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On the Incomplete $(p,q)-$Fibonacci and $(p,q)-$ Lucas Numbers

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Abstract: In this present work, the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers are defined. We examine their recurrence relations as well as some of their properties. We derive their generating functions.

Keywords: Fibonacci numbers; Lucas numbers; generating function.

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1 Introduction

The Fibonacci sequence and its generalizations offer a variety of intriguing features and uses in science and art (see, e.g., [9, 10]). The Fibonacci and Lucas numbers $\{f_h\}$ and $\{l_h\}$ are expressed as the recurrence relations, respectively, for $h \geq 0$

$$f_{h+2} = f_{h+1} + f_h \text{ with initial conditions } \quad f_0 = 0 \quad \text{and} \quad f_1 = 1,$$
$$l_{h+2} = l_{h+1} + l_h \text{ with initial conditions } \quad l_0 = 2 \quad \text{and} \quad l_1 = 1$$

Filipponi [5] introduced the incomplete Fibonacci and Lucas numbers. The incomplete Fibonacci numbers $F_h(u)$ and Lucas numbers $L_h(v)$ are expressed, respectively, by

$$F_h(u) = \sum_{i=0}^{u} \binom{h-1-i}{i}, \quad \left(\lfloor \frac{h-1}{2} \rfloor \leq u \leq h-1\right)$$

and

$$L_h(v) = \sum_{i=0}^{v} \frac{h}{h-i} \binom{h-i}{i}, \quad \left(\lfloor \frac{h}{2} \rfloor \leq v \leq h-1\right)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. It is obvious that

$$F_h(\lfloor \frac{h-1}{2} \rfloor) = f_h \quad \text{and} \quad L_h(\lfloor \frac{h}{2} \rfloor) = l_h$$

where the $h-$th Fibonacci and Lucas numbers are denoted by f_h and l_h, respectively.
The generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers were examined by Djordjevic [3]. The incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers were defined and studied by Djordjevic and Srivastava [4]. The generating functions of the incomplete Fibonacci and Lucas numbers were discovered by Pintr and Srivastava [18]. Ramrez [14] presented the bi-periodic incomplete Fibonacci sequences, the incomplete $k-$Fibonacci and $k-$Lucas numbers [15]. The incomplete Tribonacci numbers and polynomials were introduced by

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Ramirez and Sirvent [16]. The incomplete Fibonacci and Lucas $p-$numbers were defined by Tasci and Firengiz [26]. The incomplete bivariate Fibonacci and Lucas $p-$polynomials were defined by Tasci et al. [27]. We refer to other studies on incompletes of some impressive numbers and polynomials [2,11,12,13,17,20,25].
In [6,22,23,28], the $(p,q)-$Fibonacci and $(p,q)-$ Lucas sequences are defined, respectively, by

$$F_h^{(p,q)} = pF_{h-1}^{(p,q)} + qF_{h-2}^{(p,q)}, \quad F_0^{(p,q)} = 0, \quad F_1^{(p,q)} = 1, \tag{1}$$

and

$$L_h^{(p,q)} = pL_{h-1}^{(p,q)} + qL_{h-2}^{(p,q)}, \quad L_0^{(p,q)} = 2, \quad L_1^{(p,q)} = p.$$

where p and q are real coefficients.
In [8,19,24], the $(p,q)-$Fibonacci and $(p,q)-$ Lucas sequences are also given by the well-known formulas

$$F_h^{(p,q)} = \sum_{j=0}^{\lfloor\frac{h-1}{2}\rfloor} \binom{h-j-1}{j} p^{h-2j-1} q^j, \quad h \geq 1$$

and

$$L_h^{(p,q)} = \sum_{j=0}^{\lfloor\frac{h}{2}\rfloor} \frac{h}{h-j} \binom{h-j}{j} p^{h-2j} q^j, \quad h \geq 1.$$

Note that $F_h^{(p,q)}$ and $L_h^{(p,q)}$ reduce to the Fibonacci and Lucas sequences F_h and L_h, respectively, when $p = q = 1$; see, respectively, sequences A000045 and A000032 in [29].
According to Filipponi, the specific use of well-known combinatorial expressions for Fibonacci and Lucas numbers yields two interesting classes of integers (specifically, $F_n(k)$ and $L_n(k)$) ruled by the integral parameters n and k [5]. In this paper, we examine how the specific application of combinatorial phrases for $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers yields to two interesting classes of integers governed by the integral parameters n and k. Moreover, we derive some identities and the generating functions of the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers.

