\mathbb{Z}_{p-1})\rtimes(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})$. Hence, $Aut_K(G_9)\simeq(((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)\times\mathbb{Z}_{p-1})\rtimes(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})$.

The group G_{10}. Let $H=\langle a,b,c\mid a^p=b^p=c^p=1, ab=ba, bc=cb, ac=ca\rangle$ and $K=\langle d\mid d^p=1\rangle$. Then, $G_{10}\simeq H\rtimes_\phi K$, where $\phi: K\longrightarrow Aut(H)$ is given by $\phi(d)(a)=a, \phi(d)(b)=ab$, and $\phi(d)(c)=bc$.

Note that $[H,K]=\langle a,b\rangle\simeq\mathbb{Z}_p\times\mathbb{Z}_p$. By the similar argument as above, C is the trivial group. Note that, $Aut(H)\simeq GL(3,p)$ and $Aut(K)\simeq\mathbb{Z}_{p-1}$. So, any automorphism $\alpha\in Aut(H)$ can be identified as an element $\begin{pmatrix} i & j & k\\ l & m & n\\ \lambda & \mu & \rho\end{pmatrix}$ in $GL(3,p)$. Let $\alpha\in Aut(H)$ and $\delta\in Aut(K)$ be defined as, $\alpha(a)=a^ib^jc^k, \alpha(b)=a^lb^mc^n, \alpha(c)=a^\lambda b^\mu c^\rho$, and $\delta(d)=d^r$, where $1\le i,m,\rho,r\le p-1$ and $0\le j,k,l,n,\lambda,\mu\le p-1$. Now, if $(\alpha,\delta)\in W$, then (i) $\alpha(a^d)=\alpha(a)^{\delta(d)}$, (ii) $\alpha(b^d)=\alpha(b)^{\delta(d)}$ and (iii) $\alpha(c^d)=\alpha(c)^{\delta(d)}$.

Note that, $d^rbd^{-r}=a^rb$ and $d^rcd^{-r}=a^{\frac{r(r-1)}{2}}b^rc$. By (i), $a^ib^jc^k=\alpha(a)=\alpha(a^d)=\alpha(a)^{\delta(d)}=(a^ib^jc^k)^{d^r}=a^i(d^rbd^{-r})^j(d^rcd^{-r})^k=a^i(a^rb)^j(a^{\frac{r(r-1)}{2}}b^rc)^k=a^{i+rj+k\frac{r(r-1)}{2}}b^{j+rk}c^k$. Thus $k=0$ and $j=0$. Now, by (ii), $a^{i+l}b^mc^n=\alpha(ab)=\alpha(b^d)=\alpha(b)^{\delta(d)}=(a^lb^mc^n)^{d^r}=a^{l+rm+n\frac{r(r-1)}{2}}b^{m+rn}c^n$. Thus, $n=0$ and $i=rm$. By (iii), $a^{l+\lambda}b^{m+\mu}c^\rho=\alpha(bc)=\alpha(c^d)=\alpha(c)^{\delta(d)}=(a^\lambda b^\mu c^\rho)^{d^r}=a^{\lambda+r\mu+\rho\frac{r(r-1)}{2}}b^{\mu+r\rho}c^\rho$. Thus, $m=r\rho$ and $l=r\mu+\rho\frac{r(r-1)}{2} \pmod p$. So, we have, $\alpha=\begin{pmatrix} r^2\rho & 0 & 0\\ l & r\rho & 0\\ \lambda & \mu & \rho\end{pmatrix}$, where $l=r\mu+\rho\frac{r(r-1)}{2} \pmod p$. Let t be a primitive root of $1 \pmod p$ and $x=\left(\begin{pmatrix} t&0&0\\0&t&0\\0&0&t\end{pmatrix},\delta_1\right)$, $y=\left(\begin{pmatrix} t^2&0&0\\0&t&0\\0&0&1\end{pmatrix},\delta_t\right)$ and $z=\left(\begin{pmatrix} 1&0&0\\1&1&0\\1&1&1\end{pmatrix},\delta_1\right)$, where $\delta_s(d)=d^s$. Note that,

$\langle z \rangle = \langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rangle$ is an abelian group of order p^2. Therefore, $W \simeq \langle x,y,z \mid x^{p-1} = y^{p-1} = z^p, xy = yx, xz = zx, yzy^{-1} = z^u \rangle$, where $z^u = \begin{pmatrix} 1 & 0 & 0 \\ t^{-1} & 1 & 0 \\ t^{-2} & t^{-1} & 1 \end{pmatrix}$. Thus $W \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1})$ and so, $E \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1})$. Hence, $Aut_K(G_{10}) \simeq \mathbb{Z}_{p-1} \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$.

Declarations

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