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Solutions of the equation $d(kn) = \varphi(\varphi(n))$

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Abstract: Let $d(n)$ and $\varphi(n)$ denote the number of positive integers dividing the positive integer n and the Euler's phi function representing the numbers less than and prime to n, respectively. In this paper, we determine all solutions of the equation $d(n) = \varphi(\varphi(n))$ and we prove that the equation $d(kn) = \varphi(\varphi(n))$ has a finite number of solutions for any $k \geq 1$. Further, we characterize all solutions of the last equation when k is prime.

Keywords: Arithmetic functions; diophantine equations; primes; factorization.

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1 Introduction

Let $\mathbb{N}$ be the set of all positive integers and let $n \in \mathbb{N}$. Let $d(n)$ be the divisor function, which counts the number of positive divisors of n, i.e., if n has the prime factorization $n = q_1^{a_1} q_2^{a_2} ... q_k^{a_k}$ with distinct primes $q_1, q_2, ..., q_k$ and positive integers $a_1, a_2, ..., a_k$, then

$$d(n) = (a_1 + 1)(a_2 + 1)\dots(a_k + 1).$$

Let $\varphi(n)$ be the Euler function, which counts the number of positive integers $m \leq n$ with $\gcd(m,n) = 1$. From now on $\gcd(m,n)$ will be denoted by (m,n). It is well-known that

$$\varphi(n) = q_1^{a_1-1}(q_1 - 1) q_2^{a_2-1}(q_2 - 1) ... q_k^{a_k-1}(q_k - 1).$$

In our main results, we will use the following inequalities (for details, we refer the reader to [5, Problem 522] and [6, pages 110, 116, 117, 183]). At first, for all positive integers m and n we have

$$d(n) \leq 2\sqrt{n} \tag{1}$$

and

$$d(mn) \leq d(m)\, d(n). \tag{2}$$

Moreover, if $(m,n) > 1$, then

$$d(mn) < d(m)\, d(n). \tag{3}$$

If m divides n, then

$$d(n) \geq d(m). \tag{4}$$

Next, for $n \neq 2$ and $n \neq 6$, we have

$$\varphi(n) \geq \sqrt{n}. \tag{5}$$

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For all positive integers m and n, we also have

$$\varphi(mn) = \varphi(m)\varphi(n)\frac{d}{\varphi(d)}, \tag{6}$$

where $d = (m,n)$. From which it follows that

$$\varphi(m)\varphi(n) \leq \varphi(mn) \tag{7}$$

In particular, if $d > 1$ then (6) and (7) give us the inequality

$$\varphi(m)\varphi(n) < \varphi(mn). \tag{8}$$

Various diophantine equations involving the divisor function and Euler's phi function were investigated by many authors. For example, see [2], [3] and [4]. In [5, Problem 705, page 78], it is shown that $\varphi(d(n)) = d(\varphi(n))$ has infinitely many solutions; while in [6, pages 110-111], it is shown that $d(n) = \varphi(n)$ has the only solutions $1, 3, 8, 10, 24$ and 30, where $d(n) < \varphi(n)$ for $n \geq 31$. In the same context, in [7] it is shown that the equation $\varphi(n) + d(n) = n$ has the only solutions $n = 8$ and $n = 9$. Other similar problems have been discussed in publications such as Sándor [8] and [9].

The present work is a continuation of the authors' articles [1] and [3]. We first define for any positive integer k the following sets:

$$\mathbb{E}_k = \{n \in \mathbb{Z}^+ : d(kn) = \varphi(\varphi(n))\}, \tag{9}$$

$$\mathbb{L}_k = \{n \in \mathbb{Z}^+ : d(kn) < \varphi(\varphi(n))\}, \tag{10}$$

$$\mathbb{G}_k = \{n \in \mathbb{Z}^+ : d(kn) > \varphi(\varphi(n))\}. \tag{11}$$

So, the main focus of this paper is to examine the set $\mathbb{E}_k$ of solutions n of the equation $d(kn) = \varphi(\varphi(n))$ and related inequalities. In fact, we characterize all the elements of $\mathbb{E}_1$, $\mathbb{L}_1$ and $\mathbb{G}_1$, respectively. Then we deduce that the sets $\mathbb{G}_k$ and $\mathbb{E}_k$ are finite, while $\mathbb{L}_k$ is infinite. Moreover, we prove that if p is a prime number with $p \geq 23$ and $p \neq 31$, then $\mathbb{E}_p = \{11,$ 13, 33, 34, 35, 39, 62, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840$\}$.

