JJMS **17**, No. 2, 259-266 (2024)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.2.7

On Recursive Computation of Moments of Generalized Order Statistics for a Transmuted Pareto Distribution and Characterization

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Received: Feb. 23, 2023 Accepted: Oct. 29, 2023

Abstract: In this paper, some relations are obtained for recursive computation of moments of generalized order statistics for a modified Pareto distribution. The recursive methods are obtained for raw and product moments. We have also obtained the recursive methods to compute the moments of specific situations which include order statistics and record values. We have also given some characterization results for the modified Pareto distribution.

Keywords: Transmuted Pareto Distribution, Generalized Order Statistics, Recursive Computation, Characterization.

2010 Mathematics Subject Classification. 60E05, 62E10, 62E15.

1 Introduction

A single combined method for random variables arranged in increasing order is known as the generalized order statistics (*gos*). This method is proposed by [1] and provides certain other methods for ordered data as special case. The density function of all of the variables in a *gos* is given by [1] as

$$g_{1,\ldots,n:n,t,\kappa}(y_1,\ldots,y_n)=\kappa\left(\prod_{j=1}^{n-1}\gamma_j\right)[1-G(y_n)]^{\kappa-1}g(y_n)\times\prod_{j=1}^{n-1}\left[\{1-G(y_j)\}^t g(y_j)\right], \quad (1)$$

where the quantities n, t and κ are the parameters of the density function such that $\gamma_h=\kappa+(n-h)(t+1)$. The *gos* produces different other methods for ordered data for different values of the parameters. The most popular of these are ordinary order statistics, kth record values; by [2]; and simple record values by [3].

The probability density function of a single *gos* is

$$g_{p:n,t,\kappa}(y)=\frac{C_{p-1}}{(p-1)!}g(y)[1-G(y)]^{\gamma_p-1}f_t^{p-1}[G(y)], \quad (2)$$

where $C_{p-1}=\prod_{j=1}^{p}\gamma_j$ and

$$f_t(u)=h_t(u)-h_t(0)=\begin{cases}\left[1-(1-u)^{t+1}\right]\Big/(t+1)\ ;\ t\neq-1\\ -\ln(1-u)\ ;\ t=-1.\end{cases}$$

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The probability density of any two *gos* is given as

$$g_{p,q:n,t,\kappa}(y_1,y_2)=\frac{C_{q-1}}{(p-1)!\,(q-p-1)!}g(y_1)\,g(y_2)\,[1-G(y_1)]^t f_t^{p-1}[G(y_1)] \times[1-G(y_2)]^{\gamma_q-1}[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}, \tag{3}$$

for $-\infty<y_1<y_2<\infty$ and

$$h_t(u)=\begin{cases}-(1-u)^{t+1}\big/(t+1)\ ;\ t\neq-1\\ -\ln(1-u)\ ;\ t=-1.\end{cases}$$

The *gos* is a single general method to model the data arranged in increasing order. Different simple methods are the special cases of this method for specific values of the parameters. The simple order statistics is obtained from *gos* if $t=0$ and $\kappa=1$. The model reduces to kth record values of [2] for $t=-1$. The simple upper record values; [3]; are obtained from *gos* for $t=-1$ and $\kappa=1$. Some more details can be found in [4] and [5].

Several authors have used different distributions to study their properties in context of the *gos*. A lot of work has been done in obtaining expressions to compute moments of *gos* recursively for specific distributions. A general expression for relations between moments of *gos* for any parent distribution has been obtained by [6] and [7]. Pareto and related distributions have attracted several authors to obtained the recurrence relations for moments of gos and its special cases. The relations for moments of generalized Pareto distribution were obtained by [8]. The expressions for recursive computation of moments of record values for Pareto and generalized Pareto distribution were obtained by [9] and [10]. The recursive expression for moments of *gos* for Pareto distribution are developed by [10]. The relations for recursive computation of moments of *gos* for Kumaraswamy Pareto distribution are developed by [11]. Some characterizations for the distributions using *gos* have been given by [12].