2 The Incomplete $(p,q)-$Fibonacci Numbers

Definition 1.*The incomplete $(p,q)-$Fibonacci numbers $F_{(h,k)}^{(p,q)}$ are defined as*

$$F_{(h,k)}^{(p,q)} = \sum_{j=0}^{k} \binom{h-j-1}{j} p^{h-2j-1} q^j, \quad \left(1 \leq h; 0 \leq k \leq \lfloor\frac{h-1}{2}\rfloor = \hat{h}\right). \tag{2}$$

The numbers $F_{(h,k)}^{(p,q)}$ are displayed in Table 1. It shows the first few h values and the corresponding permissible k values:

Table 1: The first few values of the incomplete $(p,q)-$Fibonacci Numbers

$h \backslash k$	0	1	2	3
1	1			
2	p			
3	p^2	p^2+q		
4	p^3	p^3+2pq		
5	p^4	p^4+3p^2q	$p^4+3p^2q+q^2$	
6	p^5	p^5+4p^3q	$p^5+4p^3q+3pq^2$	
7	p^6	p^6+5p^4q	$p^6+5p^4q+6p^2q^2$	$p^6+5p^4q+6p^2q^2+q^3$
8	p^7	p^7+6p^5q	$p^7+6p^5q+10p^3q^2$	$p^7+6p^5q+10p^3q^2+4pq^3$
9	p^8	p^8+7p^6q	$p^8+7p^6q+15p^4q^2$	$p^8+7p^6q+15p^4q^2+10p^2q^3$
10	p^9	p^9+8p^7q	$p^9+8p^7q+21p^5q^2$	$p^9+8p^7q+21p^5q^2+20p^3q^3$

The relation (2) has some special cases as follows:

1. $F_{(h,0)}^{(p,q)} = p^{h-1}, \quad (h \geq 1)$
2. $F_{(h,1)}^{(p,q)} = p^{h-1} + (h-2)p^{h-3}q, \quad (h \geq 3)$
3. $F_{(h,2)}^{(p,q)} = p^{h-1} + (h-2)p^{h-3}q + \dfrac{(h-4)(h-3)}{2}p^{h-5}q^2, \quad (h \geq 5)$
4. $F_{(h,\hat{h})}^{(p,q)} = F_h^{(p,q)}, \quad (h \geq 1)$
5. $F_{(h,\hat{h}-1)}^{(p,q)} = \begin{cases} F_h^{(p,q)} - \dfrac{h}{2}pq^{(\frac{h}{2}-1)} & (h \quad even) \\ F_h^{(p,q)} - q^{(\frac{h-1}{2})} & (h \quad odd) \end{cases}, \quad (h \geq 3)$

2.1 Some identities of the numbers $F_{(n,k)}^{(p,q)}$

Proposition 1. *The incomplete $(p,q)-$Fibonacci numbers $F_{(h,k)}^{(p,q)}$ can be given by the recurrence relation*

$$F_{(h+2,k+1)}^{(p,q)} = pF_{(h+1,k+1)}^{(p,q)} + qF_{(h,k)}^{(p,q)}, \quad 0 \leq k \leq \hat{h}. \tag{3}$$

Proof. Using Definition (3), we obtain the desired equality as follows:

$$\begin{aligned} pF_{(h+1,k+1)}^{(p,q)} + qF_{(h,k)}^{(p,q)} &= \sum_{j=0}^{k+1} \binom{h-j}{j} p^{h-2j+1}q^j + \sum_{j=0}^{k} \binom{h-j-1}{j} p^{h-2j-1}q^{j+1} \\ &= \sum_{j=0}^{k+1} \binom{h-j}{j} p^{h-2j+1}q^j + \sum_{j=1}^{k+1} \binom{h-j}{j-1} p^{h-2j+1}q^j \\ &= \sum_{j=0}^{k+1} \left[\binom{h-j}{j} + \binom{h-j}{j-1} \right] p^{h-2j+1}q^j - \binom{h}{-1} \\ &= \sum_{j=0}^{k+1} \binom{h-j+1}{j} p^{h-2j+1}q^j - 0 \\ &= F_{(h+2,k+1)}^{(p,q)} \end{aligned}$$

