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Computing Nash Optimal Strategies for a Two-Player Positive Game

Ivan G. Ivanov[(1),*], *Ivelin G. Ivanov*[(2)]

[1] Faculty of Economics and Business Administration, Sofia University St.Kl.Ohridski, Sofia, Bulgaria
[2] Kolej Dobrich, Konstantin Preslavsky University of Shumen, Dobrich, Bulgaria

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Abstract: We consider a linear quadratic differential game on an infinite time horizon with two types of an information structure. The game models are considered in both information structures: the open loop design and feedback design. The Newton solver for computing the stabilizing solution of the associated Nash-Riccati equations has been established. Moreover, a convergent linearized iterative method depending on a negative constant is introduced for each information structure. The linearized iteration has a linear convergence rate, however there are cases where the iteration is faster than Newton's method. Numerical experiments are implemented to explain the computational advantages of the introduced solvers.

Keywords: Game modelling; Nash Equilibrium; Stabilizing Solution.

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1 Introduction

There is a correlation between the behaviour of economic agents and their profits on a market. Game theory has used to model and investigate the equilibrium of a market. The market price is defined via a dynamic system equation. Typical applications of game models are in different branches in economics [10,13] and specially in modelling the energy markets [17], gas network optimization [1].

The Nash equilibrium theory is an effective instrument for the analysis of the equilibrium states in game models. We analyze the problem of computation the optimal strategies to the Nash equilibrium in linear quadratic differential games. Considering a linear dynamics upon the quadratic cost, the problem lead us to solve the coupled Riccati-like differential equations.

Nash equilibrium (or optimal) strategies for differential games are studied in many papers and applications. Nash equilibrium strategies depending on a special solution of coupled algebraic Riccati equations [10] - [13].

The Nash equilibrium and its applicability in the machine learning classification via support vector machines was investigated recently in many papers, for example [5,14]. It is important to find the corresponding equilibrium fast and effective.

We consider a dynamic system of the type

$$\dot{x} = Ax + B_1u_1 + B_2u_2\,, \quad x(0) = x_0\,. \tag{1}$$

In Equation (1) the state vector is denoted by x, the initial vector is $x_0 \in \mathbb{R}^{n\times 1}$, and matrices A, B_1, B_2 belong to $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m_1}, \mathbb{R}^{n\times m_2}$, where $\mathbb{R}^{p\times q}$ denotes a set of $p \times q$ matrices with real entries. Control vectors are u_1, u_2. Each player has to choose its control in order to increase its profit. If for all nonnegative vectors x_0, u_1, u_2 the state function $x(t)$ takes only nonnegative values, then system (1) is a positive one. Moreover, system (1) is positive if and only if matrices B_1 and B_2 are nonnegative ones and the matrix (-A) is a Z-matrix [2].

We consider an infinite time horizon game model for a positive system in two cases: (a) with an open loop information design and (b) with a feedback information one. The Newton method and its computer realization for computing the

* Corresponding author e-mail: i_ivanov@feb.uni-sofia.bg

Nash equilibrium for the same problem was presented and analyzed in [3]. The Newton algorithm solves a Lyapunov matrix equation at each iteration step via Kronecker product, which approach increase double the size of computational problem [3]. In this paper we explore a problem to find Nash equilibrium strategies for a two-player infinite horizon linear quadratic differential game in these two cases. We propose new faster iterations to determine the stabilizing solution of the corresponding Nash-Riccati equations. Based on the stabilizing solution the optimal controls of each player are established. The computational algorithm of the new iteration needs to compute only two matrix inverses of each iteration step. Numerical experiments are implemented to explain the computational advantages of the introduced solvers.

1.1 A feedback design game model

The theory of the Nash equilibrium in a feedback design was established in [15,16] and computationally investigated in [2,3,8]. The goal of each player is to maximize the corresponding cost function. Cost functionals J_1, J_2 for players are defined

$$J_i(F_1,F_2,x_0) = +\int_0^\infty x^T \left(Q_i + \sum_{j=1}^{2} F_j^T R_{ij} F_j \right) x\, dt\,, \tag{2}$$

for $i=1,2$. Matrices Q_i and R_{ij} are symmetric ones with $Q_i \in \mathbb{R}^{n\times n}$ and $R_{ij} \in \mathbb{R}^{m_j\times m_j}$ and $i,j=1,2$. The following additional requirements are assumed:

(a) Q_1, Q_2, R_{12}, and R_{21} are symmetric and nonnegative matrices;

(b) R_{ii}^{-1} is nonpositive, $i=1,2$.