The area of recursive computation of moments for transmuted distributions is yet to be explored. This paper deals with developing some recursive methods to compute moments of *gos* for a transmuted Pareto distribution. A brief about the distribution is first given in the following section.

2 The Transmuted Pareto Distribution

The Pareto distribution; [13]; has tremendous applications in economics and finance. The distribution has been proposed as a suitable distribution for modeling income. The distribution has following density and distribution function

$$f(y;k,\alpha)=\frac{\alpha c^\alpha}{y^{\alpha+1}}\ ;\ y\geq c\ ,\ (\alpha,c)>0$$

and

$$F(y;k;\alpha)=1-\left(\frac{c}{y}\right)^\alpha\ ;\ y\geq c\ ,\ (\alpha,c)>0.$$

The distribution is studied extensively by several authors. Various authors has given different modifications of the distribution. A modification of the Pareto distribution has been given by [14] by using the technique of [15] and is referred to as the transmuted Pareto distribution. The density and distribution functions of this transmuted Pareto distribution are

$$g(y)=\frac{\alpha c^\alpha}{y^{\alpha+1}}\left[1+\lambda-2\lambda\left\{1-\left(\frac{c}{y}\right)^\alpha\right\}\right]\ ;\ y\geq c\ ,\ (\alpha,c)>0, \tag{4}$$

and

$$G(y)=\left[1-\left(\frac{c}{y}\right)^\alpha\right]+\lambda\left(\frac{c}{y}\right)^\alpha\left[1-\left(\frac{c}{y}\right)^\alpha\right]\ ;\ y\geq c\ ,\ (\alpha,c)>0, \tag{5}$$

where λ is the transmutation parameter such that $-1\leq\lambda\leq1$. The transmuted Pareto distribution has wide spread applications in modeling of financial and geological data. It is easy to see that (4) and (5) are related as

$$1-G(y)=\frac{y}{\alpha}g(y)-\lambda\left(\frac{c}{y}\right)^{2\alpha}. \tag{6}$$

This paper deals with obtaining recursive expressions to compute moments of *gos* for the transmuted Pareto distribution. The distribution has also been characterized on the basis of these recursive expressions of single and joint moments. The recursive expressions are obtained in Sections 3 and 4 below.

3 Relations for Simple and Reciprocal Moments

This section deals with obtaining recursive expressions to compute simple and reciprocal moments of *gos* for a transmuted Pareto distribution are given. These recursive expressions are obtained in the Theorem and the resulting corollaries, below.

Theorem 1. *The simple moments of gos for transmuted Pareto distribution can be recursively computed as*

$$\mu^r_{p:n,t,\kappa} = \frac{\alpha\gamma_p}{\alpha\gamma_p - r}\left[\mu^{r-1}_{p:n,t,\kappa} - \frac{\lambda c^{2\alpha} r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left\{\mu^{r-2\alpha}_{p:n,t,\kappa-1} - \mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right\}\right], \tag{7}$$

where $\gamma_{h(\kappa-1)} = (\kappa-1)+(n-h)(t+1)$ *and* $C_{p-1(\kappa-1)} = \prod_{h=1}^{p}\gamma_{h(\kappa-1)}$.

Proof. It is shown by [7] that the recursive expression for moments of *gos* for any distribution can be obtained by using

$$\mu^r_{p:n,t,\kappa} - \mu^r_{p-1:n,t,\kappa} = \frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{-\infty}^{\infty} y^{r-1}[1-G(y)]^{\gamma_p} f_t^{p-1}[G(y)]\,dy, \tag{8}$$

where $\mu^r_{p:n,t,\kappa} = E\left(Y^r_{p:n,t,\kappa}\right)$ and $Y^r_{p:n,t,\kappa}$ is the pth *gos*. The relation (8) can be written as

$$\mu^r_{p:n,t,\kappa} - \mu^r_{p-1:n,t,\kappa} = \frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{-\infty}^{\infty} y^{r-1}[1-G(y)][1-G(y)]^{\gamma_p-1} \times f_t^{p-1}[G(y)]\,dy.$$

