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Two-Parameter Exponential Distribution with Randomly Censored and Outlier Data

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Abstract: The two-parameter exponential distribution is an important statistical distribution, widely used in medicine, engineering, economics, demography, and longevity data. In the present work, the two-parameter exponential distribution under random censoring with the presence of outlier data is presented, and its parameters are estimated using the moment, maximum likelihood, and Bayesian methods. In parameter estimation, using maximum likelihood estimators, asymptotic confidence intervals are determined; while, by considering appropriate prior distributions, Bayesian estimation under squared error and LINEX loss functions is presented for the parameters. Next, using simulation, the estimators are compared by using statistical measures. Finally, utilizing a real data set, the suitability of the fitness of the model is evaluated according to different estimation methods.

Keywords: Estimation; Outliers; Random censoring; LINEX loss function; Lindleys approximation; Newton-Raphson method.

2010 Mathematics Subject Classification. 26A25; 26A35.

1 Introduction

In examining life data, the lifetime of each test substance is important. The application fields of lifetime data are medicine, biology, clinical trials, public health, epidemiology, engineering, economics, demography, criminology, industries, social relations, and business. Various statistical distributions are used in the analysis of lifetime data, including the Weibull distribution, Burr distribution, inverse Lomax distribution, and geometric inverse Lomax distribution. Collecting complete lifetime data is a costly and time-consuming task, due to which different types of censoring schemes are used in life tests. On the other hand, we often encounter situations where an item is randomly omitted from the test. For example, in clinical trials, a patient may die before the completion of treatment, or similarly, in engineering, there may be a reason to stop a project before its complete failure to save time and money. Gilbert (1962), first introduced random censoring. Chi-square goodness-of-fit tests for randomly censored data were studied by Kim (1993). Ghitany (2001) studied Rayleigh's mixed survival model and its application in randomly censored data and estimated the parameters of the model using the maximum likelihood method. Ghitany and Alawadhi (2002) considered the maximum likelihood estimation of the parameters of Burr XII distribution using randomly right-censored data. Saleem and Raza (2011) studied the Bayesian analysis of the exponential survival time with the assumption of exponential censoring time and estimated the parameters of this distribution using the maximum likelihood and Bayesian method. Dealt with Bayesian estimation in an exponential distribution under random censorship, Krishna and Vivekanand (2015) estimated the parameters of the Maxwell distribution with random censored data using maximum likelihood and Bayesian methods, Garg et al. (2016) studied the generalized inverted exponential distribution under random censoring. They studied different estimation procedures parameters of a generalized inverted exponential distribution. One of the important statistical topics is outlier data, which in practice has deviations from observations compared to other data, provoking the idea that it may fundamentally belong to a different distribution. In this case, examining the distribution of the data and estimating the parameters by considering the outlying data has particular importance. Let, the random variables $X_1, X_2, \ldots, X_{n-k}$ are independent,each having the probability density function $f_2(x;\theta_2)$, and the remaining random variables (k) are also independent, each having the probability density function

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$f_1(x;\theta_1)$, and the two sets of the random variables are also independent; The joint density function of the $X_1, X_2, \ldots, X_n$ is as follows:

$$f(x_1, x_2, \ldots, x_n; \Theta) = \frac{1}{\binom{n}{k}} \prod_{i=1}^{n} f_2(x_i;\theta_2) \sum_{\underline{A}} \prod_{j=1}^{k} \frac{f_1(x_{A_j};\theta_1)}{f_2(x_{A_j};\theta_2)}, \tag{1}$$

where $\sum_{\underline{A}} = \sum_{A_1=1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2} \cdots \sum_{A_k=A_{k-1}+1}^{n}$.
It can be shown that the marginal distribution of X_i is given by:

$$f(x;\Theta) = \frac{k}{n} f_1(x;\theta_1) + \frac{n-k}{n} f_2(x;\theta_2). \tag{2}$$

For more details see Dixit et al. (1996) and Dixit and Nasiri (2001). For k=2, the joint density function can be written as follow:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n; \Theta) &= \frac{1}{\binom{n}{2}} \prod_{i=1}^{n} f_2(x_i;\theta_2) \sum_{A_1=1}^{n-1} \sum_{A_1+1}^{n} \prod_{j=1}^{2} \frac{f_1(x_{A_j};\theta_1)}{f_2(x_{A_j};\theta_2)} \\ &= \frac{1}{\binom{n}{2}} \prod_{i=1}^{n} f_2(x_i;\theta_2) \sum_{A_1=1}^{n-1} \sum_{A_2=A_1+1}^{n} \frac{f_1(x_{A_1};\theta_1) f_1(x_{A_2};\theta_1)}{f_2(x_{A_1};\theta_2) f_2(x_{A_2};\theta_2)}. \end{aligned}$$