2 The equation $d(n) = \varphi(\varphi(n))$

This section is devoted to investigate the elements of (9), (10) and (11) for $k = 1$.

Theorem 1. *The numbers* 1, 5, 7, 15, 22, 26 , 40, 56, 66, 70, 78, 108, 120, 126, 168, 210 *are the only solutions of the equation* $d(n) = \varphi(\varphi(n))$*. Moreover, the numbers* 2, 3, 4, 6, 8, 9, 10, 12, 14, 16, 18, 20, 24, 28, 30, 36, 42, 48, 54, 60, 72, 84, 90, 180 *are the only solutions of the inequality* $d(n) > \varphi(\varphi(n))$.

For the proof, we start by proving the following results in which, if the solution $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ with $q_1 < q_2 < \ldots < q_s$ and α_i are positive integers, then $s \leq 5$ and $\alpha_i \leq 3$ for $1 \leq i \leq s$ and $q_s \leq 17$. This means that the number of solutions is finite. Let us start with square-free solutions.

2.1 Square-free solutions

Proposition 1. *The only prime numbers that satisfy the equation* $d(n) = \varphi(\varphi(n))$ *are* 5 *and* 7.

Proof. Let p be a prime number. If $p \in \mathbb{E}_1$, then $d(p) = 2 = \varphi(p-1)$. If $p - 1 \neq 2, 6$, then by (5), $p \leq 5$. In this case, $p = 5$ is the only solution. If $p - 1 = 2$ or 6, it follows that $p = 3$ or 7, where $7 \in \mathbb{E}_1$ while $3 \notin \mathbb{E}_1$. Finally, we conclude that if $p \in \mathbb{E}_1$, then p is either 5 or 7.

We deduce the following corollary.

Corollary 1. *Let p be a prime number. We have:*

- *If p is either* 5 *or* 7*, then* $p \in \mathbb{E}_1$.
- *If p is either* 2 *or* 3*, then* $p \in \mathbb{G}_1$.
- *If* $p \geq 11$*, then* $p \in \mathbb{L}_1$.

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Proof. By the same way of the proof of Proposition 1, we conclude that if $n = p$ is prime with $p \geq 11$, then the inequality $d(n) \geq \varphi(\varphi(n))$ cannot be true, and so $p \in \mathbb{L}_1$.

Proposition 2. *The only square-free solutions of the form q_1q_2, where q_1 and q_2 are distinct primes, are $2\cdot 11$, $2\cdot 13$, $3\cdot 5$ and $3\cdot 7$.*

Proof. Suppose that $q_1q_2 \in \mathbb{E}_1$, where q_1, q_2 are distinct primes with $2 \leq q_1 < q_2$. We obtain from (9) that

$$\varphi((q_1-1)(q_2-1)) = 4.$$

If $(q_1-1)(q_2-1) \neq 2, 6$, then by (5) we have $(q_1-1)(q_2-1) \leq 16$. Here, q_1 cannot be ≥ 5. There are two possibilities:
1. If $q_1 = 2$, then q_2 must be in $\{5, 7, 11, 13, 17\}$.
2. If $q_1 = 3$, then q_2 must be in $\{5, 7\}$.
Thus, since $2\cdot 5$, $2\cdot 7$ and $2\cdot 17$ are in $\mathbb{G}_1$, we get $(q_1, q_2) \in \{(2,11),(2,13),(3,5),(3,7)\}$.