Now, using (6) in above equation, we have

$$\begin{aligned}\mu^r_{p:n,t,\kappa} - \mu^r_{p-1:n,t,\kappa} &= \frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{c}^{\infty} y^{r-1}\left[\frac{y}{\alpha}g(y) - \lambda\left(\frac{c}{y}\right)^{2\alpha}\right] \times [1-G(y)]^{\gamma_p-1} f_t^{p-1}[G(y)]\,dy.\\ &= \frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{c}^{\infty} y^{r} g(y)[1-G(y)]^{\gamma_p-1} f_t^{p-1}[G(y)]\,dy\\ &\quad - \frac{\lambda c^{2\alpha} rC_{p-1}}{\gamma_p(p-1)!}\int_{c}^{\infty} y^{r-2\alpha-1}[1-G(y)]^{\gamma_p-1} f_t^{p-1}[G(y)]\,dy,\end{aligned}$$

or

$$\mu^r_{p:n,t,\kappa} - \mu^r_{p-1:n,t,\kappa} = \frac{r}{\alpha\gamma_p}\mu^r_{p:n,t,\kappa} - \frac{\lambda c^{2\alpha} r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\frac{(r-2\alpha)C_{p-1(\kappa-1)}}{\gamma_{p(\kappa-1)}(p-1)!} \times \int_{c}^{\infty} y^{r-2\alpha-1}[1-G(y)]^{\gamma_{p(k-1)}} f_t^{p-1}[G(y)]\,dy,$$

where $\gamma_{h(\kappa-1)} = (\kappa-1)+(n-h)(t+1)$ and $C_{p-1(\kappa-1)} = \prod_{h=1}^{p}\gamma_{h(\kappa-1)}$. Now, again using (6), we have

$$\mu^r_{p:n,t,\kappa} - \mu^r_{p-1:n,t,\kappa} = \frac{r}{\alpha\gamma_p}\mu^r_{p:n,t,\kappa} - \frac{\lambda c^{2\alpha} r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left[\mu^{r-2\alpha}_{p:n,t,\kappa-1} - \mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right]$$

or

$$\mu^r_{p:n,t,\kappa} = \frac{\alpha\gamma_p}{\alpha\gamma_p - r}\left[\mu^r_{p-1:n,t,\kappa} - \frac{\lambda c^{2\alpha} r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left\{\mu^{r-2\alpha}_{p:n,t,\kappa-1} - \mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right\}\right],$$

which is (7) and the proof is complete.

The recursive expression for simple moments of *gos* for Pareto distribution, given by [10], is readily obtained from (7) by using $\lambda = 0$.

Following corollaries are immediately obtained from Theorem 1.

Corollary 1. *Using* $-r$ *instead of* r *in (7), we have following recursive expression for the reciprocal moments of gos for the transmuted Pareto distribution*

$$\mu^{-r}_{p:n,t,\kappa} = \frac{\alpha\gamma_p}{\alpha\gamma_p + r}\left[\mu^{-r}_{p-1:n,t,\kappa} - \frac{\lambda c^{2\alpha} r}{(r+2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left\{\mu^{-(r+2\alpha)}_{p:n,t,\kappa-1} - \mu^{-(r+2\alpha)}_{p-1:n,t,\kappa-1}\right\}\right]. \tag{9}$$

Corollary 2.*Using $t=-1$ in (7), following relation for moments of kth upper record value for transmuted Pareto distribution is obtained*

$$\mu^{r}_{K(p)}=\frac{\alpha\kappa}{\alpha\kappa-r}\left[\mu^{r}_{K(p-1)}-\frac{\kappa^{p-1}\lambda c^{2\alpha}r}{(\kappa-1)^{p-1}(r-2\alpha)}\left\{\mu^{r-2\alpha}_{K-1(p)}-\mu^{r-2\alpha}_{K-1(p-1)}\right\}\right]. \tag{10}$$

The recursive expression for moments of kth record value for Pareto distribution; given by [9]; is obtained as a special case of (10) by using $\lambda=0$.