Proposition 2. *The following identity holds:*

$$F_{(h+2,k)}^{(p,q)} = pF_{(h+1,k)}^{(p,q)} + qF_{(h,k)}^{(p,q)} - \binom{h-k-1}{k} p^{h-2k-1}q^{k+1} \tag{4}$$

Proof. it is clear that

$$\begin{aligned} F_{(h+2,k)}^{(p,q)} &= \sum_{j=0}^{k} \binom{h-j+1}{j} p^{h-2j+1}q^j \\ &= \sum_{j=0}^{k} \left[\binom{h-j}{j} + \binom{h-j}{j-1} \right] p^{h-2j+1}q^j \\ &= \sum_{j=0}^{k} \binom{h-j}{j} p^{h-2j+1}q^j + \sum_{j=0}^{k} \binom{h-j}{j-1} p^{h-2j+1}q^j \\ &= pF_{(h+1,k)}^{(p,q)} + \sum_{j=-1}^{k-1} \binom{h-j-1}{j} p^{h-2j-1}q^{j+1} \\ &= pF_{(h+1,k)}^{(p,q)} + \binom{h}{-1} p^{h+1} + q\sum_{j=0}^{k} \binom{h-j-1}{j} p^{h-2j-1}q^j - \binom{h-k-1}{k} p^{h-2k-1}q^{k+1} \\ &= pF_{(h+1,k)}^{(p,q)} + qF_{(h,k)}^{(p,q)} - \binom{h-k-1}{k} p^{h-2k-1}q^{k+1} \end{aligned}$$

Proposition 3.*The following identity holds:*

$$\sum_{j=0}^{r}\binom{r}{j}p^{j}q^{r-j}F_{(h+j,k+j)}^{(p,q)}=F_{(h+2r,k+r)}^{(p,q)},\quad 0\leq k\leq\frac{h-r-1}{2} \tag{5}$$

*Proof.*We use induction on r. The sum (5) is plainly valid for $r=1$; Assume it is true for a specific case $r>1$. In order to perform the inductive step $r\rightarrow r+1$, we get

$$\begin{aligned}
F_{(h+2(r+1),k+(r+1))}^{(p,q)} &= \sum_{j=0}^{r+1}\left[\binom{r}{j}+\binom{r}{j-1}\right]p^{j}q^{r-j+1}F_{(h+j,k+j)}^{(p,q)} \\
&= q\sum_{j=0}^{r}\binom{r}{j}p^{j}q^{r-j}F_{(h+j,k+j)}^{(p,q)}+\binom{r}{r+1}p^{r+1}q^{2}F_{(h+r+1,k+r+1)}^{(p,q)} \\
&+\sum_{j=0}^{r+1}\binom{r}{j-1}p^{j}q^{r-j+1}F_{(h+j,k+j)}^{(p,q)} \\
&= qF_{(h+2r,k+r)}^{(p,q)}+0+\sum_{j=-1}^{r}\binom{r}{j}p^{j+1}q^{r-j}F_{(h+j+1,k+j+1)}^{(p,q)} \\
&= qF_{(h+2r,k+r)}^{(p,q)}+p\sum_{j=0}^{r}\binom{r}{j}p^{j}q^{r-j}F_{(h+j+1,k+j+1)}^{(p,q)}+\binom{r}{-1}p^{-1}q^{r+1}F_{(h,k)}^{(p,q)} \\
&= qF_{(h+2r,k+r)}^{(p,q)}+pF_{(h+2r+1,k+r+1)}^{(p,q)}+0
\end{aligned}$$