By the same way we can prove the following proposition:

Proposition 3. *If a solution n is square-free and has 3 or 4 distinct primes, then n is one the numbers $2\cdot 3\cdot 11$, $2\cdot 5\cdot 7$, $2\cdot 3\cdot 13$ and $2\cdot 3\cdot 5\cdot 7$.*

Proposition 4. *If a solution n is square-free and has more than 4 distinct primes, then $n \in \mathbb{L}_1$.*

For the proof we need the following lemma:

Lemma 1. *Let $k \geq 6$ and let $q_1, q_2, \ldots, q_k$ be distinct primes. Then*

$$(q_1-1)(q_2-1)\ldots(q_k-1) > 2^{2k}.$$

Proof. The proof holds by induction on k, since $(q_1-1)(q_2-1)\ldots(q_6-1) \geq 2^{2\cdot 6}$ for every 6-tuple $(q_1, q_2, \ldots, q_6)$ of distinct primes.

Proof(Proposition 4). Let $n = q_1q_2\ldots q_k$, where $q_1, q_2, \ldots, q_k$ are distinct primes. We assume further that $n \in \mathbb{E}_1 \cup \mathbb{G}_1$. In the case when $k = 5$. We see that

$$32 = d(q_1q_2\ldots q_5) \geq \varphi((q_1-1)(q_2-1)\ldots(q_5-1)).$$

By (5), we obtain

$$\varphi((q_1-1)(q_2-1)\ldots(q_5-1)) \geq \sqrt{(q_1-1)(q_2-1)\ldots(q_5-1)},$$

and hence $(q_1-1)(q_2-1)\ldots(q_5-1) \leq 1024$. Since $q_1 < q_2 < \ldots < q_5$, we distinguish the following cases:

- $q_1 = 2$ and (q_2, q_3, q_4, q_5) must belong to the set

$$\{(3,5,7,11),(3,5,7,13),(3,5,7,17),(3,5,7,19),(3,5,11,13)\}.$$

  But, the corresponding numbers $n = q_1q_2\ldots q_5$ belong to $\mathbb{L}_1$, which is impossible since $n \in \mathbb{E}_1 \cup \mathbb{G}_1$.
- $q_1 \geq 3$. We get $(q_1-1)(q_2-1)\ldots(q_5-1) \geq 2\cdot 4\cdot 6\cdot 10\cdot 12 > 1024$, which is impossible as well. Thus, q_1 cannot be greater than 3.

In the case when $k \geq 6$, we also see that

$$2^k = d(q_1q_2\ldots q_k) \geq \varphi((q_1-1)(q_2-1)\ldots(q_k-1)),$$

and by (5) we have

$$\varphi((q_1-1)(q_2-1)\ldots(q_k-1)) \geq \sqrt{(q_1-1)(q_2-1)\ldots(q_k-1)}.$$

It follows that

$$2^{2k} \geq (q_1-1)(q_2-1)\ldots(q_k-1).$$

This contradicts Lemma 1. Hence, $d(n) < \varphi(\varphi(n))$.

2.2 *Nonsquare-free odd solutions*

Proposition 5. *Let $s \geq 2$ and let $n_1, n_2, ..., n_s$ be relatively prime positive integers with $n_i \geq 3$ for $i = 1, 2, ..., s$. If $n_1, n_2, ..., n_s \in \mathbb{E}_1 \cup \mathbb{L}_1$, then $n_1 n_2 ... n_s \in \mathbb{L}_1$.*

Proof. Let $n_1, n_2, ..., n_s$ be as above and assume that $n_1, n_2, ..., n_s \in \mathbb{E}_1 \cup \mathbb{L}_1$. Since $(\varphi(n_i), \varphi(n_j)) > 1$ for $1 \leq i, j \leq s$, it follows from (8) and the multiplicativity of d and φ that

$$\begin{aligned} d(n_1 n_2 ... n_s) &= d(n_1) d(n_2) ... d(n_s) \leq \varphi(\varphi(n_1)) \varphi(\varphi(n_2)) ... \varphi(\varphi(n_s)) \\ &< \varphi(\varphi(n_1) \varphi(n_2) ... \varphi(n_s)) = \varphi(\varphi(n_1 n_2 ... n_s)). \end{aligned}$$

Hence, $n_1 n_2 ... n_s \in \mathbb{L}_1$.