Moreover, to compute a feedback Nash equilibrium point one has to solve the couple of Nash-Riccati equations [2]:

$$0 = \mathscr{R}_1(X_1,X_2) := -A^T X_1 - X_1 A - Q_1 + X_1 S_1 X_1 + X_1 S_2 X_2 + X_2 S_2 X_1 - X_2 S_{12} X_2\,, \tag{3}$$

$$0 = \mathscr{R}_2(X_1,X_2) := -A^T X_2 - X_2 A - Q_2 + X_2 S_2 X_2 + X_2 S_1 X_1 + X_1 S_1 X_2 - X_1 S_{21} X_1\,, \tag{4}$$

where the matrix coefficients are computed via:

(a) $S_i = B_i R_{ii}^{-1} B_i^T\,,\ S_i = S_i^T \le 0\,, i=1,2;$

(b) $S_{12} = B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T\,,$
$S_{12} = S_{12}^T \ge 0,\ \ (R_{12} = R_{12}^T);$

(c) $S_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T\,,$
$S_{21} = S_{21}^T \ge 0,\ R_{12} = R_{12}^T, R_{21} = R_{21}^T\,.$

We derive a faster iteration to calculate an $n\times n$ stabilizing solution $(\tilde{X}_1, \tilde{X}_2)$ of (3)-(4). The closed loop matrix $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ of system (1) is a stable one. Thus, the feedback Nash equilibrium is defined by $\tilde{F}_j = -R_{jj}^{-1} B_j^T \tilde{X}_j\,,\ j=1,2$ and optimal functional's value is $J_j(\tilde{F}_1,\tilde{F}_2,x_0) = x_0^T \tilde{X}_j x_0\,,\ j=1,2$ [15,16].

1.2 An open loop design game model

In addition, we define the cost functionals J_1, J_2 for the players in a game with an open loop design

$$J_i(u_1,u_2,x_0) = +\int_0^\infty \left(x^T Q_i x + \sum_{j=1}^{2} u_j^T R_{ij} u_j \right) dt\,. \tag{5}$$

The matrix coefficients in (5) are the same as (2). Players choose their own strategies u_1, u_2 based on the information for the initial state x_0 [2]. The Nash equilibrium point of the game is a solution of the couple Nash-Riccati equations:

$$0 = \mathscr{L}_1(X_1,X_2) := -A^T X_1 - X_1 A - Q_1 + X_1 S_1 X_1 + X_1 S_2 X_2\,, \tag{6}$$

$$0 = \mathscr{L}_2(X_1,X_2) := -A^T X_2 - X_2 A - Q_2 + X_2 S_2 X_2 + X_2 S_1 X_1\,. \tag{7}$$

A solution (X_1^*, X_2^*) has a property the closed loop matrix $(A - S_1 X_1^* - S_2 X_2^*)$ is stable. Moreover, the Nash optimal strategy (u_1^*, u_2^*) in the game is given by $u_j^* = -R_{jj}^{-1} B_j^T X_j^* x^*, j=1,2$ and x^* solves the closed loop equation $\dot{x} = (A - S_1 X_1^* - S_2 X_2^*)x,\ x(0) = x_0$.

1.3 Notations and facts

A matrix $Q=(q_{ij})$ is nonnegative one in the element wise ordering if $q_{ij}\geq 0$. A real square matrix A is called a Z-matrix if there exists a real number σ and real nonnegative matrix Q, such that $A=\sigma I-Q$. A square Z-matrix has nonpositive off-diagonal elements. If $\sigma>\rho(Q)$, the matrix A is a nonsingular M-matrix. Note $\rho(Q)$ is the spectral radius of Q.

The described two player linear-quadratic differential game is applied to positive differential system (1). We need some properties for nonnegative matrices and especially for M-matrices.

According to theory of nonnegative matrices the following allegations are equivalent for a given Z-matrix (-A):
(a) $(-A)$ is a nonsingular M-matrix;
(b) A is stable .

Lemma 1.[4]. *For a Z-matrix A, the following items are equivalent:*
(a) A is a nonsingular M-matrix;
(b) $A^{-1}\geq 0$*;*
(c) $A\,u>0$ *for some vector* $u>0$*;*
(d) All eigenvalues of A have positive real parts.

Lemma 2.[6]. *Let* $A=(a_{ij})\in\mathbb{R}^{m\times m}$ *be an M-matrix. If the elements of* $B=(b_{ij})\in\mathbb{R}^{m\times m}$ *satisfy the relations* $b_{ii}\geq a_{ii}\,,\ a_{ij}\leq b_{ij}\leq 0\,,\ \ i\neq j\,,\ i,j=1,\dots,m$ *then B also is an M-matrix.*

The paper is organized as follows. In section 2, we consider the linearized process to modify the Newton method to compute the feedback Nash equilibrium. The convergence proof is derived. In section 3, we slightly modify the introduced iteration to a game with an open loop design. In section 4, we present some numerical illustrations of the proposed iteration. Finally, we finish the paper with some conclusions.

