Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.2.9

Transmuted Fuzzy Entropy: Another Look at Generalized Fuzzy Entropy

Mohammad M. Al-Talib[1,2,*]

[1] Department of Statistics, Yarmouk University, 21163 Irbid, Jordan.
[2] Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, 31952 AlKhobar, Kingdom of Saudi Arabia.

Received: March 27, 2023 Accepted: Oct. 11, 2023

Abstract: Generalizing a fuzzy entropy presents a better measure in theory and application. In this article we propose a new concept of generalizing an entropy on the basis of transmutation distribution function. Three transmuted fuzzy entropies are proposed and compared with well established generalized fuzzy entropies in the literature.

Keywords: Fuzzy Entropy; Transmuted Distribution; Generalized Fuzzy Entropy; Fuzzy Logic.

2010 Mathematics Subject Classification. 62A86; 94A17.

1 Introduction

Fuzzy logic was established by [17], defining what is a fuzzy set and establishing most of the operations and properties. Later on, [4] extended the concept of a classical entropy to present what is known as the fuzzy entropy (FE).
In the following, we will define a fuzzy set and the fuzzy entropy:

Definition 1 *A fuzzy set A defined on a universe of discourse X is given by [17] as:*

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\}, \tag{1}$$

where, $\mu_A(x) \in [0,1]$ is the membership function of A, it describes the degree of belongingness of an element x to the set A.

Its quite important to note that, considering the elements $x_1, x_2, ..., x_n \in X$, the $\sum_{i=1}^{n} \mu_A(x_i)$ is not necessary equals to 1. Hence, $\mu_A(x)$ is not a probability.
The measure which quantify fuzzy information gained from a fuzzy set or is referred to as fuzzy entropy. In other words, a fuzzy entropy measures the average amount of knowledge or information from fuzzy data.Its different than the classical entropy which depend of a probabilistic concept, where FE is define on the basis of membership function.
[4] defined a fuzzy entropy (denoted by $H^{DT}(x)$) on the basis of shannon's entropy ([14]),

$$H^{DT}(x) = -n\sum_{i=1}^{n}\left[\mu_A(x_i)\log(\mu_A(x_i)) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]. \tag{2}$$

They presented a set of axioms needed to be satisfied by any measure to be considered as an entropy; say $H(x)$ of a fuzzy set A. The axioms are

1.$H(x) = 0\ iff\ A$ is a non-fuzzy set (crisp set), i.e. $\mu_A(x_i) = 0$ or 1 $\forall x_i \in A$.
2.$H(x)$ is maximum iff $\mu_A(x_i) = 0.5, \forall x_i \in A$.

* Corresponding author e-mail: m.altalib@yu.edu.jo, maltalib@pmu.edu.sa. ORCID: 0000-0002-7238-8839.

3.$H(x) \geq H^*(x)$, where $H^*(x)$ is the entropy of A^* a sharpened version of A.
4.$H(x) = H^c(x)$, where $H^c(x)$ is the entropy of A^c; the complement set of A. $A^c = \{\langle x, 1-\mu_A(x)\rangle | x \in X\}$.

Many researchers and many articles studied fuzzy entropies and proposing modified and generalized versions of such entropies. [11], defined what is known now as a Hybrid entropy, where the fuzzy entropy is considered to be a generalization of the classical entropy (see, [2],[7]). This kind of entropy dealt with efficiencies of the total entropy which was proposed by [4].
In the same article, [11] introduced a higher order fuzzy entropy which measures the uncertainty associated with any subset with r combination. The entropy of order r of a fuzzy set A is

$$H^r(x) = -\frac{1}{t}\sum_{i=1}^{n} \mu(S_i^r)\log(\mu(S_i^r)) + (1-\mu(S_i^r))\log(1-\mu(S_i^r)),$$

where, S denotes the Shannon function, and $\mu(S_i^r)$ is the degree of membership through the function S.
[3] introduced a generalization of fuzzy entropy of order α based on Renyi's entropy of a fuzzy set A as

$$H^{BP}(x) = \frac{1}{n(\alpha-1)} \log\left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\right]; \quad \alpha \neq 1; \alpha > 0. \tag{3}$$