Proposition 4.*For $n\geq 2k+2$, we have*

$$\sum_{j=0}^{r-1}p^{r-j-1}q^{j+2}F_{(h+j,k)}^{(p,q)}=q^{r+1}F_{(h+r+1,k+1)}^{(p,q)}-p^{r}F_{(h+1,k+1)}^{(p,q)}. \tag{6}$$

*Proof.*We use induction on r. The sum (5) is plainly valid for $r=1$; Assume it is true for a specific case $r>1$. In order to perform the inductive step $r\rightarrow r+1$, we obtain

$$\begin{aligned}
\sum_{j=0}^{r}p^{r-j}q^{j+2}F_{(h+j,k)}^{(p,q)} &= p\sum_{j=0}^{r-1}p^{r-j-1}q^{j+2}F_{(h+j,k)}^{(p,q)}+q^{r+2}F_{(h+r,k)}^{(p,q)} \\
&= p\left(q^{r+1}F_{(h+r+1,k+1)}^{(p,q)}-p^{r}F_{(h+1,k+1)}^{(p,q)}\right)+q^{r+2}F_{(h+r,k)}^{(p,q)} \\
&= q^{r+1}\left(pF_{(h+r+1,k+1)}^{(p,q)}+qF_{(h+r,k)}^{(p,q)}\right)-p^{r+1}F_{(h+1,k+1)}^{(p,q)} \\
&= q^{r+1}F_{(h+r+1,k+1)}^{(p,q)}-p^{r+1}F_{(h+1,k+1)}^{(p,q)}
\end{aligned}$$

In [24,1], note that if p and q in (1) are real variables, then $F_{h}^{(p,q)}=F_{h}(x,y)$ and hence they correspond to the bivariate Fibonacci polynomials expressed as

$$F_{h}(x,y)=xF_{h-1}(x,y)+yF_{h-2}(x,y),\quad F_{0}(x,y)=0,\quad F_{1}(x,y)=1,\quad h\geq 2.$$

Lemma 1.*In [1], the following relation holds:*

$$\frac{\partial F_{h}^{(p,q)}}{\partial p}=\frac{hF_{h}^{(p,q)}+q(h-2)F_{h-2}^{(p,q)}-2pF_{h-1}^{(p,q)}}{p^{2}+4q}.$$

Lemma 2.*For $h\in\mathbb{Z}^{+}$, the following equality is true:*

$$\sum_{j=0}^{\hat{h}}j\binom{h-j-1}{j}p^{h-2j-1}q^{j}=\frac{((h-1)(p^{2}+4q)-hp)F_{h}^{(p,q)}-pq(h-2)F_{h-2}^{(p,q)}+2p^{2}F_{h-1}^{(p,q)}}{2(p^{2}+4q)}$$

*Proof.*We are aware that

$$pF_h^{(p,q)} = \sum_{j=0}^{\hat{h}} \binom{h-j-1}{j} p^{h-2j} q^j$$

By derivating into the previous equation with respect to p, we get

$$\begin{aligned} F_h^{(p,q)} + p\frac{\partial F_h^{(p,q)}}{\partial p} &= \sum_{j=0}^{\hat{h}} (h-2j)\binom{h-j-1}{j} p^{h-2j-1} q^j \\ &= hF_h^{(p,q)} - 2\sum_{j=0}^{\hat{h}} j\binom{h-j-1}{j} p^{h-2j-1} q^j \end{aligned}$$

From Lemma 1, the proof is completed.

Proposition 5.*For $h \in \mathbb{Z}^+$, the following equality is true:*

$$\sum_{k=0}^{\hat{h}} F_{(h,k)}^{(p,q)} = \frac{((2\hat{h}-h+3)(p^2+4q)+hp)F_h^{(p,q)} + pq(h-2)F_{h-2}^{(p,q)} - 2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)}$$