2 Linearized iteration applied to a feedback equilibrium

We discuss how to compute the feedback Nash equilibrium. The Newton iteration is defined and investigated in [2,8] $(i=1,2)$:

$$-A^{(k)^T}X_i^{(k+1)}-X_i^{(k+1)}A^{(k)}+\sum_{j\neq i}\left[W_{ij}^{(k)}X_j^{(k+1)}+X_j^{(k+1)}W_{ij}^{(k)^T}\right]=Q_i^{(k)}\,, \tag{8}$$

where

$$\begin{aligned} A^{(k)}&=A-S_1X_1^{(k)}-S_2X_2^{(k)}\,,\\ W_{12}^{(k)}&=X_1^{(k)}S_2-X_2^{(k)}S_{12}\,,\\ W_{21}^{(k)}&=X_2^{(k)}S_1-X_1^{(k)}S_{21}\,,\\ Q_i^{(k)}&=Q_i+X_i^{(k)}S_iX_i^{(k)}+\sum_{j\neq i}[X_i^{(k)}S_jX_j^{(k)}+X_j^{(k)}S_jX_i^{(k)}]\,. \end{aligned} \tag{9}$$

The linearized process was effectively applied to construct iterative methods for solving the algebraic Riccati equation associated with M-matrices [7,9].

At each step of Newton iteration (8) it is necessary to find a solution of a Lyapunov matrix equation. We propose a linearized modification for the Newton method. We take $X_1^{(0)}=X_2^{(0)}=0$, and negative constant γ, and construct two matrix sequences $\{X_i^{(p)},Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 via:

$$\begin{aligned} F_1^{(p)}&=\gamma I_n+A-S_1X_1^{(p)}-S_2X_2^{(p)}\,,\\ T_1^{(p)}&=\gamma I_n-A^T+X_1^{(p)}S_1+X_2^{(p)}S_2\,,\\ Y_i^{(p)}F_1^{(p)}&=T_1^{(p)}X_i^{(p)}-Q_i(X_1^{(p)},X_2^{(p)}) \end{aligned} \tag{10}$$

$$\begin{aligned} F_2^{(p)}&=\gamma I_n+A^T-Y_1^{(p)}S_1-Y_2^{(p)}S_2\,,\\ T_2^{(p)}&=\gamma I_n-A+S_1Y_1^{(p)}+S_2Y_2^{(p)}\\ F_2^{(p)}X_i^{(p+1)}&=Y_i^{(p)}T_2^{(p)}-Q_i(Y_1^{(p)},Y_p^{(k)}) \end{aligned} \tag{11}$$

where

$$Q_i(Z_i,Z_j) = Q_i + Z_iS_iZ_i + Z_jS_{ij}Z_j\,,$$

with $(i,j=1,2; j\neq i)$.

We remark that the standard properties for the matrices of the above matrix sequences in the following Lemma:

Lemma 3. *The matrix sequences $\{X_i^{(p)},Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 are obtained applying iteration (10) - (11) with initial zero matrices $X_1^{(0)}=0\,,X_2^{(0)}=0$, and $\gamma<0$. Then, the following equalities are satisfied for $p=0,\ldots,\infty$:*

$$\begin{aligned}(i)\ (Y_i^{(p)}-X_i^{(p)})F_1^{(p)} &= (X_i^{(p)}-Y_i^{(p-1)})\,(\gamma I_n - A + S_1X_1^{(p)} + S_2X_2^{(p)})\\ &+(X_j^{(p)}-Y_j^{(p-1)})S_jX_i^{(p)} + Y_i^{(p)}S_j(X_j^{(p)}-Y_j^{(p-1)})\\ &-(X_j^{(p)}-Y_j^{(p-1)})S_{ij}X_i^{(p)} - Y_j^{(p)}S_{ij}(X_j^{(p)}-Y_j^{(p-1)})\,,\end{aligned}$$

$$\begin{aligned}(ii)F_2^{(p)}(X_i^{(p+1)}-Y_i^{(p)}) &= (\gamma I_n - A^T + Y_1^{(p)}S_1 + Y_2^{(p)}S_2)(Y_i^{(p)}-X_i^{(p)})\\ &+Y_i^{(p)}S_j(Y_j^{(p)}-X_j^{(p)}) + (Y_j^{(p)}-X_j^{(p)})\\ &\times S_jX_i^{(p)} + (X_j^{(p)}-Y_j^{(p)})S_{ij}X_i^{(p)} + Y_j^{(p)}S_{ij}(X_j^{(p)}-Y_j^{(p)})\,.\end{aligned}$$