The two-parameter exponential distribution is used in data modeling including service time of system agents, decay of radioactive particles, lifetime or failure time, mutation distance in a DNA strand as well as medical, economic, and demographic studies, etc. The two-parameter exponential distribution is as follows:

$$f(x;\mu,\theta) = \frac{1}{\theta} exp\{\frac{-(x-\mu)}{\theta}\} \quad 0 \le \mu \le x \le \infty, \ 0 < \theta,$$

where μ and $\theta$are the location and scale parameters, respectively. By comparing the lifetime and the censorship time, the minimum time is defined as the guarantee. Therefore, the introduction of the minimum warranty period in the form of the location parameter is very important in modeling the survival data. Recently, Krishna and Goel (2018), studied the classical and Bayesian estimation for two-parameter exponential distribution having scale and location parameters with randomly censored data. They introduced the location parameter in the lifetime and censoring time models. They studied different estimation procedures for estimating the unknown location and scale parameters of a two-parameter exponential distribution. In addition, Nasiri (2022), studied estimators based on an interval shrinkage with equal weights point shrinkage estimators for all individual target points for exponentially distributed observations in the presence of outliers drawn from a uniform distribution. A useful asymmetric loss function, known as the LINEX loss function, was introduced by Varian (1975) and widely used by several authors. For example, Basu and Ebrahimi (1991), derived Bayesian estimators of average lifetime and reliability function in the exponential model under asymmetric loss functions and showed that the overestimation of the reliability function or average failure time is usually much more serious than the underestimation of reliability function or mean failure time. Also, an underestimation of the failure rate results in more serious consequences than an overestimation of the failure rate. However, Bayesian estimation under the LINEX loss function is often not discussed due to the integral terms being analytically solvable. Therefore, special numerical techniques such as Lindley's approximation technique should be used for solving. Singh and Singh (2008), obtained the Bayes estimator of generalized exponential scale and shape parameter using Lindleys approximation under the LINEX loss function. The present paper estimates the parameters of the two parameters of exponential distribution with censored data in the presence of outliers by methods of the moment, maximum likelihood, and Bayes estimators under LINEX and square error loss functions using the Lindley approximation technique. In the continuation of the article, in section 2, the two-parameter exponential distribution with outlier data with the title of failure and data censoring time distribution is introduced. In section 3, first, the methods of the moment and maximum likelihood estimates, and then the variance and asymptotic distance of the parameters are discussed by using the maximum likelihood estimation of the parameters. In section 4, the formulation of the Bayesian estimation method under the LINEX and squared error loss functions with inverse gamma prior distribution are presented. In Section's 5 and 6, the parameters are estimated and checked for real data set. Finally, Section 7 presents the conclusions. Throughout the article, R statistical software is used for calculations.

2 The model and its assumptions

Two types of tires, R-4 and F-2M, are suitable tires for tractors F-2M tires are the best tractor tires, and R-4 tires have an appropriate tread, which makes them a good choice because its performance in various situations. An agricultural machinery tire manufacturing company produces these two types. We assume that a set of random variables $X_1, X_2, \dots, X_n$ represent the lifetime of the tires. Some of the observations are from the F-2M series tires, which have a two-parameter exponential distribution with the parameters (μ, α), and other observations from n random variables (for example, k), are for R-4 series tires, which have a two-parameter exponential distribution with the parameters (μ, θ). We assume that the random variables$X_1, X_2, \dots, X_n$ are such that k of with pdf,

$$f_1(x;\mu,\theta) = \frac{1}{\theta} exp\{\frac{-(x-\mu)}{\theta}\} I_{(\mu,\infty)} \quad \theta > 0\ ,\ 0 \le \mu \le x \le \infty, \tag{3}$$

and the remaining $(n-k)$ random variables with pdf,

$$f_2(x;\mu,\alpha) = \frac{1}{\alpha} exp\{\frac{-(x-\mu)}{\alpha}\} I_{(\mu,\infty)} \quad \alpha > 0\ ,\ 0 \le \mu \le x \le \infty, \tag{4}$$