Proposition 6. *Let $\alpha \geq 2$ and let $n = q^\alpha$ be a prime power with q is odd. Then $n \in \mathbb{L}_1$ except for $n = 3^2 \in \mathbb{G}_1$.*

For the proof we need the following lemma.

Lemma 2. *Let n be a positive integer with $n \geq 3$ and let q be an odd prime number. Then*

$$(n+1)^2 < q^{n-1}(q-1).$$

Proof. This follows immediately using mathematical induction.

Proof(Proposition 6). Assume by way of contradiction that $d(n) \geq \varphi(\varphi(n))$. Therefore, $\alpha + 1 \geq \varphi(q^{\alpha-1}(q-1))$. Also, by (5), we get

$$\varphi(q^{\alpha-1}(q-1)) \geq \sqrt{q^{\alpha-1}(q-1)},$$

and so $(\alpha+1)^2 \geq q^{\alpha-1}(q-1)$, which is impossible by Lemma 2 except for $(q, \alpha) = (3, 2)$.

Proposition 7. *Let $s \geq 2$ and let $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ be an odd number with* $\max(\alpha_1, \alpha_2, ..., \alpha_s) \geq 2$. *Then $n \in \mathbb{L}_1$.*

Proof. We distinguish three cases:

Case 1. $s = 2$. There are two subcases:

Subcase 1.1. $q_1 = 3$. This means that $n = 3^{\alpha_1} q^{\alpha_2}$, and we have three possibilities:

- $\alpha_1 = 1$, so α_2 must be > 1 because $q_2 \geq 5$. Thus, it is clear that

$$2^2(\alpha_2+1)^2 < 2q_2^{\alpha_2-1}(q_2-1).$$

If we assume that $n \in \mathbb{E}_1 \cup \mathbb{G}_1$, this means that $2(\alpha_2+1) \geq \varphi(2q_2^{\alpha_2-1}(q_2-1))$, and by (5) we obtain

$$\varphi(2q_2^{\alpha_2-1}(q_2-1)) \geq \sqrt{2q_2^{\alpha_2-1}(q_2-1)}.$$

Thus, $2^2(\alpha_2+1)^2 \geq 2q_2^{\alpha_2-1}(q_2-1)$. This is a contradiction. Hence, $n \in \mathbb{L}_1$.

- $\alpha_1 = 2$. Since $q_2 \geq 5$, we conclude that $3^2(\alpha_2+1)^2 < 2 \cdot 3 \cdot q_2^{\alpha_2-1}(q_2-1)$. Hence, $n \in \mathbb{L}_1$.
- $\alpha_1 > 2$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$, so by Proposition 5, $n \in \mathbb{L}_1$.

Subcase 1.2. $q_1 > 3$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$, so it is clear by Proposition 5 that $n \in \mathbb{L}_1$.

Case 2. $s = 3$. Here, we also have two subcases:

Subcase 2.1. $q_1 = 3$. We also distinguish three possibilities:

- $\alpha_1 = 1$. That is, $\alpha_2 > 1$ or $\alpha_3 > 1$. Then we can easily check that

$$2^2(\alpha_2+1)^2(\alpha_3+1)^2 < 2(q_2-1)q_2^{\alpha_2-1}(q_3-1)q_3^{\alpha_3-1},$$

and hence $n \in \mathbb{L}_1$.

- $\alpha_1 = 2$. By the same way we find $n \in \mathbb{L}_1$.
- $\alpha_1 > 2$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. So by Proposition 5, $n \in \mathbb{L}_1$.

Subcase 2.2. $q_1 > 3$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. So by Proposition 5, $n \in \mathbb{L}_1$.

Case 3. $s \geq 4$. Here, we distinguish two subcases:

Subcase 3.1. $q_1 = 3$. By induction on s we can easily prove the following inequality

$$(\alpha_1+1)^2(\alpha_2+1)^2 ... (\alpha_s+1)^2 < 2 \cdot 3^{\alpha_1-1}(q_2-1)q_2^{\alpha_2-1} ... (q_s-1)q_s^{\alpha_s-1},$$

which gives $n \in \mathbb{L}_1$.

