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Uncertainty Inequalities for Continuous Laguerre Wavelet Transform

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Abstract: This paper deals with the Continuous Laguerre Wavelet Transform CLWT, and we prove several versions of the uncertainty inequalities. More precisely, we get the analogue of Heisenberg inequality for CLWT. Moreover, dealing with concentration in time and frequency, we find an L^p local type uncertainty principle. Finally, we provide the analogue of Benedicks Amrein Berthier's type theorem in the case of CLWT.

Keywords: Continuous Laguerre wavelet transform; Heisenberg inequality; uncertainty principle; Time frequency concentration theorems; local uncertainty principle.

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1 Introduction

The theory of wavelets and continuous wavelet transforms has garnered increased interest due to the limitations of Fourier transform in providing complete information about a signal. In particular, Fourier transform can not be a suitable tool for non stationary signals, in which frequency changes with respect to time. Hence appears the importance of the wavelet and the continuous wavelet transform CWT. For an overview of CWT, we refer the reader to [5,27]. Motivated by the works of [31,14,1], we consider in this paper, time-frequency localization problems in the case of continuous Laguerre wavelet transform CLWT. The interest of studing Laguerre transform comes from Heisenberg group which replace the euclidean space in quantum mechanics. Roughly speaking, Fourier Laguerre transform is non other than the Fourier transform of radial functions in this occurence. Studying the uncertainty principle for $\mathscr{F}_L$ was subject of several works by the authors and many more, one can cite for instance [9,10,20,22,26]. However studying the uncertainty principle for CLWT still less aborded. Note that the harmonic analysis associated to CLWT was initiated in [23], where the Plancherel and the inversion formulas were established for CLWT. Recently Mejjaoli and Trimèche in [16,15] considered such problems in the case of two-wavelets in Laguerre occurence. In this paper, we improve the litterature by giving uncertainty inequalities for CLWT.

The uncertainty principle is one of the most interesting result which gives us an overview on the positioning of a function and its Fourier transform. This principle states, in quantum mechanics, that an observer cannot determine simultaneously the values of position and momentum of a quantum particule with precision. A precise quantitative formulation of the uncertainty principle, usually called Heisenberg inequality [11,30] is stated for $f \in L^2(\mathbb{R})$, as follows:

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \ . \ \int_{\mathbb{R}} \xi^2 \left|\widehat{f}(\xi)\right|^2 d\xi \geq \ \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2, \tag{1}$$

where $\widehat{f}$ is the Fourier transform, given for suitable functions by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\xi x} dx.$$

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Another version of the uncertainty principle concems with concentration of f and its Fourier transform. We reference two results: the first one was studied by Faris [17] and Price [18,19] in the classical Fourier setting, known as the local uncertainty principle. The second one goes to Benedicks and Amrein-Berthier. Benedicks [3] first introduced this theorem, stating that if a function f has a subset S of finite measure as its support, and its Fourier transform $\hat{f}$ has a subset Σ of finite measure as its support, then f must be the null function. A stronger formulation of this principle was provided by Amrein and Berthier in [2] for the classical Fourier occurence. In this paper we prove the analogue of all previous uncertainty principle theorems when considering the CLWT.

Our paper is structured as follows.

In section 2, we start by giving some useful background evoking Laguerre hypergroup $\mathbb{K}$ and Fourier Laguerre transform $\mathscr{F}_L$. Section 3 summarizes key facts about basic Laguerre wavelet theory. Section 4 is devoted to our main results. First, we prove Heisenberg-type uncertainty inequalities, analogous of inequality (1), considering the product of dispersions with both position and scale as variables, for the CLWT. Second, we prove two theorems dealing with concentration in the support of a given function and its CLWT. The first is a local uncertainty principle and the second deals with a Benedicks-Amrein-Berthier's uncertainty principle.

2 Laguerre hypergroup and Fourier Laguerre transform

Laguerre hypergroup emerges as the fundamental manifold of the radial function space in the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$, where the multiplication operator is given by

$$(z_1,t_1).(z_2,t_2) = (z_1+z_2, t_1+t_2-Im(z_1 z_2)).$$

A function f on $\mathbb{H}^n$ is considered radial if it remains invariant under the action of the unitary group $\mathscr{U}(n)$ via $u.(z,t) = (u.z,t)$. For additional details we refere the reader to [6,28,29]. Let $\alpha \geq 0$. The Laguerre hypergroup $\mathbb{K} = [0,+\infty)\times\mathbb{R}$ is equiped with the convolution product $*_\alpha$. This product is defined for two bounded Radon measures μ_1 and μ_2 on $\mathbb{K}$ as:

$$< \mu_1 *_\alpha \mu_2, f >= \int_{\mathbb{K}\times\mathbb{K}} T^\alpha_{x,t} f(y,s)\, d\mu_1 d\mu_2,$$

where $T^\alpha_{x,t}$ is the generalized translation operator on $\mathbb{K}$ given, for $\alpha = 0$, by

$$T^\alpha_{x,t} f(y,s) = \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta}, t+s+xy\sin\theta)\, d\theta \tag{2}$$

and, for $\alpha > 0$, by

$$T^\alpha_{x,t} f(y,s) = \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f(\sqrt{x^2+y^2+2xyr\cos\theta}, t+s+xyr\sin\theta) r(1-r^2)^{\alpha-1} dr d\theta. \tag{3}$$

Remark that if μ_1 and μ_2 are equal to Dirac measure at (x,t) and $(y,s)\in\mathbb{K}$ then

$$(\delta_{(x,t)} *_\alpha \delta_{(y,s)})(f) = T^\alpha_{x,t} f(y,s).$$

We find in [23] that $(\mathbb{K}, *_\alpha)$ has a commutative hypergroup structure in the sense of Jewett. The involution is defined by the homeomorphism $i(x,t) = (x,-t)$ and the Haar measure is given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi\Gamma(\alpha+1)} dx dt. \tag{4}$$

$e = (0,0)$ is the unit element of $(\mathbb{K}, *_\alpha)$ since $\delta_{(x,t)} *_\alpha \delta_{(0,0)} = \delta_{(0,0)} *_\alpha \delta_{(x,t)} = \delta_{(x,t)}$. In the case of Laguerre hypergroup, the dual space, the space of all bounded functions $\chi : \mathbb{K}\to\mathbb{C}$ satisfying for $(x,t)\in\mathbb{K}$, $\tilde{\chi}(x,t) = \overline{\chi(x,-t)} = \chi(x,t)$, is described by

$$\{\varphi_{\lambda,m};\, (\lambda,m)\in\mathbb{R}^*\times\mathbb{N}\}\cup\{\varphi_\rho;\, \rho\geq 0\},$$

where

$$\varphi_\rho = j_\alpha(\rho x) \qquad \text{and} \qquad \varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathscr{L}^{(\alpha)}_m(|\lambda| x^2). \tag{5}$$