*Proof.*From Lemma 1, we obtain

$$\begin{aligned} \sum_{k=0}^{\hat{h}} F_{(h,k)}^{(p,q)} &= F_{(h,0)}^{(p,q)} + F_{(h,1)}^{(p,q)} + \cdots + F_{(h,\hat{h})}^{(p,q)} \\ &= \binom{h-1}{0} p^{h-1} + \left[\binom{h-1}{0} p^{h-1} + \binom{h-2}{1} p^{h-3} q\right] \\ &+ \cdots + \left[\binom{h-1}{0} p^{h-1} + \binom{h-2}{1} p^{h-3} q + \cdots + \binom{h-\hat{h}-1}{\hat{h}} p^{h-2\hat{h}-1} q^{\hat{h}}\right] \\ &= (\hat{h}+1)\binom{h-1}{0} p^{h-1} + \hat{h}\binom{h-3}{1} p^{h-3} q + \cdots + \binom{h-\hat{h}-1}{\hat{h}} p^{h-2\hat{h}-1} q^{\hat{h}} \\ &= \sum_{j=0}^{\hat{h}} (\hat{h}-j+1)\binom{h-j-1}{j} p^{h-2j-1} q^j \\ &= (\hat{h}+1)\sum_{j=0}^{\hat{h}} \binom{h-j-1}{j} p^{h-2j-1} q^j - \sum_{j=0}^{\hat{h}} j\binom{h-j-1}{j} p^{h-2j-1} q^j \\ &= (hath+1)F_h^{(p,q)} - \frac{((h-1)(p^2+4q)-hp)F_h^{(p,q)} - pq(h-2)F_{h-2}^{(p,q)} + 2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)} \\ &= \frac{((2\hat{h}-h+3)(p^2+4q)+hp)F_h^{(p,q)} + pq(h-2)F_{h-2}^{(p,q)} - 2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)} \end{aligned}$$

3 The Incomplete $(p,q)-$Lucas Numbers

Definition 2.*The incomplete $(p,q)-$Lucas numbers $L_{(h,k)}^{(p,q)}$ are defined by*

$$L_{(h,k)}^{(p,q)} = \sum_{j=0}^{k} \frac{h}{h-j}\binom{h-j}{j} p^{h-2j} q^j, \quad \left(1 \le h; 0 \le k \le \lfloor\frac{h}{2}\rfloor = \tilde{h}\right). \tag{7}$$

The numbers $L_{(h,k)}^{(p,q)}$ are displayed in Table 2. It shows the first few h values and the corresponding permissible k values:

Table 2: The first few values of the incomplete $(p,q)-$Lucas Numbers

$n\setminus k$	0	1	2	3
1	p			
2	p^2	p^2+2q		
3	p^3	p^3+3pq		
4	p^4	p^4+4p^2q	$p^4+4p^2q+2q^2$	
5	p^5	p^5+5p^3q	$p^5+5p^3q+5pq^2$	
6	p^6	p^6+6p^4q	$p^6+6p^4q+9p^2q^2$	$p^6+6p^4q+9p^2q^2+2q^3$
7	p^7	p^7+7p^5q	$p^7+7p^5q+14p^3q^2$	$p^7+7p^5q+14p^3q^2+7pq^3$
8	p^8	p^8+8p^6q	$p^8+8p^6q+20p^4q^2$	$p^8+8p^6q+20p^4q^2+16p^2q^3$
9	p^9	p^9+9p^7q	$p^9+9p^7q+27p^5q^2$	$p^9+9p^7q+27p^5q^2+30p^3q^3$
10	p^10	$p^{10}+10p^8q$	$p^{10}+10p^8q+35p^6q^2$	$p^{10}+10p^8q+35p^6q^2+50p^4q^3$

The relation (7) has some special cases as follows:

–$L_{(h,0)}^{(p,q)}=p^h, \quad (h\geq 1)$
–$L_{(h,1)}^{(p,q)}=p^h+hp^{h-2}q, \quad (h\geq 2)$
–$L_{(h,2)}^{(p,q)}=p^h+hp^{h-2}q+\dfrac{h(h-3)}{2}p^{h-4}q^2, \quad (h\geq 5)$
–$L_{(h,\tilde{h})}^{(p,q)}=L_h^{(p,q)}, \quad (h\geq 1)$
–$L_{(h,\tilde{h}-1)}^{(p,q)}=\begin{cases} L_h^{(p,q)}-2q^{(\frac{h}{2})} & (h \quad even) \\ L_h^{(p,q)}-hpq^{(\frac{h-1}{2})} & (h \quad odd) \end{cases}, \quad (h\geq 2)$