[9], [10] Introduced a fuzzy entropy based on the exponential behavior of information gain, of a fuzzy set A as

$$H^{PP}(x) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \left[\mu_A(x_i)e^{(1-\mu_A(x_i))} + (1-\mu_A(x_i))e^{\mu_A(x_i)-1}\right], \tag{4}$$

Later on, [16] generalized the exponential fuzzy entropy of order–α, of a fuzzy set A is given as

$$H^{VS}(x) = \frac{1}{n(e^{(1-0.5^{\alpha})}-1)} \sum_{i=1}^{n} \left[\mu_A(x_i)e^{(1-\mu_A^{\alpha}(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x_i))^{\alpha})-1}\right]; \quad \alpha > 0. \tag{5}$$

[1] proposed a fuzzy entropy of order α with a promising application in decision making. The measure of a fuzzy set A is given by

$$H^{NT}(x) = \sum_{i=1}^{n} \left[\frac{\mu_A^{\alpha/2}(x_i)(1-\mu_A(x_i))^{\alpha/2}}{\mu_A(x_i)e^{-\alpha(1-\mu_A(x_i))} + (1-\mu_A(x_i))e^{-\alpha\mu_A(x_i)}}\right]^{1/\alpha} ; \quad \alpha > 0. \tag{6}$$

The reader may refer to [5], [6], [8] for more details.

2 Transmuted Fuzzy entropy

2.1 Quadratic Transmuted Fuzzy entropy

As noted earlier, generalizing fuzzy entropies is a common custom in fuzzy theory. Adding extra parameter(s) to an existing fuzzy entropy make it more flexible and hence secure all information from losing due fuzziness.
In a similar fashion, [15] introduced the quadratic transmuted family of distributions, where it enhances an existing distribution by adding additional variable, for solving drawbacks in financial mathematics field. The cdf for a distribution in the quadratic transmuted family is

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \qquad x \in \mathbb{R},$$

where $\lambda \in [-1,1]$, and $G(x)$ is the cdf of baseline distribution.
Motivated by this family of distributions, we propose the Quadratic Transmuted Fuzzy Entropy (QTFE) defined below, also we study and prove that QTFE satisfies the axiomatic properties of [4].

Definition 2 *For a fuzzy entropy $H(x)$ of the Fuzzy set A, the transmuted fuzzy entropy of A is given by*

$$H_Q^T(x) = (1+\lambda)H(x) - \lambda H^2(x), \qquad x \in A, \lambda \in [-1,1]. \tag{7}$$

Theorem 1. *The Quadratic Transmuted Fuzzy Entropy $H_Q^T(x)$ is a fuzzy measure and satisfies The axiomatic properties [4].*

Proof. The set of four axiomatic properties are checked as follows

• $H_Q^T(x) = 0\ iff$ A is a non-fuzzy set (crisp set), i.e. $\mu_A(x_i) = 0$ or $1\ \forall x_i \in A$.
When, $\mu_A(x_i) = 0$ or 1 ,$\forall x_i \in A$ then it straight forward that $H(x) = 0$. Hence, $H_Q^T(x) = 0$. on the other hand, when $H_Q^T(x) = 0$, then $(1+\lambda)H(x) - \lambda H^2(x) = 0$, i.e.,

$$H(x) \times \left[(1+\lambda) - \lambda H(x)\right] = 0,$$

hence at least one of the factors is zero

$$H(x) = 0 \textbf{ or } (1+\lambda) - \lambda H(x) = 0, \forall x \in A.$$

If $H(x) = 0$, then $\mu_A(x_i) = 0$ or $1\ \forall x_i \in A$, as $H(x)$ is a fuzzy entropy and satisfies this axiomatic property.
Now, when $(1+\lambda) - \lambda H(x) = 0$, so, $H(x) = \frac{1+\lambda}{\lambda}$. This conclusion is false, as it means that the fuzzy entropy is always constant and ranges between $[0,2]$.