Note that j_α is the normalized Bessel function of order α and $\mathscr{L}^{(\alpha)}_m$ is the Laguerre function given on $\mathbb{R}_+$ by

$$\mathscr{L}^{(\alpha)}_m(x) = e^{-\frac{x}{2}}\frac{L^\alpha_m(x)}{L^\alpha_m(0)}, \tag{6}$$

where L_m^α is the Laguerre polynomial of order α and degree m,

$$L_m^\alpha(x) = \sum_{k=0}^{m} (-1)^k \frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{1}{k!(m-k)!} x^k. \tag{7}$$

Topologically, the dual space can be identified to the Heisenberg fan, the set

$$\bigcup_{m\in\mathbb{N}} \{(\lambda,\mu)\in\mathbb{R}^2; \mu = |\lambda|(2m+\alpha+1)\} \bigcup \{(0,\mu)\in\mathbb{R}^2; \mu \geq 0\}.$$

The subset $\{(0,\mu)\in\mathbb{R}^2; \mu \geq 0\}$ is usually disregarded since it has zero Plancherel measure. Therefore, it is natural to concentrate on the characters $\varphi_{\lambda,m}$. For $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$, $\varphi_{\lambda,m}$ is the unique solution to the problem

$$\begin{cases} D_1 u = i\lambda\, u, \\ D_2\, u = -4|\lambda|(m+\dfrac{\alpha+1}{2})u, \end{cases} \tag{8}$$

with the initial condition

$$u(0,0) = 1,\ \frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{for all } t\in\mathbb{R},$$

where, for all $\alpha \geq 0$,

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t} \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2 \dfrac{\partial^2}{\partial t^2}. \end{cases} \tag{9}$$

For $(\lambda,m)\in\hat{\mathbb{K}} = \mathbb{R}\times\mathbb{N}$, the function $\varphi_{\lambda,m}$ satisfies, for all $(x,t),(y,s)\in\mathbb{K}$,

$$\varphi_{\lambda,m}(x,t)\,\varphi_{\lambda,m}(y,s) = T_{x,t}^\alpha\,\varphi_{\lambda,m}(y,s). \tag{10}$$

Furthermore, the Laguerre kernel is bounded function, and we have

$$\forall(\lambda,m)\in\hat{\mathbb{K}}, \quad \sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1.$$

Denote $L^p(\mathbb{K}) = L^p(\mathbb{K}, dm_\alpha)$ the space of measurable functions f satisfying

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(x,t)|^p dm_\alpha(x,t)\right)^{\frac{1}{p}} < +\infty.$$

The Fourier Laguerre transform of a function f in $L^1(\mathbb{K})$ is defined by

$$\mathscr{F}_L f(\lambda,m) = \int_{\mathbb{K}} f(x,t)\,\varphi_{-\lambda,m}(x,t) dm_\alpha(x,t). \tag{11}$$

The $\mathscr{F}_L$ is bounded operator from $L^1(\mathbb{K})$ to $L^\infty(\hat{\mathbb{K}})$ and it satisfies $\|\mathscr{F}_L f\|_\infty \leq \|f\|_{1,m_\alpha}$. Moreover, the Fourier Laguerre transform can be inverted by

$$\mathscr{F}_L^{-1} f(x,t) = \int_{\hat{\mathbb{K}}} f(\lambda,m)\,\varphi_{\lambda,m}(x,t) d\gamma_\alpha(\lambda,m), \tag{12}$$

where $d\gamma_\alpha$ is the unique positive Radon measure on $\hat{\mathbb{K}}$ for which the Fourier Laguerre transform becomes an L^2-isometry. This measure is given by

$$d\gamma_\alpha(\lambda,m) = L_m^\alpha(0)\delta_m \otimes |\lambda|^{\alpha+1} d\lambda. \tag{13}$$

To simplify we will denote, when needed, $d\gamma_\alpha$ to state $d\gamma_\alpha(\lambda,m)$. $\mathscr{F}_L$ transform satisfies the following Plancherel Formula

$$\|\mathscr{F}_L f\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}, \tag{14}$$

where

$$\|g\|_{p,\gamma_\alpha} = \left(\int_{\hat{\mathbb{K}}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}} < +\infty.$$

By Riesz Thorin interpolation, we can expand the definition of $\mathscr{F}_L f$ on $L^p(\mathbb{K})$ for $1 \leq p \leq 2$. Consequently, we obtain the Hausdorff-Young inequality, for $\frac{1}{p}+\frac{1}{p'}=1$,

$$\|\mathscr{F}_L f\|_{p',\gamma_\alpha} \leq \|f\|_{p,m_\alpha}. \tag{15}$$

If $f \in L^p(\mathbb{K})$ then, for all $(x,t) \in \mathbb{K}$, $T^\alpha_{x,t} f \in L^p(\mathbb{K})$ and verifies

$$\|T^\alpha_{x,t} f\|_{p,m_\alpha} \leq \|f\|_{p,m_\alpha}. \tag{16}$$

Moreover

$$\mathscr{F}_L(T^\alpha_{x,t} f)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathscr{F}_L f(\lambda,m). \tag{17}$$

The generalized convolution product of two functions f and g in $L^1(\mathbb{K})$ is defined by

$$f \star_\alpha g(x,t) = \int_{\mathbb{K}} T^\alpha_{x,t} f(y,s).g(y,-s)dm_\alpha(y,s), \quad (x,t) \in \mathbb{K}. \tag{18}$$

Young's inequality allows to extend the definition of $\star_\alpha$ to $L^p(\mathbb{K}) \times L^q(\mathbb{K})$, where $p,q,r \geq 1$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$. For $f \in L^p(\mathbb{K})$ and $g \in L^q(\mathbb{K})$, where $1 \leq p,q,r \leq 2$ with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$, we get

$$\|f \star_\alpha g\|_{r,m_\alpha} \leq \|f\|_{p,m_\alpha}\|g\|_{q,m_\alpha}, \tag{19}$$

and

$$\mathscr{F}_L(f \star_\alpha g) = \mathscr{F}_L(f)\mathscr{F}_L(g). \tag{20}$$

3 Basic Laguerre wavelet theory

In this section, we gather some background related to CLWT. First and foremost, we shall adapt the definition of the dilation operator in order to get formulas that can be compared to the classical Fourier Wavelets. We consider as in [21, 22] the dilated of $(x,t) \in \mathbb{K}$ defined by $\delta_r(x,t) = (rx, r^2 t)$. For $f_r(x,t) = r^{-(2\alpha+4)} f(\delta_{\frac{1}{r}}(x,t))$, we have

$$\int_{\mathbb{K}} f_r(x,t)\, dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t). \tag{21}$$

We define, for $a>0$, the **dilation operator** Δ_a by

$$\Delta_a \psi(x,t) = \frac{1}{a^{\alpha+2}}\psi(\frac{x}{a},\frac{t}{a^2}) = \frac{1}{a^{\alpha+2}}\psi(\delta_{\frac{1}{a}}(x,t)). \tag{22}$$

We can easily deduce the following properties.