Moreover, if the couple $(\tilde X_1,\tilde X_2)$ is an exact solution of (3)-(4) the identities can be verified (i=1,2):

$$\begin{aligned}(iii)\ (Y_i^{(p)}-\tilde X_i)F_1^{(p)} &= (\gamma I_n - A^T)(X_i^{(p)}-\tilde X_i) + \tilde X_iS_i(X_i^{(p)}-\tilde X_i)\\ &+\tilde X_iS_j(X_j^{(p)}-\tilde X_j) + (X_j^{(p)}-\tilde X_j)S_j\tilde X_i + \tilde X_jS_j(X_i^{(p)}-\tilde X_i)\\ &+(\tilde X_j - X_j^{(p)})S_{ij}\tilde X_j + X_j^{(p)}S_{ij}(\tilde X_j - X_j^{(p)})\,,\\ (iv)\ F_2^{(p)}(X_i^{(p+1)}-\tilde X_i) &= +(Y_i^{(p)}-\tilde X_i)(\gamma I_n - A)\\ &+(Y_i^{(p)}-\tilde X_i)S_i\tilde X_i + (Y_j^{(p)}-\tilde X_j)S_j\tilde X_j\\ &+(Y_i^{(p)}-\tilde X_i)S_j\tilde X_j + \tilde X_iS_j(Y_j^{(p)}-\tilde X_j)\\ &+(\tilde X_j - Y_j^{(p)})S_{ij}\tilde X_j + Y_j^{(p)}S_{ij}(\tilde X_j - Y_j^{(p)})\,.\end{aligned}$$

Based on the proved Lemma, we confirm the convergence of the proposed iteration (10) - (11) in the following Theorem:

Theorem 1. *Assume matrices A,S_1,S_2, and $Q_1\,,Q_2$ are coefficients of the set of matrix equations $\mathscr{R}_j(X_1,X_2)=0$, j=1,2. There exists a negative $\gamma<0$, such that $-(\gamma I_n + A)$ is an M-matrix and $\gamma I_n - A\le 0$.*

The sequences $\{X_i^{(p)},Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 obtained via (10) - (11) satisfy the properties:

(i) *$\tilde X_i \ge X_i^{(p+1)} \ge Y_i^{(p)} \ge X_i^{(p)}$ for $p=0,1,\ldots$, i=1,2 for any exact nonnegative solution $\tilde X_1,\tilde X_2$ of $\mathscr{R}_i(X_1,X_2)=0$, i=1,2. ;*

(ii) *The matrices $(-F_1^{(p)})$ and $(-F_2^{(p)})$ are M-matrices for any positive p.*

(iii) *The matrix sequences $\{X_i^{(p)},Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 converge to the stabilizing nonnegative solution $\hat X_1,\hat X_2$ to couple of Nash-Riccati equations (3)-(4).*

Proof. We provide a proof by mathematical induction on the number p of the iteration step. In the beginning, we prove theorem's statements for $p=0$. We take $X_1^{(0)}=X_2^{(0)}=0$, and construct the couple of sequences $\{X_i^{(p)},Y_i^{(p)}\}_{p=0}^{\infty}, i=1,2$ applying recursive equations (10) - (11) with $X_1^{(0)}=0\,,X_2^{(0)}=0$ and $\gamma<0$.

For $p=0$ we have $F_1^{(0)}=\gamma I_n + A$, i.e. $(\gamma I_n + A)^{-1}\le 0$. This means that $(-F_1^{(0)})$ and $(-F_2^{(0)})$ are M-matrices. and $Q_1(X_1^{(0)},X_2^{(0)})\ge 0$. Thus $Y_j^{(0)}\ge 0; Y_j^{(0)}\ge X_j^{(0)},\ j=1,2$.

In the second step, we formulate the inductive hypothesis, i.e. we assume that the statements are true for the a positive value of p. We assume that $X_i^{(p)}\ge Y_i^{(p-1)}\ge X_i^{(p-1)}\ge 0$ for some integer p and i=1,2. It is true that $X_i^{(p)}-Y_i^{(p-1)}\ge 0$, and $Y_i^{(p-1)}-X_i^{(p-1)}\ge 0$. In addition, $(-F_1^{(p)})$ and $(-F_2^{(p)})$ are M-matrices.

The next is the induction step, where we prove the statements for $p+1$. We have to prove inequalities $X_i^{(p+1)} \geq Y_i^{(p)} \geq X_i^{(p)} \geq 0$ and $(-F_1^{(p+1)})$ and $(-F_2^{(p+1)})$ are M-matrices.