• $H(x)$ is maximum iff $\mu_A(x_i) = 0.5, \forall x_i \in A$.
by differentiating $H_Q^T(x)$ with respect to $\mu_A(x_i)$, we get

$$\begin{aligned}
\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} &= (1+\lambda)\frac{\partial H(x)}{\partial \mu_A(x_i)} - 2\lambda H(x) \cdot \frac{\partial H(x)}{\partial \mu_A(x_i)} \\
&= \frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[(1+\lambda) - 2\lambda H(x)\Big] \\
&= \frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[1 + \lambda \cdot \big(1 - 2H(x)\big)\Big]
\end{aligned}$$

Let, $0 \le \mu_A(x_i) < 0.5$.
Notice that $\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always greater than 0 as $H(x)$ satisfies this particular axiomatic property, since its a fuzzy measure.
i.e., the statement $1 + \lambda \cdot \big(1 - 2H(x)\big)$ should be positive in order to have $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} > 0$.
Case 1: $\lambda \ge 0, H(x) \le 0.5$. The statement is valid.
Case 2: $\lambda \le 0, H(x) \ge 0.5$. The statement is valid.
Case 3: $\lambda \ge 0, H(x) \ge 0.5$.
Now, we have

$$\begin{aligned}
0 \ge &(1 - 2H(x)) \ge -1 \\
0 \ge &\lambda \cdot (1 - 2H(x)) \ge -\lambda > -1, \qquad \text{(multipling by } \lambda) \\
1 \ge &1 + \lambda \cdot (1 - 2H(x)) \ge 1 - \lambda > 0, \qquad \text{(adding 1)}
\end{aligned}$$

hence, $1 + \lambda \cdot (1 - 2H(x)) \ge 0$, The statement is valid.
Case 4: $\lambda \le 0, H(x) \le 0.5$.

$$\begin{aligned}
1 \ge &(1 - 2H(x)) \ge 0 \\
-1 < &\lambda \le \lambda \cdot (1 - 2H(x)) \le 0, \qquad \text{(multipling by } \lambda) \\
0 < &1 + \lambda \le 1 + \lambda \cdot (1 - 2H(x)) \le 1, \qquad \text{(adding 1)}
\end{aligned}$$

hence, $1 + \lambda \cdot (1 - 2H(x)) \ge 0$, The statement is valid.
Hence, $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} > 0$ when $0 < \mu_A(x_i) \le 0.5$.
Now, let, $0.5 < \mu_A(x_i) \le 1.0$,
$\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always less than 0 as $H(x)$ satisfies this particular axiomatic property, since its a fuzzy measure. On the other hand and as explained earlier, the statement $1 + \lambda \cdot \big(1 - 2H(x)\big)$ is positive. i.e.,

$$\frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[1 + \lambda \cdot \big(1 - 2H(x)\big)\Big]$$

Hence, $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} < 0$.
Thus, $H_Q^T(x)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$.

• $H_Q^T(x) \geq H^*(x)$, where $H^*(x)$ is the entropy of A^* a sharpened version of A.

As shown in the last point, $H_Q^T(x)$ is increasing on the interval $[0,0.5)$ and decreasing on the interval $(0.5,1]$. It follows that $\mu_A^*(x_i)$; the membership function entropy of A^*, is less the $\mu_A(x_i)$ in the interval $[0,0.5)$ and its greater than $\mu_A(x_i)$ in the interval $(0.5,1]$. Hence, $H_Q^T(x) \geq H^*(x)$

• $H_Q^T(x) = H_Q^{Tc}(x)$, where $H_Q^{Tc}(x)$ is the entropy of A^c; the complement set of A.

$$H_Q^T(x) = (1+\lambda)H(x) - \lambda H^2(x) = (1+\lambda)H^c(x) - \lambda H^{c2}(x),$$

as $H(x)$ is a fuzzy entropy. So,

$$(1+\lambda)H^c(x) - \lambda H^{c2}(x) = H_Q^{Tc}(x).$$

Hence the theorem is proved, the QTFE satisfies the axiomatic properties, i.e., $H_Q^T(x)$ is indeed a fuzzy entropy.