Proposition 1. *Let $a>0$, we have*

1. For all $a,b>0$ $\quad \Delta_a\Delta_b = \Delta_{ab}$.
2. For all $\psi \in L^2(\mathbb{K})$, the function $\Delta_a(\psi)$ belongs to $L^2(\mathbb{K})$ and satisfies

$$\|\Delta_a \psi\|_{2,m_\alpha} = \|\psi\|_{2,m_\alpha}. \tag{23}$$

3. For all $\psi \in L^2(\mathbb{K})$, the Fourier Laguerre of $\Delta_a(\psi)$ is well defined and we have

$$\mathscr{F}_L \Delta_a \psi = \hat{\Delta}_{\frac{1}{a}} \mathscr{F}_L \psi, \tag{24}$$

where

$$\hat{\Delta}_a f(\lambda,m) = a^{-(\alpha+2)} f(\delta'_{\frac{1}{a}}(\lambda,m)),$$

and $\delta'_r(\lambda,m) = (r^2\lambda, m)$ is the dilated of $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$.
4. Let $h,g \in L^2(\mathbb{K})$, we have

$$< \Delta_a h, g >_{L^2(\mathbb{K})} = < h, \Delta_{\frac{1}{a}} g >_{L^2(\mathbb{K})}.$$

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5.*For all $a>0$ and $(x,t)\in\mathbb{K}$, $\Delta_a T^{\alpha}_{x,t}=T_{\delta_a}(x,t)\Delta_a$, where $T^{\alpha}_{x,t}$ is the translation operator associated to Laguerre hypergroup given by (2) and (3).*

*Proof.*1. For all $a,b>0$,

$$\Delta_a\Delta_b\psi(x,t)=\Delta_a\left(\frac{1}{b^{\alpha+2}}\psi\left(\frac{x}{b},\frac{t}{b^2}\right)\right)=\frac{1}{(ab)^{\alpha+2}}\psi\left(\frac{x}{ab},\frac{t}{(ab)^2}\right)=\Delta_{ab}\psi(x,t).$$

2. The result is obvious by considering the substitutions $y=\frac{x}{a}$ and $u=\frac{t}{a^2}$.
3. Considering $y=\frac{x}{a}$ and $u=\frac{t}{a^2}$ in (11), we get

$$\mathscr{F}_L\Delta_a f(\lambda,m)=\int_{\mathbb{K}}f(y,u)\,\varphi_{-\lambda,m}(ay,a^2u)\,a^{\alpha+2}\,dm_\alpha(y,u).$$

Now using (5), we observe that $a^{\alpha+2}\varphi_{-\lambda,m}(ay,a^2u)=\varphi_{-a^2\lambda,m}(y,u)$, which gives the wanted result.
4. By the same change of variables, we obtain

$$<\Delta_a h,g>=a^{\alpha+2}\int_{\mathbb{K}}h(y,u)g(ay,a^2u)dm_\alpha(y,u).$$

Hence, the result holds from (22).
5. The last point follows by remarking, in (2) and (3), that

$$f\left(\sqrt{x^2+\left(\frac{y}{a}\right)^2+2x\frac{y}{a}r\cos\theta},\frac{t}{a^2}+s+x\frac{y}{a}r\sin\theta\right)=f\left(\frac{\sqrt{(ax)^2+y^2+2(ax)yr\cos\theta}}{a},\frac{t+a^2s+(ax)yr\sin\theta}{a^2}\right).$$

Definition 1.*Let $\psi\in L^2(\mathbb{K})$. We say that ψ is an admissible Laguerre wavelet on $\mathbb{K}$ if there exists a constant c_ψ satisfying, for all $m\in\mathbb{N}$ and $\lambda\in\mathbb{R}$,*

$$0<c_\psi=\int_0^{+\infty}|\mathscr{F}_L\psi(\delta'_a(\lambda,m)|^2\frac{da}{a}<+\infty. \tag{25}$$

According to [23], such admissible wavelet in Laguerre hypergroup exists. For instance, we cite the following function in $L^2(\mathbb{K})$: $\psi=\mathscr{F}_L^{-1}(\Theta)$, where

$$\forall(\lambda,m)\in\hat{\mathbb{K}},\quad \Theta(\lambda,m)=\lambda(m+\frac{\alpha+1}{2})e^{-\lambda^2(m+\frac{\alpha+1}{2})^2}. \tag{26}$$

Now, let ψ be a Laguerre wavelet on $\mathbb{K}$ in $L^2(\mathbb{K})$. We consider the family $\psi^{a,x,t}$, of Laguerre wavelets on $\mathbb{K}$, defined by

$$\forall(x',t')\in\mathbb{K},\quad \psi^{a,x,t}(x',t')=T^{\alpha}_{x,t}(\Delta_a\psi(x',-t')). \tag{27}$$

By virtue of (16) and (23), we get immediately, for all $a>0$ and $(x,t)\in\mathbb{K}$,

$$\|\psi^{a,x,t}\|_{2,m_\alpha}\leq\|\psi\|_{2,m_\alpha}. \tag{28}$$

Definition 2.*The continuous Laguerre wavelet transform CLWT, W^L_ψ is defined for a regular function f on $\mathbb{K}$ by*

$$\forall(a,x,t)\in(0,+\infty)\times\mathbb{K},\quad W^L_\psi f(a,x,t)=\int_{\mathbb{K}}f(x',t')\overline{\psi^{a,x,t}(x',t')}dm_\alpha(x',t'). \tag{29}$$

We can also write

$$W^L_\psi f(a,x,t)=<f,\psi^{a,x,t}>_{L^2(\mathbb{K})}=<f,T^{\alpha}_{x,t}\Delta_a\psi>_{L^2(\mathbb{K})}. \tag{30}$$

Moreover, relation (29) can be written as:

$$W^L_\psi f(a,x,t)=f\star_\alpha\overline{\Delta_a\psi}(x,t). \tag{31}$$

By Young's inequality, the CLWT can be defined for a function $f \in L^p(\mathbb{K})$, where $p \in [1,+\infty]$, and an admissible wavelet $\psi \in L^{p'}(\mathbb{K})$, where $p' = \frac{p}{p-1}$. Consequently, for all $(a,x,t) \in (0,+\infty) \times \mathbb{K}$,

$$|W^L_\psi f(a,x,t)| \leq a^{\frac{2\alpha+4}{p'}-(\alpha+2)} \|\psi\|_{p',m_\alpha} \, \|f\|_{p,m_\alpha}. \tag{32}$$