Corollary 3.*The recursive expression for simple moments of order statistics is derived by using $t=0$ and $\kappa=1$ in (7) and is*

$$\mu^{r}_{p:n}=\frac{\alpha(n-p+1)}{\alpha(n-p+1)-p}\left[\mu^{r}_{p-1:n}-\frac{n\lambda c^{2\alpha}r}{(r-2\alpha)(n-p+1)}\left\{\mu^{r-2\alpha}_{p:n}-\mu^{r-2\alpha}_{p-1:n}\right\}\right]. \tag{11}$$

The recursive expression for moments of order statistics for Pareto distribution is obtained by setting $\lambda=0$ in (11).

Corollary 4.*The recursive expression for reciprocal moments of transmuted Pareto distribution are obtained by using $t=-1$ in (9) and is*

$$\mu^{-r}_{K(p)}=\frac{\alpha\kappa}{\alpha\kappa-r}\left[\mu^{-r}_{K(p-1)}-\frac{\kappa^{p-1}\lambda c^{2\alpha}r}{(\kappa-1)^{p-1}(r+2\alpha)}\left\{\mu^{-(r+2\alpha)}_{K-1(p)}-\mu^{-(r+2\alpha)}_{K-1(p-1)}\right\}\right]. \tag{12}$$

Corollary 5.*The recursive expression for reciprocal moments of order statistics is obtained by using $t=0$ and $\kappa=1$ in (9) and is*

$$\mu^{-r}_{p:n}=\frac{\alpha(n-p+1)}{\alpha(n-p+1)+p}\left[\mu^{-r}_{p-1:n}-\frac{n\lambda c^{2\alpha}r}{(r+2\alpha)(n-p+1)}\left\{\mu^{-(r+2\alpha)}_{p:n}-\mu^{-(r+2\alpha)}_{p-1:n}\right\}\right]. \tag{13}$$

We will now obtain recursive expression for joint and ratio moments of *gos* for transmuted Pareto distribution.

4 Recursive Computation of Joint and Ratio Moments

The recursive relations for joint moments of *gos* for a transmuted Pareto distribution is obtained in the following theorem.

Theorem 2.*The joint moments of gos for transmuted Pareto distribution can be recursively computed by using*

$$\begin{aligned}\mu^{p,q}_{r,s:n,t,\kappa}=\frac{\alpha\gamma_s}{\alpha\gamma_s-q}&\left[\mu^{p,q}_{r,s-1:n,t,\kappa}-\frac{\lambda c^{2\alpha}q}{(q-2\alpha)}\frac{\gamma_{s(\kappa-1)}C_{s-1}}{\gamma_s C_{s-1(\kappa-1)}}\right.\\&\left.\times\left\{\mu^{p,q-2\alpha}_{r,s:n,t,\kappa-1}-\mu^{p,q-2\alpha}_{r,s-1:n,t,\kappa-1}\right\}\right],\end{aligned} \tag{14}$$

where $\mu^{r,s}_{p,q:n,t,\kappa}=E\left(Y^{r}_{p:n,t,\kappa}Y^{s}_{q:n,t,\kappa}\right)$ and $p<q$.