according to the equation (2), the marginal distribution of X_i is given by

$$f(x;\mu,\theta,\alpha) = \frac{k}{n\theta} exp\{\frac{-(x-\mu)}{\theta}\} + \frac{n-k}{n\alpha} exp\{\frac{-(x-\mu)}{\alpha}\}, \tag{5}$$

also, let $T_1, T_2, \dots, T_n$ be the (i.i.d.) random censoring times of these items. Let the pdf of $T_i s$ be $f_T(t, \mu, 1)$. such that X_i and T_i are independent and inconsistent. According the definition of random variables X_i, T_i, We define the variables Y_i and D_i as: $Y_i = min(X_i, T_i)$ and

$$D_i = \begin{cases} 1 & X_i \le T_i \\ 0 & T_i < X_i \end{cases}, \tag{6}$$

it is assumed that the survival time X and the censoring time T, independently follow the random has two-parameter exponential distribution with the presence of outliers, then variable X it has pdf (5) and the censoring time random variable T has the following density function:

$$f(t;\mu) = exp\{-(t-\mu)\} \qquad 0 \le \mu \le t \le \infty, \tag{7}$$

where μ is threshold parameter of the minimum life time or warranty period. The probability of an item failing before being censored is given by:

$$p = P[D=1] = P[X \le T] = \int_0^\infty P[X \le T | T = t] f_T(t) d(t) \tag{8}$$

$$= \int_0^\infty \int_\mu^t (\frac{k}{n\theta} exp\{\frac{-(x-\mu)}{\theta}\} + \frac{n-k}{n\alpha} exp\{\frac{-(x-\mu)}{\alpha}\}) exp\{-(t-\mu)\} dx dt = 1 - \frac{(n-k)\alpha}{n(\alpha+1)} - \frac{k\theta}{n(\theta+1)}. \tag{9}$$

As we know that the randoms variables Y_is and D_is are independent, therefore the joint probability function of (Y_i, D_i) is as follows:

$$f_{Y,D}(y_i, d_i; \mu, \theta, \alpha) = [f_X(y_i;\mu,\theta,\alpha)(1 - F_T(y_i;\mu))]^{d_i} [f_T(y;\mu)(1 - F_X(y;\mu,\theta,\alpha))]^{(1-d_i)}$$

$$= [\frac{k}{n\theta} exp\{\frac{-(y_i-\mu)(\theta+1)}{\theta}\} + \frac{n-k}{n\alpha} exp\{\frac{-(y_i-\mu)(\alpha+1)}{\alpha}\}]^{d_i} [\frac{k}{n} exp\{\frac{-(y_i-\mu)(\theta+1)}{\theta}\} + \frac{n-k}{n} exp\{\frac{-(y_i-\mu)(\alpha+1)}{\alpha}\}]^{(1-d_i)}. \tag{10}$$

In which Y has a two-parameter exponential distribution in the presence of outlier with parameters $(\frac{\theta+1}{\theta})$ and $(\frac{\alpha+1}{\alpha})$, hence

$$f_Y(y;\mu,\theta,\alpha) = \frac{k(\theta+1)}{n\theta} exp\{-\frac{(y-\mu)(\theta+1)}{\theta}\} + \frac{(n-k)(\alpha+1)}{n\alpha} exp\{-\frac{(y-\mu)(\alpha+1)}{\alpha}\}, \tag{11}$$

and

$$f_D(d|\mu,\theta,\alpha) = \begin{cases} \frac{(n-k)}{n(\alpha+1)} + \frac{k}{n(\theta+1)} & d = 1 \\ \frac{(n-k)\alpha}{n(\alpha+1)} + \frac{k\theta}{n(\theta+1)} & d = 0 \end{cases}. \tag{12}$$