Subcase 3.2. $q_1 > 3$. By Corollary 1 and Proposition 6, $q_i^{\alpha_i} \in \mathbb{E}_1 \cup \mathbb{L}_1$ for $1 \leq i \leq s$. Similarly, by Proposition 5, $n \in \mathbb{L}_1$.

2.3 *Nonsquare-free even solutions*

In this subsection, we can prove the following proposition as those appearing with the case when n is nonsquare-free odd.

Proposition 8. *Let* $n = 2^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ *be an even number such that* $\max(\alpha_1, \alpha_2, ..., \alpha_s) \geq 2$. *There are four possibilities:*

- $s = 1$. *Here n is a prime power, where* $n = 2^{\alpha_1}$ *with* $\alpha_1 \geq 2$. *We have:*
  - *If n is either* 2^2 *or* 2^3 *or* 2^4, *then* $n \in \mathbb{G}_1$.
  - *If* $\alpha_1 \geq 5$, *then* $n \in \mathbb{L}_1$.
- $s = 2$. *We have:*
  - *If* $n = 2^2 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 5, 2^3 \cdot 3, 2^2 \cdot 7, 2^2 \cdot 3^2, 2^4 \cdot 3, 2 \cdot 3^3, 2^3 \cdot 3^2$, *then* $n \in \mathbb{G}_1$.
  - *If* $n = 2^3 \cdot 5, 2^3 \cdot 7, 2^2 \cdot 3^3$, *then* $n \in \mathbb{E}_1$.
  - *If n is different from the previous mentioned numbers, then* $n \in \mathbb{L}_1$.
- $s = 3$. *We have:*
  - *If* $n = 2^2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 7, 2 \cdot 3^2 \cdot 5, 2^2 \cdot 3^2 \cdot 5$, *then* $n \in \mathbb{G}_1$.
  - *If* $n = 2^3 \cdot 3 \cdot 5, 2 \cdot 3^2 \cdot 7, 2^3 \cdot 3 \cdot 7$, *then* $n \in \mathbb{E}_1$.
  - *If n is different from the previous mentioned numbers, then* $n \in \mathbb{L}_1$.
- $s \geq 4$. *Here, we have* $n \in \mathbb{L}_1$.

Now, we are ready to prove Theorem 1.

Proof(Theorem 1). Clearly, $1 \in E_1$. Assume that $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_s^{\alpha_s}$ with $n \neq$ 1, 2, 3, 4, 5, 6, 7 , 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26 , 28, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 84, 90, 108, 120, 126, 168, 180 and 210, where $q_1 < q_2 < ... < q_s$ are primes and $\alpha_1, \alpha_2, ..., \alpha_s$ are positive integers. There are five cases to consider:

Case 1. $s = 1$. There are two possibilities:

- $\alpha_1 = 1$. Since $n \neq 2, 3, 5, 7$, it follows from Corollary 1 that $n \in \mathbb{L}_1$.
- $\alpha_1 \geq 2$. Since $n \neq 2^2, 2^3, 3^2, 2^4$, it follows from Propositions 6 and 8 that $n \in \mathbb{L}_1$.

Case 2. $s = 2$. There are two possibilities:

- $\alpha_1 = \alpha_2 = 1$. Since $n \neq 10, 14, 15, 21, 22, 26, 34$, by Proposition 2 we conclude that $n \in \mathbb{L}_1$.
- α_1 or $\alpha_2 \geq 2$. Since $n \neq 12, 18, 20, 24, 28, 36, 40, 48, 54, 56, 72, 108$, it follows from Proposition 7 and Proposition 8 that $n \in \mathbb{L}_1$.

Case 3. $s = 3$.

- n is square-free. Since $n \neq 30, 42, 66, 70, 78$, by Propositions 3 and 4 we have $n \in \mathbb{L}_1$.
- n is not square-free. Since $n \neq 60, 84, 90, 120, 126, 168, 180$, it follows from Propositions 7 and 8 that $n \in \mathbb{L}_1$.

Case 4. $s = 4$.

- n is square-free. Since $n \neq 210$, it follows from Propositions 3 and 4 that $n \in \mathbb{L}_1$.
- n is not square-free. It follows from Propositions 7 and 8 that $n \in \mathbb{L}_1$.