*Proof.*The joint moments of *gos* for any distribution are related as; see [7];

$$\begin{aligned}\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=&\frac{sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)\\&\times[1-G(y_1)]^t f_t^{p-1}[G(y_1)][1-G(y_2)]^{\gamma_q}\\&\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1,\end{aligned} \tag{15}$$

where $\mu^{r,s}_{p,q:n,t,\kappa}=E\left(Y^{r}_{p:n,t,\kappa}Y^{s}_{q:n,t,\kappa}\right)$. The relation (15) can also be written as

$$\begin{aligned}\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=&\frac{sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)\\&\times f_t^{p-1}[G(y_1)][1-G(y_1)]^t\\&\times[1-G(y_2)][1-G(y_2)]^{\gamma_q-1}\\&\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1.\end{aligned}$$

Using (6), we have

$$\mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = \frac{sC_{q-1}}{\gamma_q (p-1)!(q-p-1)!} \int_c^\infty \int_{y_1}^\infty y_1^r y_2^{s-1} g(y_1)$$
$$\times f_t^{p-1}[G(y_1)][1-G(y_1)]^t [1-G(y_2)]^{\gamma_q -1}$$
$$\times \left[\frac{y_2}{\alpha} g(y_2) - \lambda \left(\frac{c}{y_2}\right)\right]$$
$$\times [h_t\{G(y_2)\} - h_t\{G(y_1)\}]^{q-p-1} dy_2 dy_1$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = \frac{sC_{q-1}}{\gamma_q (p-1)!(q-p-1)!} \int_c^\infty \int_{y_1}^\infty y_1^r y_2^{s-1} g(y_1) g(y_2)$$
$$\times [1-G(y_1)]^t f_t^{p-1}[G(y_1)] [1-G(y_2)]^{\gamma_q -1}$$
$$\times [h_t\{G(y_2)\} - h_t\{G(y_1)\}]^{q-p-1} dy_2 dy_1$$
$$- \frac{\lambda c^{2\alpha} sC_{q-1}}{\gamma_q (p-1)!(q-p-1)!} \int_c^\infty \int_{y_1}^\infty y_1^r y_2^{s-1} g(y_1)$$
$$\times [1-G(y_1)]^t f_t^{p-1}[G(y_1)] [1-G(y_2)]^{\gamma_q -1}$$
$$\times [h_t\{G(y_2)\} - h_t\{G(y_1)\}]^{q-p-1} dy_2 dy_1$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = \frac{s}{\alpha\gamma_q} \mu^{r,s}_{p,q:n,t,\kappa} - \frac{\lambda c^{2\alpha} sC_{q-1}}{\gamma_q (p-1)!(q-p-1)!}$$
$$\times \int_c^\infty \int_{y_1}^\infty y_1^r y_2^{s-1} g(y_1) [1-G(y_1)]^t$$
$$\times f_t^{p-1}[G(y_1)] [1-G(y_2)]^{\gamma_q -1}$$
$$\times [h_t\{G(y_2)\} - h_t\{G(y_1)\}]^{q-p-1} dy_2 dy_1$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = \frac{s}{\alpha\gamma_q} \mu^{r,s}_{p,q:n,t,\kappa} - \frac{\lambda c^{2\alpha} s\gamma_{q(\kappa-1)} C_{q-1}}{(s-2\alpha)\gamma_q C_{q-1(\kappa-1)}}$$
$$\times \frac{(s-2\alpha) C_{q-1(\kappa-1)}}{\gamma_{q(\kappa-1)} (p-1)!(q-p-1)!} \int_c^\infty \int_{y_1}^\infty y_1^r y_2^{s-1}$$
$$\times g(y_1) [1-G(y_1)]^t f_t^{p-1}[G(y_1)] [1-G(y_2)]^{\gamma_q -1}$$
$$\times [h_t\{G(y_2)\} - h_t\{G(y_1)\}]^{q-p-1} dy_2 dy_1,$$

where $\gamma_{h(\kappa-1)} = (\kappa-1) + (n-h)(t+1)$ and $C_{q-1(\kappa-1)} = \prod_{h=1}^{q} \gamma_{h(\kappa-1)}$. Again using (15), we have