2.2 Cubic Transmuted Fuzzy Entropy

[12] and [13] introduced the idea of the cubic transmuted distribution function, The cdf for a distribution in the cubic transmuted family is

$$F(x) = (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x), \qquad x \in \mathbb{R},$$

where $\lambda_1 \in [-1,1]$, $\lambda_2 \in [-1,1]$ and $-2 < \sum_i^2 \lambda_i < 1$. $G(x)$ is the cdf of baseline distribution.
On the basis of the defined family of distribution, we define the Cubic Transmuted Fuzzy Entropy (CTFE).

Definition 3 *For a fuzzy entropy $H(x)$ of the Fuzzy set A, the Cubic Transmuted Fuzzy Entropy of A is given by*

$$H_C^T(x) = (1+\lambda_1)H(x) + (\lambda_2 - \lambda_1)H^2(x) - \lambda_2 H^3(x), \tag{8}$$

where, $\lambda_1, \lambda_2 \in [-1,1]$, and $-2 < \sum_i^2 \lambda_i < 1$, $x \in A$.

Theorem 2. *The Cubic Transmuted Fuzzy Entropy $H_C^T(x)$ is a fuzzy measure and satisfies The axiomatic properties [4].*

Proof. The axiomatic properties are checked as follows,

• $H_C^T(x) = 0 \; iff$ A is a non-fuzzy set (crisp set), i.e. $\mu_A(x_i) = 0$ or $1 \; \forall x_i \in A$.

When, $\mu_A(x_i) = 0$ or $1 \; \forall x_i \in A$ then $H(x) = 0$. and hence $H_C^T(x) = 0$. on the other hand, when $H_C^T(x) = 0$, then $(1+\lambda_1)H(x) + (\lambda_2 - \lambda_1)H^2(x) - \lambda_2 H^3(x) = 0$, i.e.,

$$H(x) \times \left[(1+\lambda_1) + (\lambda_2 - \lambda_1)H(x) - \lambda_2 H^2(x)\right] = 0,$$

hence at least one of the factors is zero

$$H(x) = 0 \textbf{ or } (1+\lambda_1) + (\lambda_2 - \lambda_1)H(x) - \lambda_2 H^2(x) = 0, \forall x \in A.$$

If $H(x) = 0$, then $\mu_A(x_i) = 0$ or $1 \; \forall x_i \in A$, as $H(x)$ satisfies this axiomatic property.
Now when $(1+\lambda_1) + (\lambda_2 - \lambda_1)H(x) - \lambda_2 H^2(x) = 0$, this will give a specific value of the entropy dependent on the choice of λ_1 and λ_2. And hence this factor is not zero.

• $H_C^T(x)$ is maximum iff $\mu_A(x_i) = 0.5, \forall x_i \in A$.
by differentiating $H_C^T(x)$ with respect to $\mu_A(x_i)$, we get

$$\begin{aligned}
\frac{\partial H_C^T(x)}{\partial \mu_A(x_i)} &= (1+\lambda_1)\frac{\partial H(x)}{\partial \mu_A(x_i)} + 2(\lambda_2 - \lambda_1)H(x) \cdot \frac{\partial H(x)}{\partial \mu_A(x_i)} - 3\lambda_2 H^2(x) \cdot \frac{\partial H(x)}{\partial \mu_A(x_i)} \\
&= \frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)H(x) \cdot -3\lambda_2 H^2(x) \cdot \Big] \\
&= \frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[1 + \lambda_1 \cdot (1 - 2H(x)) + \lambda_2 H(x) \cdot (2 - 3H(x))\Big]
\end{aligned}$$