Let $\mathbb{U} = (0,+\infty) \times \mathbb{K}$. For $p \geq 1$, we equiped this space by the "affine" measure

$$d\nu_\alpha(a,x,t) = \frac{da\, dm_\alpha(x,t)}{a^{2\alpha+5}}. \tag{33}$$

Denote by $L^p(\mathbb{U})$ the space of measurable functions f on $\mathbb{U}$ that satisfies

$$\|f\|_{p,\nu_\alpha} = \left(\int_0^{+\infty} \int_{\mathbb{K}} |f(a,x,t)|^p d\nu_\alpha(a,x,t) \right)^{\frac{1}{p}} < +\infty. \tag{34}$$

According to (31) we assert that if ψ is an admissible Laguerre wavelet on $\mathbb{K}$, and $f \in L^2(\mathbb{K})$ then the following Plancherel's formula for CLWT holds.

$$\|W^L_\psi f\|^2_{2,\nu_\alpha} = c_\psi \|f\|^2_{2,m_\alpha}. \tag{35}$$

Furthermore, we can deduce the following Parseval's relation for f and g in $L^2(\mathbb{K})$,

$$c_\psi < f,g >_{L^2(\mathbb{K})} = \int_0^{+\infty} \int_{\mathbb{K}} W^L_\psi f(a,x,t) \overline{W^L_\psi g(a,x,t)}\, d\nu_\alpha(a,x,t). \tag{36}$$

According to (32) and (35), we derive from Riesz Thorin interpolation theorem that the definition of CLWT can be extended to $L^p(\mathbb{K})$ when $1 < p < 2$. We get that $W^L_\psi f$ belongs to $L^{p'}(\mathbb{U})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\|W^L_\psi f\|_{p',\nu_\alpha} \leq c_\psi^{\frac{1}{p'}} \left(a^{-(\alpha+2)} \|\psi\|_{\infty,m_\alpha} \right)^{1-\frac{2}{p'}} \|f\|_{p,m_\alpha}. \tag{37}$$

4 Main results : CLWT uncertainty inequalities

We shall introduce the following notations. For all $(x,t) \in \mathbb{K}$, the homogeneous norm on $\mathbb{K}$ is given by

$$|(x,t)| = |(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{\frac{1}{4}}. \tag{38}$$

$\mathbb{R} \times \mathbb{N}$ is equiped with the quasinorm defined, for all $(\lambda,m) \in \mathbb{R} \times \mathbb{N}$, by

$$|(\lambda,m)| = 4|\lambda|(m + \frac{\alpha+1}{2}). \tag{39}$$

4.1 Heisenberg type inequalites for CLWT

From [9,26], the Heinsenberg inequality for $\mathscr{F}_L$ states that for $b \geq 1$ and $f \in L^2(\mathbb{K})$,

$$\| \, |(x,t)|^b f\|_{2,m_\alpha} . \| \, |(\lambda,m)|^{\frac{b}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \geq C \, \|f\|^2_{2,m_\alpha}. \tag{40}$$

In the case of CLWT, Heisenberg type inequality dealing with dispersion on position (x,t), is given by the following theorem.

Theorem 1. *Let ψ be an admissible Laguerre Wavelet on $\mathbb{K}$ and $b \geq 1$. Then for all $f \in L^2(\mathbb{K})$,*

$$\| |(x,t)|^b W^L_\psi f\|_{2,\nu_\alpha} . \| |(\lambda,m)|^{\frac{b}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \geq C \sqrt{c_\psi} \|f\|^2_{2,m_\alpha}, \tag{41}$$

where C is the same constant given in (40).

*Proof.*By virtue of relations (31) and (20), we have

$$|\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 = |\mathscr{F}_L f(\lambda,m)|^2 |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2.$$

Relation (24) and the admissible condition (25) lead to

$$\int_0^{+\infty} |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2 \frac{da}{a^{2\alpha+5}} = c_\psi. \tag{42}$$

Therefore, using Fubini's theorem, we get

$$\int_0^{+\infty} \int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 d\gamma_\alpha \frac{da}{a^{2\alpha+5}} = c_\psi \int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha.$$

On the other hand, since f belongs to $L^2(\mathbb{K})$ then we deduce that the function $W_\psi^L f(a,.,.)$ belongs to $L^2(\mathbb{K})$. Applying Heinsenberg type inequality (40) to $W_\psi^L f(a,.,.)$, we get, for all $a \in (0,+\infty)$,

$$\left(\int_{\mathbb{K}} |(x,t)|^{2b} |W_\psi^L f(a,x,t)|^2 dm_\alpha(x,t)\right)^{\frac{1}{2}} \left(\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2}} \geq C \int_{\mathbb{K}} |W_\psi^L f(a,x,t)|^2 dm_\alpha(x,t).$$

Integrating with respect to $\frac{da}{a^{2\alpha+5}}$, the left hand side is given by

$$\sqrt{c_\psi} \left(\int_{\mathbb{U}} |(x,t)|^{2b} |W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t)\right)^{\frac{1}{2}} \left(\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2}},$$

and the right hand side is written as multiple of

$$\int_0^{+\infty} \int_{\mathbb{K}} |W_\psi^L f(a,x,t)|^2 dm_\alpha(x,t) \frac{da}{a^{2\alpha+5}}.$$

Using Plancherel formula, this integral equals to

$$X = \int_0^{+\infty} \int_{\hat{\mathbb{K}}} |\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \frac{da}{a^{2\alpha+5}}.$$

Therefore, relation (42) leads to

$$\begin{aligned} X &= \int_0^{+\infty} \int_{\hat{\mathbb{K}}} |\mathscr{F}_L f(\lambda,m)|^2 |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \frac{da}{a^{2\alpha+5}} \\ &= c_\psi \int_{\hat{\mathbb{K}}} |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\ &= c_\psi \|f\|_{2,m_\alpha}^2. \end{aligned}$$

Consequently

$$\sqrt{c_\psi} \left(\int_{\mathbb{U}} |(x,t)|^{2b} |W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t)\right)^{\frac{1}{2}} \left(\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha\right)^{\frac{1}{2}} \geq C c_\psi \|f\|_{2,m_\alpha}^2.$$

which allows to deduce inequality (41).