Case 5. $s \geq 5$. By Propositions 4, 7 and 8 we have $n \in \mathbb{L}_1$.
The proof is finished.

3 On the equation $d(k \cdot n) = \varphi(\varphi(n))$ with $k \geq 2$

Proposition 9. *Let* $s, k \geq 2$ *and let* $n_1, n_2, ..., n_s$ *be relatively prime positive integers with* $n_i \geq 3$ *for* $i = 1, 2, ..., s$. *If* $n_1, n_2, ..., n_s \in \mathbb{E}_k \cup \mathbb{L}_k$, *then* $n_1 n_2 ... n_s \in \mathbb{L}_k$.

Proof. This is similar to the proof of Proposition 5.

Theorem 2. *Let* $k \geq 2$. *The sets* $\mathbb{G}_k$ *and* $\mathbb{E}_k$ *are finite, while* $\mathbb{L}_k$ *is infinite.*

Proof. At first, we prove that for any prime p there exists an exponent α_0 such that $p^x \in \mathbb{L}_k$ for every $x \geq \alpha_0$. Let $n = p^\alpha$ be a prime power such that $p^\alpha \in \mathbb{E}_k \cup \mathbb{G}_k$. Then

$$d(k \cdot p^\alpha) \geq \varphi(p^{\alpha-1}(p-1)).$$

Put $k = p^a m$, where $a \geq 0$, $m \geq 1$ and $(p,m) = 1$. Then

$$d(m)(a+\alpha+1) \geq p^{\alpha-2}(p-1)\varphi(p-1)$$

and by (1) and (5) we have

$$2\sqrt{m}(a+\alpha+1) \geq d(m)(a+\alpha+1) \geq p^{\alpha-2}(p-1)\varphi(p-1) \geq p^{\alpha-2}(p-1)\sqrt{p-1}. \tag{12}$$

Therefore,

$$4m(a+\alpha+1)^2 \geq p^{2\alpha-4}(p-1)^3.$$

Since a and m are fixed and α is the exponent of p, the last inequality has only finite number of solutions. Thus, the set $\mathbb{E}_k \cup \mathbb{G}_k$ contains only finitely many prime powers, namely $l_1^{\gamma_1}, l_2^{\gamma_2}, ..., l_s^{\gamma_s}$. Consequently, it suffices to choose $\alpha_0 = \max(\gamma_1+1, \gamma_2+1, ..., \gamma_s+1)$.

Now, let $n = q_1 q_2 ... q_m$, where $q_1 < q_2 < ... < q_m$ are primes ($m \geq 2$). Assume that $n \in \mathbb{E}_k \cup \mathbb{G}_k$. Therefore,

$$d(k \cdot q_1 q_2 ... q_m) \geq \varphi(\varphi(q_1 q_2 ... q_m)).$$

By (1), (2) and (5) we obtain

$$(2\sqrt{k})(2^{m+1}) \geq d(k)d(q_1...q_m) \geq d(k \cdot q_1...q_m) \geq \varphi(\varphi(q_1...q_m)) \geq \sqrt{(q_1-1)...(q_m-1)},$$

from which it follows that

$$2^{2m+2}4k \geq (q_1-1)(q_2-1)...(q_m-1). \tag{13}$$

Since k is fixed and $(q_1-1)(q_2-1)...(q_m-1)$ can be sufficiently large, we deduce that there exists a positive integer m_0 for which the inequality (13) is not true for every $m \geq m_0$. Thus the inequality $d(k \cdot n) \geq \varphi(\varphi(n))$ holds for finitely many square-free integers.