Let, $0 \le \mu_A(x_i) < 0.5$,
Notice that $\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always greater than 0 as $H(x)$ a fuzzy measure and hence it satisfies this axiomatic property, i.e., the statement $1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 H(x) \cdot (2-3H(x))$ should be positive in order to have $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} > 0$.
Case 1: $\lambda_1, \lambda_2 \ge 0, H(x) \le 0.5$. The statement is valid.
Case 2: $\lambda_1, \lambda_2 \ge 0, 0.5 \le H(x) \le 2/3$. The statement is valid.
Case 3: $\lambda_1, \lambda_2 \le 0, H(x) \le 2/3$. The statement is valid.
Case 4: $\lambda_1, \lambda_2 \ge 0, H(x) \ge 2/3$.
we have

$$\begin{array}{rcl} 0 \ge (1-2H(x)) \ge -1 & , & 0 \ge H(x)(2-3H(x)) \ge -1 \\ 0 \ge \lambda_1 \cdot (1-2H(x)) \ge -\lambda_1 > -1 & , & 0 \ge \lambda_2 \cdot H(x)(2-3H(x)) \ge -\lambda_2 > 0 \end{array}$$

hence,

$$0 \ge \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x)) > -1,$$

then,

$$1 \ge 1 + \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x)) > 0$$

hence, The statement is valid.
Case 5: $\lambda_1, \lambda_2 \le 0, H(x) \le 0.5$.
we have

$$\begin{array}{rcl} 1 \ge (1-2H(x)) \ge 0 & , & 1 \ge H(x)(2-3H(x)) \ge 0 \\ -1 < \lambda_1 \ge \lambda_1 \cdot (1-2H(x)) \ge 0 & , & -1 \ge \lambda_2 \ge \lambda_2 \cdot H(x)(2-3H(x)) < 0 \end{array}$$

hence,

$$-1 \ge \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x)) < 0,$$

then,

$$0 < 1 + \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x))$$

hence, The statement is valid.
Case 6: $\lambda_1, \lambda_2 \le 0, 0.5 \le H(x) \le 2/3$.
we have

$$\begin{array}{rcl} 1 \le (1-2H(x)) \le 0 & , & H(x)(2-3H(x)) \ge 0 \\ 1 > \lambda_1 \ge \lambda_1 \cdot (1-2H(x)) \ge 0 & , & -1 \le \lambda_2 \le \lambda_2 \cdot H(x)(2-3H(x)) < 0 \\ 1 > \lambda_1 \ge \lambda_1 \cdot (1-2H(x)) \ge 0 & , & 0 \le 1+\lambda_2 \le 1 + \lambda_2 \cdot H(x)(2-3H(x)) < 1 \end{array}$$

then,

$$0 < 1 + \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x))$$

hence, The statement is valid, and hence $\frac{\partial H_C^T(x)}{\partial \mu_A(x_i)} > 0$.

Now, let, $0.5 < \mu_A(x_i) \le 1.0$,
$\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always less than 0 as $H(x)$ satisfies this particular axiomatic property, since its a fuzzy measure. On the other hand and as shown above, the statement $1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x))$ is positive. Hence, $\frac{\partial H_C^T(x)}{\partial \mu_A(x_i)} < 0$.
Thus, $H_C^T(x)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$.

- $H_C^T(x) \ge H^*(x)$, where $H^*(x)$ is the entropy of A^* a sharpened version of A.

As shown in the last point, $H_C^T(x)$ is increasing on the interval $[0, 0.5)$ and decreasing on the interval $(0.5, 1]$. It follows that $\mu_A^*(x_i)$; the membership function of A^*, is less the $\mu_A(x_i)$ in the interval $[0, 0.5)$ and its greater than $\mu_A(x_i)$ in the interval $(0.5, 1]$. Hence, $H_C^T(x) \ge H^*(x)$.

- $H_C^T(x) = H_C^{T^c}(x)$, where $H_C^{T^c}(x)$ is the entropy of A^c; the complement set of A.

$$H_C^T(x) = (1+\lambda_1)H(x) + (\lambda_2 - \lambda_1)H^2(x) - \lambda_2 H^3(x),$$

as $H(x)$ is a fuzzy entropy; i.e., $H(x) = H^c(x)$. So,

$$(1+\lambda_1)H^c(x) + (\lambda_2 - \lambda_1)H^{2^c}(x) - \lambda_2 H^{3^c}(x) = H_C^{T^c}(x).$$

Hence the theorem is proved, the Cubic Transmuted Fuzzy entropy satisfies the axiomatic properties, i.e., $H_C^T(x)$ is a fuzzy measure.