Applying Lemma 3 (i), we get

$$Y_i^{(p)} - X_i^{(p)} = W_i^{(p)} \, (F_1^{(p)})^{-1} \, ,$$

where

$$\begin{aligned} W_i^{(p)} &:= (X_i^{(p)} - Y_i^{(p-1)})(\gamma I_n - A + S_1 X_1^{(p)} + S_2 X_2^{(p)}) \\ &+ (X_j^{(p)} - Y_j^{(p-1)}) S_j X_i^{(p)} + Y_i^{(p)} S_j (X_j^{(p)} - Y_j^{(p-1)}) - (X_j^{(p)} - Y_j^{(p-1)}) S_{ij} X_i^{(p)} \\ &- Y_j^{(p)} S_{ij} (X_i^{(p)} - Y_i^{(p-1)}) \, . \end{aligned}$$

We note the following $S_1 \leq 0, S_2 \leq 0, \gamma I_n - A \leq 0, S_{12} \geq 0, S_{21} \geq 0$. Thus $W_i^{(p)} \leq 0$. Therefore $Y_i^{(p)} - X_i^{(p)} \geq 0$, because $(F_1^{(p)})^{-1} \leq 0$ for $i = 1,2$.

Further on, according to Lemma 3 (ii) we have

$$X_i^{(p+1)} - Y_i^{(p)} = (F_2^{(p)})^{-1} \, H^{(p)} \, ,$$

where

$$\begin{aligned} H^{(p)} &:= (\gamma I_n - A^T + Y_1^{(p)} S_1 + Y_2^{(p)} S_2)\,(Y_i^{(p)} - X_i^{(p)}) + Y_i^{(p)} S_j (Y_j^{(p)} - X_j^{(p)}) \\ &+ (Y_j^{(p)} - X_j^{(p)}) S_j X_i^{(p)} + (X_j^{(p)} - Y_j^{(p)}) S_{ij} X_i^{(p)} + Y_j^{(p)} S_{ij} (X_i^{(p)} - Y_i^{(p)}) \, . \end{aligned}$$

With similar arguments we arrive at the conclusion $X_j^{(p+1)} - Y_j^{(p)} \geq 0 \, , \; j = 1,2$.

We compute (i=1,2) $X_i^{(p+1)}$ via (11) and $Y_i^{(p+1)}$ via (10). Consider the matrices $F_1^{(p+1)} = \gamma I_n + A - S_1 X_1^{(p+1)} - S_2 X_2^{(p+1)}$ and $F_2^{(p+1)} = \gamma I_n + A^T - Y_1^{(p+1)} S_1 - Y_2^{(p+1)} S_2$. According to Lemma 2 and properties $X_i^{(p+1)} \geq X_i^{(p)}$ and $Y_i^{(p+1)} \geq X_i^{(p+1)}, i = 1,2$ we derive the conclusion $(-F_1^{(p+1)})$ and $(-F_2^{(p+1)})$ are M-matrices and therefore $(F_1^{(p+1)})^{-1} \leq 0$ and $(F_2^{(p+1)})^{-1} \leq 0$.

Thus, the sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}, i = 1,2$ are monotone increasing. We have to prove that they are bonded above. Consider any exact nonnegative solution $(\tilde{X}_1, \tilde{X}_2)$ of $\mathscr{R}_j(X_1, X_2) = 0$, j=1,2. We shall prove that the solution is an upper bound of the matrix sequences.

For $p = 0$, we have $\tilde{X}_i \geq X_i^{(0)} = 0$, and according to Lemma 3 (iii)

$$(Y_i^{(0)} - \tilde{X}_i) F_1^{(0)} = -(\gamma I_n - A^T)\tilde{X}_i - \tilde{X}_i S_i \tilde{X}_i - \tilde{X}_i S_j \tilde{X}_j - \tilde{X}_j S_j \tilde{X}_i - \tilde{X}_j S_j \tilde{X}_i + \tilde{X}_j S_{ij} \tilde{X}_j \; \geq 0 \, ,$$

we infer $Y_i^{(0)} - \tilde{X}_i \leq 0, i = 1,2$.

Moreover, for $p > 0$ we have

$$\begin{aligned} (Y_i^{(p)} - \tilde{X}_i) F_1^{(p)} &= (\gamma I_n - A^T)(X_i^{(p)} - \tilde{X}_i) + \tilde{X}_i S_i (X_i^{(p)} - \tilde{X}_i) + \tilde{X}_i S_j (X_j^{(p)} - \tilde{X}_j) \\ &+ (X_j^{(p)} - \tilde{X}_j) S_j \tilde{X}_i + \tilde{X}_j S_j (X_i^{(p)} - \tilde{X}_i) + (\tilde{X}_j - X_j^{(p)}) S_{ij} \tilde{X}_j + X_j^{(p)} S_{ij} (\tilde{X}_j - X_j^{(p)}) \; \geq \; 0 \, , \end{aligned}$$

we have $Y_i^{(p)} - \tilde{X}_i \leq 0, i = 1,2$.