3.1 Some identities of the numbers $L_{(h,k)}^{(p,q)}$

Proposition 6.*The following identity holds:*

$$L_{(h,k)}^{(p,q)}=qF_{(h-1,k-1)}^{(p,q)}+F_{(h+1,k)}^{(p,q)}, \quad 0\leq k\leq \tilde{h}. \tag{8}$$

*Proof.*Using Definition (2), we obtain the desired equality as follows:

$$\begin{aligned} qF_{(h-1,k-1)}^{(p,q)}+F_{(h+1,k)}^{(p,q)} &= q\sum_{j=0}^{k-1}\binom{h-j-2}{j}p^{h-2j-2}q^j+\sum_{j=0}^{k}\binom{h-j}{j}p^{h-2j}q^j \\ &= q\sum_{j=1}^{k}\binom{h-j-1}{j-1}p^{h-2j}q^{j-1}+\sum_{j=0}^{k}\binom{h-j}{j}p^{h-2j}q^j \\ &= \sum_{j=0}^{k}\left[\binom{h-j-1}{j-1}+\binom{h-j}{j}\right]p^{h-2j}q^j-\binom{h-1}{-1}p^h \\ &= \sum_{j=0}^{k}\frac{h}{h-j}\binom{h-j}{j}p^{h-2j}q^j-0=L_{(h,k)}^{(p,q)} \end{aligned}$$

Proposition 7.*The incomplete $(p,q)-$Lucas numbers $L_{(h,k)}^{(p,q)}$ can be given by the recurrence relation*

$$L_{(h+2,k+1)}^{(p,q)}=pL_{(h+1,k+1)}^{(p,q)}+qL_{(h,k)}^{(p,q)}, \quad 0\leq k\leq \tilde{h}. \tag{9}$$

*Proof.*Relation (9) can be proved by using (8).

Proposition 8.*The following identity holds:*

$$L_{(h+2,k)}^{(p,q)}=pL_{(h+1,k)}^{(p,q)}+qL_{(h,k)}^{(p,q)}-\frac{h}{h-k}\binom{h-k}{k}p^{h-2k}q^{k+1} \tag{10}$$

*Proof.*Relation (10) can be proved by using (4) and (8).

Proposition 9.*The following identity holds:*

$$\sum_{j=0}^{r} \binom{r}{j} q^{r-j} p^j L_{(h+j,k+j)}^{(p,q)} = L_{(h+2r,k+r)}^{(p,q)}, \quad 0 \leq k \leq \frac{h-r}{2} \tag{11}$$

*Proof.*Relation (11) can be proved by using (5) and (8).

4 Generating Functions of the Incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas Numbers

The generating functions of the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers are given in this section.

Lemma 3.*Assume $\{T_h\}_{h=0}^{\infty}$ is a complex sequence that obeys the non-homogeneous second-order recurrence relation:*

$$T_h = \alpha T_{h-1} + \beta T_{h-2} + R_h, \quad h > 1,$$

where $\alpha, \beta \in \mathbb{C}$ (the field of complex numbers) and $R_h : \mathbb{N} \to \mathbb{C}$ is a sequence. Then the generating function $U(t)$ of T_h is

$$U(t) = \frac{G(t) + T_0 - R_0 + (T_1 - \alpha S_0 - R_1)t}{1 - \alpha t - \beta t^2}$$

where the generating function of $\{R_h\}$ is denoted by $G(t)$ (See [18]).