2.3 k-Transmuted Fuzzy Entropy

[12] have introduced the k-Transmuted families of distributions, a generalization of transmuted families, defined as

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^{k} \lambda_i G^i(x),$$

where, $\lambda_i \in [-1,1]$ and $-k < \sum_i^k \lambda_i < 1$. $G(x)$ is the cdf of baseline distribution.
Based on this family of distribution we define the kTransmuted Fuzzy Entropy.

Definition 4 *For a fuzzy entropy $H(x)$ of the Fuzzy set A, the k-Transmuted Fuzzy Entropy of A is given by*

$$H_k^T(x) = H(x) + [1 - H(x)] \sum_{i=1}^{k} \lambda_i H^i(x), \qquad x \in A. \tag{9}$$

Where, $\lambda_i \in [-1,1]$ and $-k < \sum_i^k \lambda_i < 1$, k=2,3,. . . .

It follows that we have another version of cubic Transmuted Fuzzy Entropy, given as

$$H_3^T(x) = H(x) + [1 - H(x)] \sum_{i=1}^{3} \lambda_i H^i(x),$$

which after simplifying turned out to be equivalent to CTFE.
Lets define a Quartic Transmuted Fuzzy Entropy; a k-TFE at $k = 4$, denoted by $H_4^T(x)$ given by

$$H_4^T(x) = H(x) + [1 - H(x)] \sum_{i=1}^{4} \lambda_i H^i(x). \tag{10}$$

Theorem 3. *The k–Transmuted Fuzzy Entropy $H_k^T(x)$ is a fuzzy measure and satisfies The axiomatic properties [4].*

The proof is straight forward as done in the previous theorems.

3 Transmuted entropies of well known fuzzy measures

Fuzzy entropy literature is rich in many versions and generalizations of FE, among many of these measures, we presented the most known and applied measures in equations (2) - (6). The Quadratic, Cubic and K–Transmuted Fuzzy entropies of fourth order are generalizations of the fuzzy measures mentioned above are found by applying equation (7), (8) and (10), respectively.
Table 1 presents the values of Deluca and Terminin 1972 performance at different values of $\mu_A(x_i)$ (refereed to as Normalized Values) alongside its Transmuted Fuzzy entropies.

$\mu_A(x_i)$	H^{DT}	H_Q^{DT}	H_C^{DT}	H_4^{DT}
0	1.71×10^{-6}	3.08×10^{-6}	8.56×10^{-7}	8.56×10^{-7}
0.1	0.325	0.501	0.279	0.284
0.2	0.501	0.700	0.488	0.501
0.3	0.611	0.801	0.622	0.640
0.4	0.673	0.849	0.696	0.716
0.5	0.693	0.863	0.719	0.740
0.6	0.673	0.849	0.696	0.716
0.7	0.611	0.801	0.622	0.640
0.8	0.501	0.700	0.488	0.501
0.9	0.325	0.501	0.279	0.284
1.0	1.71×10^{-6}	3.08×10^{-6}	8.56×10^{-7}	8.56×10^{-7}

Table 1: Normalized Values of $H^{DT}(A)$ and its respected transmuted generalizations

All transmuted fuzzy entropies are performing better than the original entropy, as we expected. There is a significant increase in the values of the generalized entropies in comparison with De-Luca and Termini measure. QTFE enhances the performance of the entropy measure more than the others.
The difference in performance between QTFE and CTFE, k–TFE is noticeable. But does this remark going against what we believe in?, introducing more parameters will end up in a better performance of a measure?
In fact, parameter λ in QTFE is acting as an additional parameter to the measure, but in the cases of CTFE and k–TFE, more λ's is not considered more parameters, it is just a division of one parameter.
As shown below in Table 2, TFE is performing better than the original measure, with best performance for QTFE.