We evaluate the matrix difference $X_i^{(p+1)} - \tilde{X}_i, i = 1,2$. Applying Lemma 3 (iv) we obtain:

$$\begin{aligned} F_2^{(p)} (X_i^{(p+1)} - \tilde{X}_i) &= (Y_i^{(p)} - \tilde{X}_i)(\gamma I_n - A) + (Y_i^{(p)} - \tilde{X}_i) S_i \tilde{X}_i + (Y_j^{(p)} - \tilde{X}_j) S_j \tilde{X}_j \\ &+ (Y_i^{(p)} - \tilde{X}_i) S_j \tilde{X}_j + \tilde{X}_i S_j (Y_j^{(p)} - \tilde{X}_j) \\ &+ (\tilde{X}_j - Y_j^{(p)}) S_{ij} \tilde{X}_j + Y_j^{(p)} S_{ij} (\tilde{X}_j - Y_j^{(p)}) \; \geq \; 0 \, . \end{aligned}$$

Thus, $X_j^{(p+1)} - \tilde{X}_j \leq 0, j = 1,2$.

The matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 of nonnegative matrices converge to the couple of nonnegative matrices $(\hat{X}_1, \hat{X}_2)$. By taking the limits in (10) - (11) it follows that the couple of matrices is a solution to Nash-Riccati equations

(3)-(4). Moreover, the limit matrix has the property $\hat{X}_i \leq \tilde{X}_i, i = 1,2$ (in the element wise ordering). The matrix $-A + S_1\hat{X}_1 + S_2\hat{X}_2$ is an M-matrix because $(-F_1^{(p)})$ is an M-matrix for all positive p. Therefore, the matrix $A - S_1\hat{X}_1 + S_2\hat{X}_2$ is stable. The solution $(\hat{X}_1, \hat{X}_2)$ is a stabilizing one.

Corollary 1*The stabilizing solution $(\hat{X}_1, \hat{X}_2)$ of Nash-Riccati equations (3)-(4) derived in Theorem 1 is the minimal one to (3)-(4).*

3 Linearized iteration applied to a open loop design

In this section, we change iteration formula (10) - (11) to obtain a new iteration to compute the stabilizing solution of the set of Nash-Riccati equations in case of a game with open loop design. In formula (10) - (11), we change the matrices $Q_i(Z_i, Z_j)$, $i, j = 1,2; j \neq i$ as follows:

$$Q_i(Z_i, Z_j) = Q_i + Z_i S_i Z_i + Z_j S_j Z_i \,, \tag{12}$$

with $i, j = 1,2; j \neq i$.

Applying Theorem 1, we derive a proof for the convergence of iteration (10) - (11) in the next Theorem:

Theorem 2.*Assume matrices A, S_1, S_2, and Q_1, Q_2 are coefficients of the set of matrix equations $\mathscr{L}_i(X_1, X_2) = 0$, i=1,2 defined with (6) - (7). There exists negative $\gamma < 0$, such that $-(\gamma I_n + A)$ is an M-matrix and $\gamma I_n - A \leq 0$.*

The sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 constructed by (10) - (11) with $Q_i(Z_i, Z_j)$ defined in (12) fulfill the properties:

(i) *$\tilde{X}_i \geq X_i^{(p+1)} \geq Y_i^{(p)} \geq X_i^{(p)}$ for $p = 0,1,\ldots$, i=1,2 for any exact nonnegative solution $\tilde{X}_1, \tilde{X}_2$ of $\mathscr{L}_i(X_1, X_2) = 0$, i=1,2.;*

(ii) *The matrices $(-F_1^{(p)})$ and $(-F_2^{(p)})$ are M-matrices for any positive p.*

(iii) *The matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 converge to the stabilizing nonnegative solution $(\hat{X}_1, \hat{X}_2)$ to couple of Nash-Riccati equations $\mathscr{L}_i(X_1, X_2) = 0$, i=1,2. In fact the matrix $A - S_1\hat{X}_1 - S_2\hat{X}_2$ is stable.*