As an application, we proceed in similar way as in [1], we deduce the following result:

Corollary 1.*For all $s,\beta \geq 1$ and for all $f \in L^2(\mathbb{K})$, we have*

$$\||(x,t)|^s W_\psi^L f\|_{2,\nu_\alpha}^\beta . \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^s \geq C(\sqrt{c_\psi})^{1-\beta(s-1)} \|f\|_{2,m_\alpha}^{s+\beta}. \tag{43}$$

*Proof.*Let $s,\beta > 1$. For $f \in L^2(\mathbb{K})$, assume that

$$\||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^\beta,\ \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^s < +\infty.$$

Applying Hölder's inequality, we have

$$\||(x,t)|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha} \leq \||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{1/s} \|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{1/s'}$$

and

$$\||(\lambda,m)|^{\frac{1}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \leq \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^{1/\beta} \|\mathscr{F}_L f\|_{2,\gamma_\alpha}^{1/\beta'}.$$

Therefore

$$\||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha} \geq \frac{\||(x,t)|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^s}{\|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{s-1}}$$

and

$$\||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \geq \frac{\||(\lambda,m)|^{\frac{1}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^\beta}{\|\mathscr{F}_L f\|_{2,\gamma_\alpha}^{\beta-1}}.$$

Using Theorem 1, we derive that

$$\||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^\beta \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^s \geq \frac{C\sqrt{c_h}\|f\|_{2,m_\alpha}^{2\beta s}}{\|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{\beta(s-1)} \|\mathscr{F}_L f\|_{2,\gamma_\alpha}^{s(\beta-1)}}.$$

Plancherel formula and relation (37) allow to deduce the wanted result.

Lemma 1.*Let $\beta \in \mathbb{R}$. We consider ψ, an admissible Laguerre wavelet, satisfying*

$$\forall (\lambda,m) \in \hat{\mathbb{K}}, \quad \mathscr{F}_L \psi(\lambda,m) = \phi(|\lambda|). \tag{44}$$

If f belongs to $L^2(\mathbb{K})$ then

$$\|a^\beta W_\psi^L f\|_{2,\nu_\alpha}^2 = \mathscr{M}(|\tilde{\psi}|^2)(2\beta).\||\lambda|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^2, \tag{45}$$

where $\tilde{\phi}(\lambda) = \phi(\lambda^2)$ and $\mathscr{M}$ is the Mellin transform defined by

$$\mathscr{M} f(x) = \int_0^{+\infty} t^x f(t) \frac{dt}{t}.$$

Proof.

$$\begin{aligned}\|a^\beta W_\psi^L f\|_{2,\nu_\alpha}^2 &= \int_0^{+\infty} a^{-2\beta} \int_{\hat{\mathbb{K}}} |\mathscr{F}_L f(\lambda,m)|^2 |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \frac{da}{a^{2\alpha+5}} \\ &= \sum_{m=0}^{+\infty} L_m^\alpha \int_{\mathbb{R}} |\mathscr{F}_L f(\lambda,m)|^2 \Psi(\lambda) d\gamma_\alpha(\lambda,m),\end{aligned}$$

where

$$\Psi(\lambda) = \int_0^{+\infty} a^{2\beta} |\mathscr{F}_L \delta_a' \psi(\lambda,m)|^2 \frac{da}{a}.$$

Making a change of variable, we have

$$\Psi(\lambda) = |\lambda|^\beta \int_0^{+\infty} u^{2\beta} |\tilde{\phi}(u)|^2 \frac{du}{u} = |\lambda|^\beta \mathscr{M}(|\tilde{\phi}|^2)(2\beta).$$

Thus

$$\|a^{-\beta} W_\psi^L f\|_{2,\nu_\alpha}^2 = \mathscr{M}(|\tilde{\phi}|^2)(2\beta) \sum_{m=0}^{+\infty} L_m^\alpha \int_{\mathbb{R}} |\lambda|^\beta |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).$$

This gives the wanted result.

Theorem 2.*Let $s,\beta \geq 1$ and h an admissible Laguerre wavelet verifying (44). Then, for all f belonging to $L^2(\mathbb{K})$, we have*

$$\|a^\beta W_\psi^L f\|_{2,\nu_\alpha}^s \||(x,t)|^s W_\psi^L f\|_{2,\nu_\alpha}^\beta \geq C c_\psi \mathscr{M}(|\tilde{\phi}|^2)(2\beta).\||\lambda|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^2 \|f\|_{2,m_\alpha}^2. \tag{46}$$

*Proof.*Theorem 2 holds from Lemma 1 and Corollary 1.

4.2 L^p Local uncertainty principles for CLWT

This section is devoted to uncertainty principles of concentration type for CLWT in the L^p theory.

Theorem 3. *If $1<p\leq 2$, $q=\frac{p}{p-1}$ and , then for all nonzero $f\in L^p(\mathbb{K})$ and for all measurable subset $T\subset \mathbb{U}$ such that $0<\nu_\alpha(T)<+\infty$, we have*
(a) If $0<s<\frac{2\alpha+4}{q}$,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\leq\ C_1(s,q,\alpha)\ c_\psi^{\frac{1}{q}-\frac{s}{2\alpha+4}}\ \nu_\alpha(T)^{\frac{s}{2\alpha+4}}\left\||(x,t)|^s f\right\|_{p,m_\alpha}, \tag{47}$$

where $C_1(s,q,\alpha)$ is a constant that depends on s, q and α.
(b) If $s>\frac{2\alpha+4}{q}$,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\leq\ C_2(s,q,\alpha)\ \ \nu_\alpha(T)^{\frac{1}{q}}\left\|f\right\|_{p,m_\alpha}^{\frac{p}{q}(1-\frac{2\alpha+4}{sq})}\left\|\,|(x,t)|^s\ f\right\|_{p,m_\alpha}^{\frac{p}{q}\frac{(2\alpha+4)}{sq}}, \tag{48}$$

where $C_2(s,q,\alpha)$ is a constant that depends on s, q and α.
(c) If $s=\frac{2\alpha+4}{q}$,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\leq C_3(q,\alpha)c_\psi^{\frac{1}{q}-\frac{2}{q(2\alpha+4)}}\ \nu_\alpha(T)^{\frac{2}{q(2\alpha+4)}}\left\|f\right\|_{p,\alpha}^{1-\frac{1}{\alpha+2}}\left\|\,|(x,t)|^{\frac{2\alpha+4}{q}}\ f\right\|_{p,\alpha}^{\frac{1}{\alpha+2}}, \tag{49}$$

where $C_3(q,\alpha)=C_1(\frac{2}{q},q,\alpha)(\alpha+2)\ (\alpha+1)^{\frac{1}{\alpha+2}-1}$.