$\mu_A(x_i)$	H^{PP}	H_Q^{PP}	H_C^{PP}	H_4^{PP}
0	4.19×10^{-7}	7.54×10^{-7}	2.09×10^{-7}	2.09×10^{-7}
0.1	0.370	0.558	0.332	0.338
0.2	0.651	0.833	0.670	0.689
0.3	0.846	0.950	0.880	0.899
0.4	0.962	0.991	0.975	0.982
0.5	1.000	1.000	1.000	1.00
0.6	0.962	0.991	0.975	0.982
0.7	0.846	0.950	0.880	0.899
0.8	0.651	0.833	0.670	0.689
0.9	0.370	0.558	0.332	0.338
1.0	4.19×10^{-7}	7.54×10^{-7}	2.09×10^{-7}	2.09×10^{-7}

Table 2: Normalized Values of $H^{PP}(A)$ and its respected transmuted generalizations

$\mu_A(x_i)$	H^{BP}	H_Q^{BP}	H_C^{BP}	H_4^{BP}
0	2.00×10^{-7}	3.60×10^{-7}	1.00×10^{-7}	1.00×10^{-7}
0.1	0.198	0.325	0.147	0.149
0.2	0.385	0.575	0.349	0.356
0.3	0.545	0.743	0.542	0.557
0.4	0.654	0.835	0.674	0.693
0.5	0.693	0.863	0.719	0.740
0.6	0.654	0.835	0.674	0.693
0.7	0.545	0.743	0.542	0.557
0.8	0.385	0.575	0.349	0.356
0.9	0.198	0.325	0.147	0.149
1.0	2.00×10^{-7}	3.60×10^{-7}	1.00×10^{-7}	1.00×10^{-7}

Table 3: Normalized Values of $H^{BP}(A)$ and its respected transmuted generalizations

$\mu_A(x_i)$	H^{VS}	H_Q^{VS}	H_C^{VS}	H_4^{VS}
0	1.26×10^{-7}	5.99×10^{-7}	1.66×10^{-7}	1.66×10^{-7}
0.1	0.837	0.494	0.274	0.278
0.2	0.985	0.791	0.608	0.626
0.3	0.999	0.936	0.851	0.871
0.4	0.999	0.989	0.969	0.977
0.5	1.000	1.000	1.000	1.000
0.6	0.999	0.989	0.969	0.977
0.7	0.999	0.936	0.851	0.871
0.8	0.985	0.791	0.608	0.626
0.9	0.837	0.494	0.274	0.278
1.0	1.26×10^{-7}	5.99×10^{-7}	1.66×10^{-7}	1.66×10^{-7}

Table 4: Normalized Values of $H^{VS}(A)$ and its respected transmuted generalizations

$\mu_A(x_i)$	H^{NT}	H_Q^{NT}	H_C^{NT}	H_4^{NT}
0	3.16×10^{-4}	5.69×10^{-4}	1.58×10^{-4}	1.58×10^{-4}
0.1	0.346	0.527	0.303	0.308
0.2	0.527	0.726	0.520	0.534
0.3	0.677	0.852	0.701	0.721
0.4	0.784	0.919	0.819	0.840
0.5	0.824	0.940	0.859	0.879
0.6	0.784	0.919	0.819	0.840
0.7	0.677	0.852	0.701	0.721
0.8	0.527	0.726	0.520	0.534
0.9	0.346	0.527	0.303	0.308
1.0	3.16×10^{-4}	5.69×10^{-4}	1.58×10^{-4}	1.58×10^{-4}

Table 5: Normalized Values of $H^{NT}(A)$ and its respected transmuted generalizations

We state the following based on Table 1 and Table2;

$$H^{DT} < H_C^{DT} < H_4^{DT} < H_Q^{DT},$$

$$H^{PP} < H_C^{PP} < H_4^{PP} < H_Q^{PP}.$$

In the later tables we studied three generalized fuzzy entropies of order α, and we a reached similar conclusion, In Table 3

$$H^{BP} < H_C^{BP} < H_4^{BP} < H_Q^{BP}.$$

Also, in Table 4

$$H^{VS} < H_C^{VS} < H_4^{VS} < H_Q^{VS}.$$

And in Table 5

$$H^{NT} < H_C^{NT} < H_4^{NT} < H_Q^{NT}.$$

4 Conclusion

The proposed generalized entropy named Transmuted Fuzzy entropy is another form of generalized entropies. we presented 3 different measures; QTFE, CTFE and k–TFE with performance much better than the original FE, where QTFE presented the best enhancement.
As TPD was first introduced in financial mathematics and later applied in modeling lifetime and survival data, we are intrigued to apply TFE, and specially QTFE in these fields in future research.