4 Results

In this section, we apply the proposed iterations to compute the stabilizing solution of the couple Nash-Riccati equations which help to find the Nash equilibrium point for the games with feedback information structure and the open loop information stricture. Experiments are provided with different matrix coefficients of Nash-Riccati equations (3)-(4) and (6)-(7). In addition, we present the comparative analysis between Newton method (8) and proposed linearized iterations in the considered two cases. All experiments are executed with MATLAB R2018b on a Laptop with 1.50 GHz Intel(R) Core(TM) and 8 GB RAM, running on Windows 10. The stop criterion for each iteration is $max\left(\|\mathscr{R}_1(X_1^{(k)}, X_2^{(k)})\|_2, \|\mathscr{R}_2(X_1^{(k)}, X_2^{(k)})\|_2\right) \leq tol$ or $max\left(\|\mathscr{L}_1(X_1^{(k)}, X_2^{(k)})\|_2, \|\mathscr{L}_2(X_1^{(k)}, X_2^{(k)})\|_2\right) \leq tol$, where $\|.\|_2$ is the spectral matrix norm and $tol = 0.1e-10$.

Moreover, the property of symmetry for matrices S_1, S_2 give us possibility to improve the computational scheme of iteration (10)-(11) in order to decrease the computations for each iteration step and accelerate the algorithm based on (10)-(11).

*Example 1.*Consider the matrix coefficients of system (1) and cost functions J_1, J_2:

$$A = \begin{pmatrix} -4 & 1 & 1 & 0.5 \\ 1 & -5 & 0.8 & 1 \\ 1 & 1 & -4 & 1 \\ 0.9 & 1 & 2 & -6 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 5 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0.8 & 1 & 0 & 0.2 \\ 0.3 & 1 & 1 & 0 \\ 0.6 & 0 & 0 & 1 \end{pmatrix},$$

Q_1 = diag [5; 0; 0.5; 3], Q_2 = diag [50; 4; 5; 0], $R_{11} = -90 \in \mathbf{R}^{1\times 1}$; $R_{21} = 200 \in \mathbf{R}^{1\times 1}$, R_{12} = diag [400; 200; 500; 300], and

$$R_{22} = \begin{pmatrix} -400 & 0 & 0 & -10 \\ 0 & -100 & 0 & 0 \\ 0 & 0 & -200 & 0 \\ -10 & 0 & 0 & -400 \end{pmatrix}.$$

We compute the matrix coefficients $S_1 \leq 0, S_2 \leq 0, S_{12} \geq 0$, and $S_{21} \geq 0$.

	The proposed iteration (10)-(11)	
γ	It	CPU time seconds
-10	219	0.55
-5	118	0.30
-1	73	0.18
-1.25	42	0.105
-0.5		no result

Table 1: Results for Example 2 with $tol = .1e-10$.

To compute the stabilizing solution of the couple of Nash-Riccati equations (3)-(4) we apply linearized iteration (10) - (11). After 118 iteration steps with $\gamma = -5$ we obtain the stabilizing solution $(\hat{X}_1, \hat{X}_2)$. The matrices are nonnegative and symmetric:

$$\hat{X}_1 = \begin{pmatrix} 1.1055 & 0.3416 & 0.4615 & 0.2485 \\ 0.3416 & 0.1767 & 0.2514 & 0.1449 \\ 0.4615 & 0.2514 & 0.4074 & 0.2194 \\ 0.2485 & 0.1449 & 0.2194 & 0.3415 \end{pmatrix}, \quad \hat{X}_2 = \begin{pmatrix} 8.8338 & 2.0633 & 2.6256 & 1.2793 \\ 2.0633 & 1.2650 & 1.2269 & 0.5980 \\ 2.6256 & 1.2269 & 2.1363 & 0.8011 \\ 1.2793 & 0.5980 & 0.8011 & 0.3741 \end{pmatrix}.$$

The closed loop matrix $A - S_1\hat{X}_1 - S_2\hat{X}_2$ has the eigenvalues
$-1.1274, -4.7929, -5.6682, -6.8395$.

To compute the stabilizing solution of the couple of Nash-Riccati equations (6)-(7) we apply linearized iteration (10) - (11) with the matrices $Q_i(Z_i, Z_j)$, $i, j = 1, 2; j \neq i$ defined by (12). Matrices $\hat{X}_1$, and $\hat{X}_2$ are nonnegative and nonsymmetric:

$$\hat{X}_1 = \begin{pmatrix} 0.7775 & 0.1567 & 0.2067 & 0.1260 \\ 0.1569 & 0.0678 & 0.1007 & 0.0726 \\ 0.2079 & 0.1011 & 0.1990 & 0.1195 \\ 0.1279 & 0.0734 & 0.1202 & 0.2938 \end{pmatrix}, \quad \hat{X}_2 = \begin{pmatrix} 7.5325 & 1.4131 & 1.7341 & 0.8439 \\ 1.4048 & 0.9198 & 0.7526 & 0.3661 \\ 1.7156 & 0.7499 & 1.4809 & 0.4807 \\ 0.8248 & 0.3617 & 0.4765 & 0.2154 \end{pmatrix}.$$

The closed loop matrix has the eigenvalues $-1.3124, -4.8023, -5.6708, -6.8395$.