$$\mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = \frac{s}{\alpha\gamma_q} \mu^{r,s}_{p,q:n,t,\kappa} - \frac{\lambda c^{2\alpha} s\gamma_{q(\kappa-1)} C_{q-1}}{(s-2\alpha)\gamma_q C_{q-1(\kappa-1)}}$$
$$\left[\mu^{r,s-2\alpha}_{p,q:n,t,\kappa-1} - \mu^{r,s-2\alpha}_{p,q-1:n,t,\kappa-1}\right]$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa} = \frac{\alpha\gamma_q}{\alpha\gamma_q - s} \left[\mu^{r,s}_{p,q-1:n,t,\kappa} - \frac{\lambda c^{2\alpha} s}{(s-2\alpha)} \frac{\gamma_{q(\kappa-1)} C_{q-1}}{\gamma_q C_{q-1(\kappa-1)}}\right.$$
$$\left. \times \left\{\mu^{r,s-2\alpha}_{p,q:n,t,\kappa-1} - \mu^{r,s-2\alpha}_{p,q-1:n,t,\kappa-1}\right\}\right].$$

This is (14) and the proof is complete.

The relation (14) transforms to the relation for joint moments of *gos* from Pareto distribution, obtained by [10], for $\lambda = 0$ as it should be.

Theorem 2 provides following corollaries.

Corollary 6. *Substituting $-s$ in (14), the recursive expression for the ratio moments of gos for transmuted Pareto distribution is*

$$\mu^{r,-s}_{p,q:n,t,\kappa} = \frac{\alpha\gamma_q}{\alpha\gamma_q + s}\left[\mu^{r,-s}_{p,q-1:n,t,\kappa} - \frac{\lambda c^{2\alpha} s}{(s+2\alpha)}\frac{\gamma_{q(\kappa-1)}C_{q-1}}{\gamma_q C_{q-1(\kappa-1)}}\right.$$
$$\left.\times\left\{\mu^{r,-(s+2\alpha)}_{p,q:n,t,\kappa-1} - \mu^{r,-(s+2\alpha)}_{p,q-1:n,t,\kappa-1}\right\}\right]. \quad (16)$$

Corollary 7. *Substituting $t = -1$ in (14), following recursive expression for (r,s)th moments of (p,q)th upper record values for transmuted Pareto distribution is obtained*

$$\mu^{r,s}_{K(p,q)} = \frac{\alpha\kappa}{\alpha\kappa - s}\left[\mu^{r,s}_{K(p,q-1)} - \frac{\kappa^{q-1}\lambda c^{2\alpha} s}{(\kappa-1)^{q-1}(s-2\alpha)}\left\{\mu^{r,s-2\alpha}_{K-1(p,q)} - \mu^{r,s-2\alpha}_{K-1(p,q-1)}\right\}\right]. \quad (17)$$

Corollary 8. *Substituting $t = 0$ and $\kappa = 1$ in (14), following recursive expression for joint moments of order statistics for transmuted Pareto distribution is obtained*

$$\mu^{r,s}_{p,q:n} = \frac{\alpha(n-q+1)}{\alpha(n-q+1) - s}\left[\mu^{r,s}_{p,q-1:n} - \frac{n\lambda c^{2\alpha} s}{(n-q+1)(s-2\alpha)}\left\{\mu^{r,s-2\alpha}_{p,q:n} - \mu^{r,s-2\alpha}_{p,q-1:n}\right\}\right]. \quad (18)$$

Corollary 9. *Substituting $t = -1$ in (16), following recursive expression for the ratio moments of kth record value for transmuted Pareto distribution is obtained*

$$\mu^{r,-s}_{K(p,q)} = \frac{\alpha\kappa}{\alpha\kappa + s}\left[\mu^{r,-s}_{K(p,q-1)} - \frac{\kappa^{q-1}\lambda c^{2\alpha} s}{(\kappa-1)^{q-1}(s+2\alpha)}\left\{\mu^{r,-(s+2\alpha)}_{K-1(p,q)} - \mu^{r,-(s+2\alpha)}_{K-1(p,q-1)}\right\}\right]. \quad (19)$$