Theorem 1.*The generating function of the incomplete $(p,q)-$Fibonacci numbers $F_{(h,k)}^{(p,q)}$ is*

$$G_{p,q,k}^F(x) = \frac{\frac{x^2 q^{k+1}}{(1-px)^{k+1}} + F_{2k+1}^{(p,q)} + qF_{2k}^{(p,q)}x}{1 - px - qx^2}$$

*Proof.*Assume k is a fixed positive integer. Using (2) and (4), $F_{(h,k)}^{(p,q)} = 0$ for $0 \leq h < 2k+1$, $F_{(2k+1,k)}^{(p,q)} = F_{p,q,2k+1}$, and $F_{(2k+2,k)}^{(p,q)} = F_{2k+2}^{(p,q)}$,

$$F_{(h,k)}^{(p,q)} = pF_{(h-1,k)}^{(p,q)} + qF_{(h-2,k)}^{(p,q)} - \binom{h-k-3}{k} p^{h-2k-3} q^{k+1}$$

Now consider $T_0 = F_{(2k+1,k)}^{(p,q)}$, $T_1 = F_{(2k+2,k)}^{(p,q)}$ and $T_h = F_{(h+2k+1,k)}^{(p,q)}$.
Also, consider $R_0 = R_1 = 0$,

$$R_h = \binom{h+k-2}{h-2} p^{h-2} q^{k+1}.$$

Here,

$$G(x) = \frac{x^2 q^{k+1}}{(1-px)^{k+1}}$$

is the generating function of the sequence $\{R_h\}$ (see [21]). As a result of Lemma 3, we obtain the generating function $G_{p,q,k}^F(x)$ of the sequence $\{T_h\}$.

Theorem 2.*The generating function of the incomplete $(p,q)-$Lucas numbers $F_{(h,k)}^{(p,q)}$ is*

$$G_{p,q,k}^L(x) = \frac{\frac{x^2(2-px)q^{k+1}}{(1-px)^{k+1}} + L_{2k}^{(p,q)} + qL_{2k-1}^{(p,q)}x}{1 - px - qx^2}$$

*Proof.*Assume k is a fixed positive integer. Using (2) and (4), $L_{(h,k)}^{(p,q)} = 0$ for $0 \leq h < 2k$, $L_{(2k,k)}^{(p,q)} = L_{2k}^{(p,q)}$, and $L_{(2k+1,k)}^{(p,q)} = L_{2k+1}^{(p,q)}$,

$$L_{(h,k)}^{(p,q)} = pL_{(h-1,k)}^{(p,q)} + qL_{(h-2,k)}^{(p,q)} - \frac{h-2}{h-k-2}\binom{h-k-2}{h-2k-2}p^{h-2k-2}q^{k+1}$$

Now consider $T_0 = L_{(2k,k)}^{(p,q)}$, $T_1 = L_{(2k+1,k)}^{(p,q)}$ and $T_n = L_{(n+2k,k)}^{(p,q)}$.
Also, consider $R_0 = R_1 = 0$,

$$R_h = \frac{n+2k-2}{h+k-2}\binom{h+k-2}{h-2}p^{h-2}q^{k+1}$$

Here,

$$G(x) = \frac{x^2(2-px)q^{k+1}}{(1-px)^{k+1}}$$

is the generating function of the sequence $\{R_h\}$ (see [21]). As a result of Lemma 3, we get the generating function $G_{p,q,k}^{L}(x)$ of the sequence $\{T_h\}$.

5 Conclusion

In this paper, the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers are defined. Some properties and identities for them are given. The generating functions are derived. From these results, we can reach familiar results for some special numbers, such as Fibonacci, Lucas, Pell, and Jacobsall, as special cases

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References

[1] T. Çakmak and E. Karaduman, On the derivatives of bivariate Fibonacci polynomials, *Notes on Number Theory and Discrete Mathematics*, **24**(2018), 37-46.
[2] P. M. M. C. Catarino and A. Borges, A note on incomplete Leonardo numbers. *Integers: Electronic Journal of Combinatorial Number Theory*, **20**(2020).
[3] G. B. Djordjevic, Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers, *Fibonacci Quarterly*, **42**(2004), 106–113.
[4] G. B. Djordjevi and H. M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, *Mathematical and Computer Modelling*, **42**(2005), 1049–1056.
[5] P. Filipponi, Incomplete fibonacci and lucas numbers, *Rendiconti del Circolo Matematico di Palermo*, **45(1)**(1996), 37–56.
[6] A. İpek, On $(p,q)-$Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums, *Advances in Applied Clifford Algebras*, **27(2)**(2017), 1343–1351.
[7] C. Kızılateş, M. C. Firengiz and N. Tuglu, $q-$Generalization of Biperiodic Fibonacci and Lucas Polynomials, *Journal of Mathematical Analysis*, **8(5)**(2017).
[8] E. G. Kocer and S. Tuncez, Bivariate Fibonacci and Lucas like polynomials, *Gazi University Journal of Science*, **29(1)**(2016) 109–113.
[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications,* Volume 1. John Wiley Sons 2018.
[10] T. Koshy, *Fibonacci and Lucas Numbers with Applications,* Volume 2. John Wiley Sons 2019.