Declarations

Competing interests: The Author declares that there is no conflict of interest.
Authors' contributions: The Author confirms sole responsibility for the study methodology and design, data analysis, interpretation of results and manuscript preparation.
Funding: The author received no financial support for this manuscript.
Availability of data and materials: Availability is upon request.
Acknowledgments: The Author sincerely thank the Editor in chief of JJMS and the anonymous reviewers for taking the time to review this manuscript and providing constructive feedback to improve the manuscript.

References

[1] M. Al-Talib,A. Al-Nasser, New Fuzzy Entropy Measure of Order α, *Pakistan Journal of Statistics and Operation Research*, **14(4)** (2018), 831–838.
[2] M. Al-Rawwash, A. Al-Nasser, Longitudinal data analysis using generalized maximum entropy approach, *Jordan Journal of Mathematics and Statistics (JJMS)*, **4(1)** (2011), 47–60.
[3] D. Bhandari, and N. R. Pal,Some new information measures for fuzzy sets *Information Sciences*,**67(3)**(1993), 209–228.
[4] A. De-Luca , S. Termini, A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, *Information and Control*, **20(4)** (1972), 301–312.
[5] K.C. Deshmukh, P.G. Khot,, Generalized Measures of Fuzzy Entropy and their Properties, *International Journal of Mathematical and Computational Sciences*, **5(8)** (2011), 1176–1180.
[6] C. Hwang, M. Yang, On entropy of fuzzy sets, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **105** (2008), 519–527.
[7] Z. Giski, Some Special Features of Relative Intuitionistic Dynamical Systems, *Jordan Journal of Mathematics and Statistics (JJMS)*, **14(1)** (2021), 111–125.
[8] H. Migdady, M. Al-Talib, An enhanced fuzzy K-means clustering with application to missing data imputation, *Electronic Journal of Applied Statistical Analysis*, **11(2)** (2018), 674–686.
[9] N. R. Pal, S. K. Pal, Object-background segmentation using new definitions of entropy, *IEEE Proceedings E-Computers and Digital Techniques*, **136(4)** (1989), 284–295.
[10] N. R. Pal, S. K. Pal, Entropy: A new definition and its applications, *IEEE transactions on systems, man, and cybernetics*, **21(5)** (1991), 1260–1270.
[11] N. R. Pal, S. K. Pal, Higher order fuzzy entropy and hybrid entropy of a set, *Information Sciences*, **61(2)** (1992), 211–231.
[12] M. Rahman, B. Al-Zahrani, M. Shahbaz,A general transmuted family of distributions, *Pakistan Journal of Statistics and Operation Research*, **14(2)** (2018), 451–469.
[13] M. Rahman, B. Al-Zahrani, M. Shahbaz,New general transmuted family of distributions with applications, *Pakistan Journal of Statistics and Operation Research*, **14(4)** (2018), 807–829.
[14] C. E. Shannon, A mathematical theory of communication, *Bell system technical journal*, **27(4)** (1948), 623–656.
[15] W. Shaw, I.R. Buckley,, The alchemy of probability distributions: Beyond gram-charlier & cornish-fisher expansions, and skew-normal or kurtotic-normal distributions, *arXiv preprint arXiv:0901.0434*, (2009)
[16] R. Verma, B. D. Sharma, On generalized exponential fuzzy entropy, *World Academy of Science, Engineering and Technology*, **60** (2011), 1402–1405.
[17] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353.
[18] L. A. Zadeh, Probability measures of Fuzzy events, *Journal of Mathematical Analysis and Applications*, **23(2)** (1968), 421–427.