*Example 2.*Consider the same matrix coefficients as in Example 1. We compare the Newton iteration and the proposed linearized iteration to compute the stabilizing solution of (3)-(4).

The Newton method computes the solution for 6 iteration steps and CPU time of 0.21 seconds for 100 runs. Results from experiments with proposed iteration are given in Table 1. The execution CPU time for 100 runs is given. The convergence of the proposed method is proved in Theorem 1. The proposed iteration executes smaller number of iteration steps (It=42) for $\gamma = -1.25$. For this value of γ the proposed method is faster than Newton method which has a quadratic convergence rate. In addition, the method does not converge for $\gamma = -0.5$. Weakness of the method that one has to find a properly value of a which gives speed of the method. In addition, we check the conditions of Theorem 1 for choosing values of γ.

*Example 3.*Define the matrix coefficients of system (1) and cost functions J_1, J_2 as follows (n=8).

$$A_0 = \begin{pmatrix} -24 & 0 & 0 & 2 \\ 20 & -25 & 0 & 0 \\ 0 & 16 & -25 & 0 \\ 1.5 & 0 & 18 & -24 \end{pmatrix}, \quad B_{10} = \begin{pmatrix} 0.7 \\ 0.9 \\ 0.9 \\ 0.8 \end{pmatrix}, \quad B_{20} = \begin{pmatrix} 2.8 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 1.5 \\ 0 & 0 & 3 & 8 \end{pmatrix},$$

$$\begin{aligned} & A = diag[A_0, A_0], \quad B_1 = diag[B_{10}, B_{10}], \quad B_2 = diag[B_{20}, B_{20}], \\ & R_{11} = -1.9 \in \mathbf{R}^{1\times 1}, \quad R_{21} = 20 \in \mathbf{R}^{1\times 1}, \\ & v = diag[40, 30, 20, 10, 40, 60, 70, 80], \quad R_{12} = diag[v], \\ & R_{22} = diag[-150, -1, ..., -1, -120] \in \mathbf{R}^{n\times n}, \\ & Q_1 = diag[4, 1, ..., 1, 1.5] \in \mathbf{R}^{n\times n}, \\ & Q_2 = 0.5\, Q_1 . \end{aligned}$$

	The proposed iteration (10)-(11)	
γ	It	CPU time seconds
-20	101	0.32
-16	88	0.297
-15	85	0.292
-12	75	0.254

Table 2: Results for Example 3 with $tol = .1e-10$.

We compare the Newton iteration and the proposed linearized iteration to compute the stabilizing solution to set of matrix equations (3)-(4). Results are given in Table 2 The execution CPU time for 100 runs is given. The Newton method computes the stabilizing solution of (3)-(4) for 7 iteration steps and CPU time of 0.97 seconds for 100 runs.

*Example 4.*Define the matrix coefficients of system (1) and cost functions J_1, J_2 as follows (n=16) using notations from Example 3;
$A = 2diag[A_0, A_0, A_0, A_0]$,
$B_1 = diag[B_{10}, B_{10}, B_{10}, B_{10}]$,
$B_2 = diag[B_{20}, B_{20}, B_{20}, B_{20}]$, $R_{12} = diag[v, v]$.

The Newton method computes the stabilizing solution of (3)-(4) for 4 iteration steps and CPU time of 1.774 seconds for 10 runs. Moreover, new iteration (10)-(11) finds the stabilizing solution for 22 iteration steps, CPU time of 0.047 seconds for 10 runs and $\gamma = -20$.

5 Conclusion

The computation of the stabilizing solution of the Nash-Riccati equations is important for applications. In this paper, we applied a linearized process to modify Newton's method to compute the stabilizing solution for a set of Nash-Riccati equations. Moreover, we have proposed fast iterative methods to find this solution. Here, we were presented a convergence proof to effective iteration scheme (10)- (11). The computational simplicity of the algorithm leads to the efficiency of the proposed iteration and it makes the new iteration an alternative method for computing the stabilizing solution. Related discussions are expected to lead to new computational algorithms to similar problems. Based on the considered examples we may conclude that the proposed iteration is an effective solver for these examples. As a future research the linearized process may be extended to construct a new iteration to find the Nash equilibrium strategies of an N-player infinite horizon linear quadratic differential game.

Declarations

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