Proof. (a) For all $r>0$, we define $B_r=\{(x,t)\in\mathbb{K}\ \ ;\ \ |(x,t)|\leq r\}$. Denote by χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions. Let $f\in L^p_\alpha(\mathbb{K}), 1<p\leq 2$ and $q=\frac{p}{p-1}$. It follows using Minkowski's inequality,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\leq\left\|\chi_T W_\psi^L(f\chi_{B_r})\right\|_{q,\nu_\alpha}+\left\|\chi_T W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}.$$

Therefore

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\leq\nu_\alpha(T)^{\frac{1}{q}}\left\|W_\psi^L(f\chi_{B_r})\right\|_{\infty,\nu_\alpha}+\left\|W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}. \tag{50}$$

Using relation (32), we get

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\leq\nu_\alpha(T)^{\frac{1}{q}}a^{-(\alpha+2)}\left\|\psi\right\|_{\infty,m_\alpha}\left\|f\chi_{B_r}\right\|_{1,m_\alpha}+\left\|W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}. \tag{51}$$

Let $0<s<\frac{2\alpha+4}{q}$. By Hölder's inequality, we obtain

$$\left\|f\chi_{B_r}\right\|_{1,m_\alpha}\leq\left\|\,|(x,t)|^s\ f\right\|_{p,m_\alpha}\left\|\,|(x,t)|^{-s}\chi_{B_r}\right\|_{q,m_\alpha}. \tag{52}$$

Considering (38), let's examine polar coordinates in the Laguerre hypergroup structure:

$$\begin{cases} x=\rho\cos(\theta)^{\frac{1}{2}} \\ t=\frac{\rho^2}{2}\sin(\theta)\end{cases},\quad where\ \rho=|(x,t)|_{\mathbb{K}}.$$

The Jacobian is given by:

$$\begin{vmatrix}\cos(\theta)^{\frac{1}{2}} & \rho\sin(\theta)\\ -\frac{\rho}{2}\sin(\theta)\cos(\theta)^{-\frac{1}{2}} & \frac{\rho^2}{2}\cos(\theta)\end{vmatrix}=\frac{\rho^2}{2}\cos(\theta)^{-\frac{1}{2}}$$

and

$$\left\|\,|(x,t)|^{-s}\chi_{B_r}\right\|_{q,m_\alpha}^q=\frac{1}{2\pi\Gamma(\alpha+1)}\int_0^r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\rho^{-sq+2\alpha+3}\cos(\theta)^\alpha d\rho d\theta=A(s,q,\alpha)^q.$$

Therefore, we have

$$\left\|f\chi_{B_r}\right\|_{1,m_\alpha}\leq\ A(s,q,\alpha)\ \ r^{\frac{2\alpha+4}{q}-s}\ \left\|\,|(x,t)|^s\ f\right\|_{p,m_\alpha}, \tag{53}$$

where $A(s,q,\alpha)=\left(\dfrac{B(\frac{\alpha+1}{2},\frac{1}{2})}{2\pi\Gamma(\alpha+1)(2\alpha+4-sq)}\right)^{\frac{1}{q}}$, B is the beta function.
On the other hand, by relation (32), we obtain

$$\begin{aligned}\left\|W_{\psi}^{L}(f\chi_{B_r^c})\right\|_{q,\nu_\alpha} &\leq c_{\psi}^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}}\|f\chi_{B_r^c}\|_{p,m_\alpha}\\ &\leq c_{\psi}^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}}\|\,|(x,t)|^{s} f\|_{p,m_\alpha}\,\left\|\,|(x,t)|^{-s}\,\chi_{B_r^c}\right\|_{\infty,m_\alpha}.\end{aligned}$$

Hence

$$\left\|W_{\psi}^{L}(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}\leq c_{\psi}^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}} r^{-s}\,\|\,|(x,t)|^{s} f\|_{p,m_\alpha}. \tag{54}$$

Combining the relations (51),(53) and (54), we deduce that

$$\left\|\chi_T W_{\psi}^{L} f\right\|_{q,\nu_\alpha}\leq g_{\alpha,s}(r)\ \ \|\,|(x,t)|^{s} f\|_{p,m_\alpha}, \tag{55}$$

where $g_{\alpha,s}$is the function defined on $(0,+\infty)$ by

$$g_{\alpha,s}(r)=c_{\psi}^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}} r^{-s}+A(s,q,\alpha)\ \ a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\ \ \nu_\alpha(T)^{\frac{1}{q}} r^{\frac{2\alpha+4}{q}-s}.$$

By minimization of the right-hand side of the relation (55) over $r>0$, we get

$$\left\|\chi_T W_{\psi}^{L} f\right\|_{q,\nu_\alpha}\leq\ \ C_1(s,q,\alpha)\ c_{\psi}^{\frac{1}{q}-\frac{s}{2\alpha+4}}\ \ \nu_\alpha(T)^{\frac{s}{2\alpha+4}}\,\||(x,t)|^{s} f\|_{p,m_\alpha},$$

where

$$C_1(s,q,\alpha)=\left(\frac{2\alpha+4}{2\alpha+4-sq}\right)\left(\frac{2\alpha+4}{sq}-1\right)^{\frac{sq}{2\alpha+4}}\ \left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}+\frac{2s}{2\alpha+4}} A(s,q,\alpha)^{\frac{sq}{2\alpha+4}}.$$

(b) The inequality (48) holds if $\||(x,t)|^{s} f\|_{p,m_\alpha}=+\infty$. Assume that $\||(x,t)|^{s}\ f\|_{p,m_\alpha}<+\infty$.
From the hypothesis $s>3\alpha+2$, we derive that the function

$$(x,t)\longmapsto(1+\ |(x,t)|^{ps})^{-\frac{1}{p}}$$

belongs to $L^{q}(\mathbb{K})$. Hölder's inequality leads to

$$\|f\|_{1,m_\alpha}\leq\left\|(1+\ |(x,t)|^{ps})^{\frac{1}{p}} f\right\|_{p,m_\alpha}\ \left\|(1+\ |(x,t)|^{ps}\ )^{-\frac{1}{p}}\right\|_{q,m_\alpha}. \tag{56}$$

Since

$$\left\|(1+\ |(x,t)|^{ps})^{\frac{1}{p}} f\right\|_{p,m_\alpha}^{p}=\|f\|_{p,m_\alpha}^{p}+\|\ |(x,t)|^{s}\ f\|_{p,m_\alpha}^{p}$$

then

$$\|f\|_{1,m_\alpha}\leq N(s,q)\ \left(\|f\|_{p,\alpha}^{p}+\|\ |(x,t)|^{s}\ f\|_{p,m_\alpha}^{p}\right)^{\frac{1}{p}}. \tag{57}$$

where

$$N(s,q,\alpha)=\left\|(1+\ |(x,t)|^{ps})^{-\frac{1}{p}}\right\|_{q,m_\alpha}.$$

Using polar coordinates in the Laguerre hypergroup structure, we obtain

$$N(s,q,\alpha)=\left(\frac{B(\frac{\alpha+1}{2},\frac{1}{2})B(\frac{q}{p}-\frac{2(\alpha+2)}{sp},\frac{2(\alpha+2)}{sp})}{2\pi\ sp\Gamma(\alpha+1)}\right)^{\frac{1}{q}}.$$