Corollary 10. *Substituting $t = 0$ and $\kappa = 1$ in (16), the following expression for recursive computation of ratio moments of order statistics for transmuted Pareto distribution is obtained*

$$\mu^{r,-s}_{p,q:n} = \frac{\alpha(n-q+1)}{\alpha(n-q+1) + s}\left[\mu^{r,-s}_{p,q-1:n} - \frac{n\lambda c^{2\alpha} s}{(n-q+1)(s+2\alpha)}\left\{\mu^{r,-(s+2\alpha)}_{p,q:n} - \mu^{r,-(s+2\alpha)}_{p,q-1:n}\right\}\right]. \quad (20)$$

The above relations are useful for recursive computation of moments.

5 Some Characterizations

Some characterizations of the transmuted Pareto distribution in terms of simple and joint moments of *gos* are given in the following theorems.

Theorem 3. *For a random variable X to have the density and distribution functions given in (4) and (5) respectively, the simple moments of its gos should be related as*

$$\mu^{r}_{p:n,t,\kappa} - \mu^{r}_{p-1:n,t,\kappa} = \frac{r}{\alpha\gamma_p}\mu^{r}_{p:n,t,\kappa} - \frac{\lambda c^{2\alpha} r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}$$
$$\times\left[\mu^{r-2\alpha}_{p:n,t,\kappa-1} - \mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right].$$

Proof. The necessary part of the Theorem is easily proved from Theorem 1. The sufficient condition is proved by considering (7); with $\bar{G}(x) = 1 - G(x)$; as

$$\frac{rC_{p-1}}{\gamma_p(p-1)!}\int_c^\infty y^{r-1}\left\{\bar{G}(y)\right\}^{\gamma_p} f_t^{p-1}[G(y)]\,dy$$
$$= \frac{rC_{p-1}}{\gamma_p(p-1)!}\int_c^\infty y^{y-1}\left\{\bar{G}(y)\right\}^{\gamma_p-1} f_t^{p-1}[G(y)]$$
$$\times\left[\frac{y}{\alpha}g(y) - \lambda\left(\frac{c}{y}\right)^{2\alpha}\right]dy$$

or

$$\frac{rC_{p-1}}{\gamma_p (p-1)!} \int_c^{\infty} y^{y-1} \left\{ \bar{G}(y) \right\}^{\gamma_p - 1} f_t^{p-1} [G(y)] \left[\bar{G}(y) - \left\{ \frac{y}{\alpha} g(y) - \lambda \left(\frac{c}{y} \right)^{2\alpha} \right\} \right] dy = 0.$$

Applying *Müntz–Száz* theorem; see [16]. We have; from above equation;

$$\bar{G}(y) = \left\{ \frac{y}{\alpha} g(y) - \lambda \left(\frac{c}{y} \right)^{2\alpha} \right\}.$$

The above is (6) and hence the proof is complete.

Theorem 4. *For a random variable X to have the density and distribution functions given in (4) and (5) respectively, the joint moments of its gos should be related as*

$$\mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = \frac{s}{\alpha \gamma_q} \mu^{r,s}_{p,q:n,t,\kappa} - \frac{\lambda c^{2\alpha} s \gamma_{q(k-1)} C_{q-1}}{(s-2\alpha) \gamma_q C_{q-1(k-1)}} \left[\mu^{r,s-2\alpha}_{p,q:n,t,\kappa-1} - \mu^{r,s-2\alpha}_{p,q-1:n,t,\kappa-1} \right].$$