[11] B. Kuloğlu, E. Ozkan and A. G. Shannon, Incomplete generalized VietaPell and VietaPellLucas polynomials. *Notes on Number Theory and Discrete Mathematics*, **27(4)**(2021), 245–256.

[12] E. Özkan, M. Uysal and B. Kuloglu, Catalan transform of the incomplete Jacobsthal numbers and incomplete generalized Jacobsthal polynomials. *Asian-European Journal of Mathematics*, **15(06)**(2022), 2250119.

[13] B. K. Patel and N. Behera, On a generalization of incomplete Fibonacci quaternions. *Journal of the Indian Mathematical Society*, (2021), 88.

[14] J. L. Ramirez, Bi-periodic incomplete Fibonacci sequences, *Ann. Math. Inform* **42**(2013), 83–92.

[15] J. L. Ramirez, Incomplete $k-$Fibonacci and $k-$Lucas Numbers, *Chinese Journal of Mathematics,* (2013) 7 pp.

[16] J. L. Ramirez and V. F. Sirvent, Incomplete Tribonacci Numbers and Polynomials, *J. Integer Seq.*, **17(4)**(2014), 14–4.

[17] J. L. Ramirez, Incomplete generalized Fibonacci and Lucas polynomials, *Hacettepe journal of Mathematics and Statistics*, **44(2)**(2015), 363–373.

[18] A. Pinter and H. M. Srivastava, Generating functions of the incomplete Fibonacci and Lucas numbers, *Rendiconti del Circolo Matematico di Palermo*, **48(3)**(1999), 591–596.

[19] M. Shattuck, Combinatorial proofs of determinant formulas for the Fibonacci and Lucas polynomials, *Fibonacci Quarterly*, **51(1)**(2013), 63–71.

[20] M. Shattuck and E. Tan, Incomplete Generalized $(p,q,r)-$Tribonacci Polynomials, *Applications and Applied Mathematics: An International Journal (AAM)*, **13(1)**(2018), 1.

[21] H. Srivastava and H. Manocha, *Treatise on generating functions*, John Wiley Sons, Inc., 605 Third Ave., New York, NY 10158, USA, 1984.

[22] A. Suvarnamani, Some properties of $(p,q)-$Lucas number, *Kyungpook Mathematical Journal*, **56(2)** (2016), 367-370.

[23] A. Suvarnamani and M. Tatong, Some Properties of $(p,q)-$Fibonacci Numbers, *Science and Technology RMUTT Journal*, **5(2)**(2015), 17–21

[24] M. N. S. Swamy, Generalized Fibonacci and Lucas polynomials, and their associated diagonal polynomials, *Fibonacci Quarterly*, **37**(1999), 213–222.

[25] E. Tan, M. Dağlı and A. Belkhir, Bi-periodic incomplete Horadam numbers.*Turkish Journal of Mathematics*, **47(2)**(2023), 554–564.

[26] D. Tasci and M. C. Firengiz, Incomplete Fibonacci and Lucas $p-$numbers, *Mathematical and Computer Modelling*, **52(9-10)**(2010), 1763–1770.

[27] D. Tasci, M. Cetin Firengiz and N. Tuglu, Incomplete Bivariate Fibonacci and Lucas $p-$Polynomials. *Discrete Dynamics in Nature and Society,* (2012).

[28] B. Thongkam, K. Butsuwan and P. Bunya, Some properties of $(p,q)-$Fibonacci-like and $(p,q)-$Lucas numbers, *Notes on Number Theory and Discrete Mathematics*, **26**(2020), 216-224.

[29] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org.