For $r>0$, we consider $f_r(x,t)=r^{-(2\alpha+4)}f(\frac{x}{r},\frac{t}{r^2})$. Then we have

$$\|f_r\|_{1,m_\alpha}=\|f\|_{1,m_\alpha}, \tag{58}$$

$$\|f_r\|^p_{p,m_\alpha} = r^{\frac{-(2\alpha+4)p}{q}} \|f\|^p_{p,m_\alpha}, \tag{59}$$

and

$$\| \, |(x,t)|^s \, f_r\|^p_{p,m_\alpha} = r^{p(s-\frac{2\alpha+4}{q})} \| \, |(x,t)|^s \, f\|^p_{p,m_\alpha}. \tag{60}$$

Considering f_r in relation (57), we conclude that for all $r > 0$, we get

$$\|f\|^q_{1,m_\alpha} \leq N(s,q,\alpha)^q \left(r^{\frac{-(2\alpha+4)p}{q}} \|f\|^p_{p,m_\alpha} + r^{p(s-\frac{2\alpha+4}{q})} \| \, |(x,t)|^s \, f\|^p_{p,m_\alpha} \right)^{\frac{q}{p}}.$$

By minimizing the right-hand side of this inequality, we deduce

$$\|f\|^q_{1,m_\alpha} \leq N(s,q,\alpha)^q (\frac{sq}{2\alpha+4} - 1)^{\frac{2\alpha+4}{sq}} \left(\frac{sq}{sq-(2\alpha+4)} \right) \|f\|^{p(1-\frac{2\alpha+4}{sq})}_{p,m_\alpha} \| |(x,t)|^s f\|^{p\frac{(2\alpha+4)}{sq}}_{p,m_\alpha}. \tag{61}$$

Then, according to relation (61), the function f belongs to $L^1(\mathbb{K})$, and we have

$$\begin{aligned} \left\|\chi_T W^L_\psi f\right\|^q_{q,\nu_\alpha} &\leq \nu_\alpha(T) \left\|W^L_\psi f\right\|^q_{\infty,\nu_\alpha} \\ &\leq \nu_\alpha(T) \left(a^{-(\alpha+2)} \|\psi\|_{\infty,m_\alpha} \right)^q \|f\|^q_{1,m_\alpha}. \end{aligned}$$

Using the relation (61), we get

$$\left\|\chi_T W^L_\psi f\right\|^q_{q,\nu_\alpha} \leq \nu_\alpha(T) \, C^q_2(s,q,\alpha) \; \|f\|^{p(1-\frac{2\alpha+4}{sq})}_{p,m_\alpha} \| \, |(x,t)|^s \, f\|^{p\frac{(2\alpha+4)}{sq}}_{p,m_\alpha},$$

where

$$C^q_2(s,q,\alpha) = \left(a^{-(\alpha+2)} \|\psi\|_{\infty,m_\alpha} \right)^q N(s,q,\alpha)^q \left(\frac{sq}{2\alpha+4} - 1 \right)^{\frac{2\alpha+4}{sq}} \left(\frac{sq}{sq-(2\alpha+4)} \right).$$

(c) Consider $s = \frac{2}{q}(\alpha+2)$. Using the fact that for $\varepsilon > 0$,

$$\frac{|(x,t)|^{\frac{2}{q}}}{\varepsilon^{\frac{2}{q}}} \leq 1 + \frac{|(x,t)|^{\frac{2(\alpha+2)}{q}}}{\varepsilon^{\frac{2(\alpha+2)}{q}}},$$

it follows that

$$\left\| \, |(x,t)|^{\frac{2}{q}} \, f \right\|_{p,\alpha} \leq \; \varepsilon^{\frac{2}{q}} \|f\|_{p,\alpha} + \varepsilon^{\frac{2}{q}-\frac{2}{q}(\alpha+2)} \left\| \, |(x,t)|^{\frac{2}{q}(\alpha+2)} \, f \right\|_{p,\alpha}.$$

Optimizing in ε, we get:

$$\left\| \, |(x,t)|^{\frac{2}{q}} \, f \right\|_{p,\alpha} \leq \; (\alpha+2) \, (\alpha+1)^{\frac{1}{\alpha+2}-1} \|f\|^{1-\frac{1}{\alpha+2}}_{p,\alpha} \left\| \, |(x,t)|^{\frac{2\alpha+4}{q}} \, f \right\|^{\frac{1}{\alpha+2}}_{p,\alpha}.$$

Together with (47) for $s = \frac{2}{q} < \frac{2\alpha+4}{q}$, we get the wanted result.

Theorem 4. *Let s, p be two real numbers such that $0 < s < 2\alpha+4$ and $p \geq 1$. Then, for every function $f \in L^p(\mathbb{K})$ and for every measurable subset $T \subset \mathbb{U}$ such that $0 < \nu_\alpha(T) < +\infty$, we have*

$$\left\|\chi_T W^L_\psi f\right\|_{p,\nu_\alpha} \leq \; C \;\; \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\| |(\frac{1}{a},x,t)|^s W^L_\psi f \right\|^{\kappa}_{2,\nu_\alpha} \left(\|h\|_{2,m_\alpha} \|f\|_{2,m_\alpha} \right)^{1-\kappa}, \tag{62}$$

where $\kappa = \dfrac{2(2\alpha+4)-s}{(p+1)(2\alpha+4)}$ and C is a constant that depends on s, p and α.
Note here that

$$\left| (\frac{1}{a},x,t) \right| = (\frac{1}{a^4} + x^4 + 4t^2)^{\frac{1}{4}}.$$

*Proof.*One can assume that $\|f\|_{2,m_\alpha} = 1$, and $\|\psi\|_{2,m_\alpha} = 1$. The general formula follows by making the substitution $f := \frac{f}{\|f\|_{2,m_\alpha}}$ and $\psi := \frac{\psi}{\|\psi\|_{2,m_\alpha}}$.
For all $r > 0$, we put $V_r = \left\{(a,x,t) \in (0,+\infty)\times\mathbb{K} \;\; ; \;\; |(\frac{1}{a},x,t)| \le r\right\}$. Let $0 < s < 2\alpha+4$. By Hölder's inequality, we obtain

$$\left\|\chi_T W_\psi^L f\right\|_{p,\nu_\alpha} \le \left\|\chi_{T\cap V_r} W_\psi^L f\right\|_{p,\nu_\alpha} + \left\|\chi_{T\cap V_r^c} W_\psi^L f\right\|_{p,\nu_\alpha}.$$

Let $0 < s < 2\alpha+4$. Using Hölder's inequality and relation (32), we obtain

$$\begin{aligned}\left\|\chi_{T\cap V_r} W_\psi^L f\right\|_{p,\nu_\alpha} &\le \left\|W_\psi^L(f)\right\|_{\infty,\nu_\alpha}^{\frac{p}{p+1}} \left(\int_{\mathbb{U}} \chi_T(a,x,t)\chi_{V_r}(a,x,t)|W_\psi^L f(a,x,t)|^{\frac{p}{p+1}} d\nu_\alpha\right)^{\frac{1}{p}}\\ &\le \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\|\chi_{V_r} W_\psi^L f\right\|_{1,\nu_\alpha}^{\frac{1}{p+1}}\\ &\le \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}} \left\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}}.\end{aligned}$$