Finally, let $n = q_1^{\alpha_1} q_2^{\alpha_2} ... q_m^{\alpha_m}$, where $q_1 < q_2 < ... < q_m$ are prime numbers and $\alpha_1, \alpha_2, ..., \alpha_m$ are positive integers. Note that the number m cannot be sufficiently large as we wish such that $d(k \cdot s) \geq \varphi(\varphi(s))$ for $s = q_1 q_2 ... q_m$. Moreover, from above there exist positive integers $\alpha_1^{(0)}, \alpha_2^{(0)}, ..., \alpha_m^{(0)}$ such that the numbers $q_1^{\alpha_1'}, q_2^{\alpha_2'}, ..., q_m^{\alpha_m'}$ satisfy the inequality

$$d\left(k \cdot q_i^{\alpha_i'}\right) < \varphi\left(\varphi\left(q_i^{\alpha_i'}\right)\right)$$

for every $\alpha_i' \geq \alpha_i^{(0)}$ $(1 \leq i \leq m)$. Applying Proposition 9, the numbers $n' = q_1^{\alpha_1'} q_2^{\alpha_2'} ... q_m^{\alpha_m'}$ with $\alpha_i' \geq \alpha_i^{(0)}$ $(1 \leq i \leq m)$ satisfy the inequality $d(k \cdot n') < \varphi(\varphi(n'))$. Thus, the inequality $d(k \cdot n) \geq \varphi(\varphi(n))$ has only a finite number of solutions, while the inequality $d(n) < \varphi(\varphi(n))$ has infinitely many solutions.

Proposition 10. *We have* $\bigcap_{k \geq 1} \mathbb{G}_k = \mathbb{G}_1$.

Proof. It suffices to prove that $\mathbb{G}_1 \subset \bigcap_{k \geq 1} \mathbb{G}_k$. As we have already seen

$$\mathbb{G}_1 = \{2,3,4,6,8,9,10,12,14,16,18,20,24,28,30,36,42,48,54,60,72,84,90,180\}.$$

Then for every $k \geq 1$ and x in $\mathbb{G}_1$ we see that $d(kx) > \varphi(\varphi(x))$. This proves $\mathbb{G}_1 \subset \mathbb{G}_k$. This completes the proof.

Proposition 11. *Let* $r, s \geq 2$. *If* r *divides* s, *then* $\mathbb{G}_r \subset \mathbb{G}_s$.

Proof. Let $n \in \mathbb{G}_r$. Since r divides s, we see by (4) that

$$d(s \cdot n) \geq d(r \cdot n) > \varphi(\varphi(n)),$$

and so $n \in \mathbb{G}_s$, as required.

Theorem 3. *Let p be a prime number with $p \geq 23$ and $p \neq 31$. Then* $\mathbb{E}_p = \{11, 13, 33, 34, 35, 39, 62, 63, 76 , 88, 98, 102, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.

Proof. Let $p \geq 23$ be prime with $p \neq 31$. Clearly, if n is one of the above numbers, then we can easily check that $d(p \cdot n) = 2d(n) = \varphi(\varphi(n))$. Then the numbers mentioned in this theorem are part of the solution set.

Now, let $n \geq 1$ such that $n \in \mathbb{E}_p \cup \mathbb{G}_p$. It follows that

$$2d(n) \geq d(p \cdot n) \geq \varphi(\varphi(n)).$$

As in the proof of Theorem 1, we can show that the only solutions of $2d(n) \geq \varphi(\varphi(n))$ are $n =$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 , 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 48, 50, 52, 54 , 56,60, 62, 63, 66, 70, 72, 76, 78, 80, 84, 88, 90 , 96, 98, 100, 102, 104, 105, 108, 110, 112, 114, 120, 126, 130, 132, 140, 144, 150, 154, 156, 162, 168, 180, 186, 198, 210, 216, 228, 234, 240, 252, 264, 270, 280, 288, 294, 300, 312, 330, 336, 360, 390, 396, 420, 450, 462, 504, 540, 630 and 840. In addition, it is easy to check that the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 30, 32, 36, 38, 40, 42, 44, 45, 48, 50, 52, 54, 56,60, 66, 70, 72, 78, 80, 84, 90, 96, 100, 108, 112, 114, 120, 126, 132, 140, 144, 150, 156, 162, 168, 180, 198, 210, 216, 240, 252, 270, 288, 300, 336, 360, 396, 420 and 450 satisfy the inequality $d(p \cdot n) > \varphi(\varphi(n))$. Thus, the numbers quoted in the text of the present theorem are the only solutions of the equation $d(p \cdot n) = \varphi(\varphi(n))$, where $p \geq 23$ is prime with $p \neq 31$. This completes the proof.