Proof. The necessity is readily proved from Theorem 2. For sufficiency consider (15) as

$$\begin{aligned} \mu^{r,s}_{p,q:n,t,\kappa} - \mu^{r,s}_{p,q-1:n,t,\kappa} = & \frac{sC_{q-1}}{\gamma_q (p-1)! (q-p-1)!} \int_{-\infty}^{\infty} \int_{y_1}^{\infty} y_1^r y_2^{s-1} g(y_1) \\ & \times \left[\bar{G}(y_1) \right]^t f_t^{p-1} [G(y_1)] \left[\bar{G}(y_2) \right]^{\gamma_q} \\ & \times \left[h_t \{G(y_2)\} - h_t \{G(y_1)\} \right]^{q-p-1} dy_2 dy_1, \end{aligned}$$

Now, using above relation with (6) we have

$$\begin{aligned} & \frac{sC_{q-1}}{\gamma_q (p-1)! (q-p-1)!} \int_{-\infty}^{\infty} \int_{y_1}^{\infty} y_1^r y_2^{s-1} g(y_1) \left[\bar{G}(y_1) \right]^t f_t^{p-1} [G(y_1)] \\ & \times \left[h_t \{G(y_2)\} - h_t \{G(y_1)\} \right]^{q-p-1} \left[\bar{G}(y_2) \right]^{\gamma_q} dy_2 dy_1 \\ = & \frac{sC_{q-1}}{\gamma_q (p-1)! (q-p-1)!} \int_{-\infty}^{\infty} \int_{y_1}^{\infty} y_1^r y_2^{s-1} g(y_1) \left[\bar{G}(y_1) \right]^t f_t^{p-1} [G(y_1)] \\ & \times \left[h_t \{G(y_2)\} - h_t \{G(y_1)\} \right]^{q-p-1} \left[\bar{G}(y_2) \right]^{\gamma_q - 1} \\ & \times \left\{ \frac{y_2}{\alpha} g(y_2) - \lambda \left(\frac{c}{y_2} \right)^{2\alpha} \right\} dy_2 dy_1 \end{aligned}$$

or

$$\begin{aligned} & \frac{sC_{q-1}}{\gamma_q (p-1)! (q-p-1)!} \int_{-\infty}^{\infty} \int_{y_1}^{\infty} y_1^r y_2^{s-1} g(y_1) \left[\bar{G}(y_1) \right]^t f_t^{p-1} [G(y_1)] \\ & \times \left[h_t \{G(y_2)\} - h_t \{G(y_1)\} \right]^{q-p-1} \left[\bar{G}(y_2) \right]^{\gamma_q - 1} \\ & \times \left[\bar{G}(y_2) - \left\{ \frac{y_2}{\alpha} g(y_2) - \lambda \left(\frac{c}{y_2} \right)^{2\alpha} \right\} \right] dy_2 dy_1 \\ = & \, 0. \end{aligned}$$

Applying *Müntz–Száz* theorem; see [16]. We have; from above equation;

$$\bar{G}(y_2) = \frac{y_2}{\alpha} g(y_2) - \lambda \left(\frac{c}{y_2} \right)^{2\alpha}.$$

The above is (6) and hence the proof is complete.

6 Conclusion

This paper deals with obtaining some recursive expressions to compute the simple and joint moments of *gos* for a transmuted Pareto distribution. These expressions can be used to recursively compute the higher moments from the lower moments. We have also given the recursive expressions for the simple and joint moments of the specific cases of *gos*. The simple and joint moments are also used to obtain some characterization results. We have found that the recursive expressions for simple and joint moments of *gos* for the Pareto distribution appear as a special case. These relations are also useful in studying certain properties of the transmuted Pareto distribution.

Declarations

Competing interests: The authors declare no competing interests for this paper.
Authors' contributions: **SHS**: Mathematical developments, Writing the original draft.
MQS: Conceptualization, Project supervision, Writing the final draft.
Funding: The project receives no funding.
Availability of data and materials: The material is available with the authors.
Acknowledgments:The authors are thankful to the reviewers for healthy comments that increased quality of the paper.

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