Making the change of variables $\begin{cases} u = \frac{1}{a^2} \\ v = x^2 \\ w = 2t \end{cases}$, we get

$$\left\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\right\|_{2,\nu_\alpha}^2 = \int_{\mathbb{U}} (u^2+v^2+w^2)^{-\frac{2s}{4}} \chi_{V_r} \frac{u^{\alpha+1}v^\alpha}{8\pi\Gamma(\alpha+1)} du\,dv\,dw.$$

Applying polar coordinates in $\mathbb{R}^3$, we find

$$\left\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\right\|_{2,\nu_\alpha}^2 = \int_0^r \rho^{-2s} \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\rho\cos(\theta)\cos(\varphi))^{\alpha+1}(\rho\sin(\theta)\cos(\varphi))^\alpha}{8\pi\Gamma(\alpha+1)} \rho^2\cos(\varphi)d\rho\, d\theta\, d\varphi.$$

By a simple calculation, we get

$$\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\|_{2,\nu_\alpha} = A_1(s,\alpha) r^{2\alpha+4-s},$$

where $A_1(s,\alpha) = \left(\frac{B(\frac{\alpha+1}{2},\frac{\alpha}{2}+1)B(\alpha+\frac{3}{2},\frac{1}{2})}{16\pi\,(2\alpha+4-s)\Gamma(\alpha+1)}\right)^{\frac{1}{2}}$. Thus we obtain

$$\left\|\chi_{T\cap V_r} W_\psi^L f\right\|_{p,\nu_\alpha} \le \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}} C_1^{p+1} s^{\frac{2\alpha+4-s}{p+1}}. \tag{63}$$

On the other hand, using Hölder's inequality and relation (32), we conclude that

$$\begin{aligned}\left\|\chi_{T\cap V_r^c} W_\psi^L f\right\|_{p,\nu_\alpha} &\le \left\|W_\psi^L f\right\|_{\infty,\nu_\alpha}^{\frac{p-1}{p+1}} \left(\int_{\mathbb{U}} \chi_{T\cap V_r^c}(a,x,t)|W_\psi^L f(a,x,t)|^{\frac{2p}{p+1}} d\nu_\alpha(a,x,t)\right)^{\frac{1}{p}}\\ &\le \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left(\int_{\mathbb{U}} \chi_{V_r^c}(a,x,t)|W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t)\right)^{\frac{1}{p+1}}\\ &\le \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{2}{p+1}} r^{\frac{-s}{p+1}}.\end{aligned}$$

Hence

$$\left\|\chi_T W_\psi^L f\right\|_{p,\nu_\alpha} \le h_{\alpha,s}(r)\; \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}}, \tag{64}$$

where $h_{\alpha,s}$ is the function defined on $(0,+\infty)$ by

$$h_{\alpha,s}(r) = A_1(s,\alpha)^{p+1} s^{\frac{2\alpha+4-s}{p+1}} + \left\| |(\frac{1}{a},x,t)|^s W_\psi^L f \right\|_{2,\nu_\alpha}^{\frac{1}{p+1}} r^{\frac{-s}{p+1}}.$$

By minimizing the right-hand side of the inequality (64) with respect to $r > 0$, we obtain

$$\left\| \chi_T W_\psi^L f \right\|_{p,\nu_\alpha} \leq C(s,p,\alpha) \;\; \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\| |(\frac{1}{a},x,t)|^s W_\psi^L f \right\|_{2,\nu_\alpha}^{\frac{2}{p+1} - \frac{s}{(p+1)(2\alpha+4)}},$$

where

$$C(s,p,\alpha) = (\frac{2\alpha+4}{2\alpha+4-s}) \, (\frac{2\alpha+4-s}{s})^{\frac{s}{2\alpha+4}} A_1(s,\alpha)^{\frac{s}{p(2\alpha+4)}}.$$

4.3 Benedicks-Amrein-Berthier's uncertainty principle for CLWT

A strong formulation of Benedicks-Amrein-Berthier's result for the Laguerre Fourier transform was established by the second author in [20]. This result asserts that, for $S \subset \mathbb{K}$, $\Sigma \subset \hat{\mathbb{K}}$ a pair of measurable subsets of finite measures $\mu_\alpha(S)$, $\hat{\mu}_\alpha(\Sigma) < +\infty$, we can find a constant $C(S,\Sigma)$ such that, for all $f \in L^2(\mathbb{K})$,

$$\|f\|^2_{2,m_\alpha} \leq C(S,\Sigma) \left(\int_{\mathbb{K}\setminus S} |f(x,t)|^2 dm_\alpha(x,t) + \int_{\hat{\mathbb{K}}\setminus\Sigma} |\mathscr{F}_L f|^2 d\gamma_\alpha(\lambda,m) \right). \tag{65}$$

The constant $C(S,\Sigma)$ is called the annihilating constant, and (S,Σ) is termed a strong annihilating pair. In the context of CLWT, we obtain the following result.

Theorem 5.*Consider two measurable subsets $S \subset \mathbb{K}$, $\Sigma \subset \hat{\mathbb{K}}$ with finite measures $\mu_\alpha(S)$, $\hat{\mu}_\alpha(\Sigma) < +\infty$. Let ψ be a Laguerre wavelet on $\mathbb{K}$ in $L^2(\mathbb{K})$. For an arbitrary function $f \in L^2(\mathbb{K})$,the following uncertainty inequality holds.*

$$\frac{c_\psi \|f\|^2_{2,m_\alpha}}{C(S,\Sigma)} \leq \int_0^{+\infty} \int_{\mathbb{K}\setminus S} |W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t) + c_\psi \int_{\hat{\mathbb{K}}\setminus\Sigma} |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha, \tag{66}$$

where $C(S,\Sigma)$ is the annihilating constant given in (65).

*Proof.*We have, for all $a > 0$, $W_\psi^L f(a,.,.) \in L^2(\mathbb{K})$ whenever $f \in L^2(\mathbb{K})$. This allows using (65) to get

$$\|W_\psi^L f\|^2_{2,m_\alpha} \leq C(S,\Sigma) \left(\int_{\mathbb{K}\setminus S} |W_\psi^L f(x,t)|^2 dm_\alpha(x,t) + \int_{\hat{\mathbb{K}}\setminus\Sigma} |\mathscr{F}_L W_\psi^L f|^2 d\gamma_\alpha(\lambda,m) \right).$$

Integrating both sides with respect to $\dfrac{da}{a^{2\alpha+5}}$, we proceed similarly to the proof of Theorem 1. Consequently, (66) holds using relations (35) and (42).

Declarations

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