Remark. By the same argument as above and by a brute force search with Maple in the range $1 \leq n \leq 10^{10}$, one can show that

- $\mathbb{E}_2 = \{11, 13, 33, 35, 38, 39, 44, 52, 63, 105, 114, 132, 140, 156, 252, 270, 420\}$.
- $\mathbb{E}_3 = \{11, 13, 34, 35, 45, 62, 76, 88, 98, 104, 110, 114\ 130, 154, 198, 252, 280, 360\}$.
- $\mathbb{E}_5 = \{11, 13, 33, 34, 39, 50, 62, 63, 76, 88, 98, 102, 104, 150, 154, 186, 228, 234, 264, 270, 294, 312, 462, 504\}$.
- $\mathbb{E}_7 = \{11, 13, 33, 34, 39, 62, 76, 88, 102 , 104, 110, 130, 186, 228, 234, 264, 312, 330, 390, 540\}$.
- $\mathbb{E}_{11} = \{13, 34, 35, 39, 62, 63, 76, 98, 102 , 104, 105, 130, 186, 228, 234, 280, 294, 312, 390, 504 , 540, 630, 840\}$.
- $\mathbb{E}_{13} = \{11, 33, 34, 35, 62, 63, 76, 88, 98 , 102, 105, 110 , 154, 186, 228, 264, 280, 294, 330, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{17} = \{11, 13, 33, 35, 39, 62, 63, 76, 88 , 98, 104, 105, 110, 130, 154, 186, 228, 234, 264, 280 , 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{19} = \{11, 13, 33, 34, 35, 38, 39, 62, 63 , 88, 98, 102, 104, 105, 110, 114, 130, 154, 186, 234, 264, 280, 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.
- $\mathbb{E}_{31} = \{11, 13, 33, 34, 35, 39, 63, 76, 88, 98, 102, 104, 105, 110, 130, 154, 228, 234, 264, 280 , 294, 312, 330, 390, 462, 504, 540, 630, 840\}$.

Moreover, from the proof of Theorem 3, for any prime $p \geq 23$ with $p \neq 31$ we deduce that $\mathbb{G}_p = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21 , 22, 24, 26, 27, 28, 30, 32, 36, 38, 40, 42, 44, 45, 48, 50, 52, 54, 56,60, 66, 70, 72, 78, 80, 84, 90, 96, 100, 108, 112, 114, 120, 126, 132, 140, 144, 150, 156, 162, 168, 180, 198, 210, 216, 240, 252, 270, 288, 300, 336, 360, 396, 420\ 450\}$. Thus, if I is a finite subset of positive integers, say $I = \{1, 2, ..., N\}$ with $N \geq 100$, then $\mathbb{L}_p \cap I = I - \mathbb{E}_p - \mathbb{G}_p = \{17, 19, 23, 25, 29, 31, 37, 41, 43, 46, 47, 49, 51, 53, 55, 57, 58, 59, 61, 64, 65, 67, 68, 69, 71, 73, 74, 75, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 97, 99, ...\}$.

4 Conclusion

In the previous sections, we investigated the solutions n of the equation (*i*) $d(kn) = \varphi(\varphi(n))$ and also the respective solutions of the corresponding inequalities (*ii*) $d(kn) < \varphi(\varphi(n))$ and (*iii*) $d(kn) > \varphi(\varphi(n))$. Since the positive integers are naturally partitioned into 3 subsets according to (*i*), (*ii*) and (*iii*), we have also characterized the elements of the sets $\mathbb{E}_1$, $\mathbb{L}_1$, $\mathbb{G}_1$, $\mathbb{E}_p$, $\mathbb{L}_p$ and $\mathbb{G}_p$ whenever p is prime. As a conclusion, we gave the relation between the sizes of these sets, where $\cap\ \mathbb{E}_p = \{\emptyset\}$ as p runs through the sequence of all primes. The same technique is applied to characterize the elements $\mathbb{E}_k$, $\mathbb{L}_k$ and $\mathbb{G}_k$ whenever k is composite.

Declarations

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