In the special case, if $k = 0$, then the equation (10) is equal to:

$$f_{Y,D}(y_i, d_i; \mu, \alpha) = [f_X(y_i;\mu,\alpha)(1 - F_T(y_i;\mu,1))]^{d_i} [f_T(y_i;\mu,1)(1 - F_X(y_i;\mu,\alpha))]^{(1-d_i)}$$

$$= [\frac{1}{\alpha^{d_i}}] exp\{[\frac{-(y_i-\mu)(\alpha+1)}{\alpha}]\},$$

which is the same as the density function of the two-parameter exponential distribution given by Krishna and Goel (2018). According to the equation (11) the r-th population moment is given by

$$E(Y^r) = \int_{\mu}^{\infty} y^r [\tfrac{k(\theta+1)}{n\theta} exp\{\tfrac{-(y-\mu)(\theta+1)}{\theta}\} + \tfrac{n-k(\alpha+1)}{n\alpha} exp\{\tfrac{-(y-\mu)(\alpha+1)}{\alpha}\}]dy \tag{13}$$
$$= \tfrac{k}{n}\sum_{t=0}^{r}\binom{r}{t}\mu^{r-t}(\tfrac{\theta}{\theta+1})^t\Gamma(t+1) + \tfrac{n-k}{n}\sum_{t=0}^{r}\binom{r}{t}\mu^{r-t}(\tfrac{\alpha}{\alpha+1})^t\Gamma(t+1),$$

according to the equation (12) the r-th population moment is given by

$$E(D^r) = \sum_{d=0}^{1} d^r f_D(d|\mu,\theta,\alpha) = \tfrac{(n-k)}{n(\alpha+1)} + \tfrac{k}{n(\theta+1)}, \tag{14}$$

according to the equations (13) and (14) the mathematical expectation and variance of Y and D variables are calculated as follows:

$$E(Y) = \mu + \tfrac{k\theta}{n(1+\theta)} + \tfrac{(n-k)\alpha}{n(1+\alpha)}, \tag{15}$$

$$E(Y^2) = \mu^2 + 2\mu[\tfrac{k\theta}{n(1+\theta)} + \tfrac{(n-k)\alpha}{n(1+\alpha)}] + 2[\tfrac{k}{n}(\tfrac{\theta}{\theta+1})^2 + \tfrac{n-k}{n}(\tfrac{\alpha}{\alpha+1})^2],$$

$$E(D) = E(D^2) = \tfrac{k}{n(1+\theta)} + \tfrac{n-k}{n(1+\alpha)}, \tag{16}$$

$$V(Y) = E(Y^2) - E^2(Y) = 2[\tfrac{k}{n}(\tfrac{\theta}{\theta+1})^2 + \tfrac{n-k}{n}(\tfrac{\alpha}{\alpha+1})^2] - [\tfrac{k\theta}{n(1+\theta)} + \tfrac{(n-k)\alpha}{n(1+\alpha)}]^2, \tag{17}$$

$$V(D) = E(D^2) - E^2(D) = [\tfrac{k}{n(1+\theta)} + \tfrac{n-k}{n(1+\alpha)}][\tfrac{k\theta}{n(1+\theta)} + \tfrac{(n-k)\alpha}{n(1+\alpha)}]. \tag{18}$$

3 Classical methods of estimation

In this section, the parameters of the two-parameter exponential distribution with randomly censored and outliers are estimated by using the methods of moment and maximum likelihood estimation.

3.1 Method of moments

According to the equations (15), (16), (17), (18) and sample moments

$$s_y^2 = \tfrac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2, \quad \bar{y} = \tfrac{1}{n}\sum_{i=1}^{n} y_i, \quad s_d^2 = \tfrac{1}{n}\sum_{i=1}^{n}(d_i-\bar{d})^2, \quad \bar{d} = \tfrac{1}{n}\sum_{i=1}^{n} d_i,$$

we can write:

$$E(D) = \tfrac{k}{n(1+\theta)} + \tfrac{n-k}{n(1+\alpha)} = \bar{d},$$

on the other hand,

$$1-\bar{d} = \tfrac{k(1+\theta)(1+\alpha)-k(1+\alpha)}{n(1+\theta)(1+\alpha)} + \tfrac{(n-k)(1+\theta)(1+\alpha)-(n-k)(1+\theta)}{n(1+\theta)(1+\alpha)} = \tfrac{k\theta}{n(1+\theta)} + \tfrac{(n-k)\alpha}{n(1+\alpha)},$$

hence

$$E(Y) = \bar{y} = \mu + \tfrac{k\theta}{n(1+\theta)} + \tfrac{(n-k)\alpha}{n(1+\alpha)} = \mu + 1 - \bar{d},$$

so

$$\hat{\mu} = \bar{y} + \bar{d} - 1. \tag{19}$$

According to equation (15), the estimator of θ is given by

$$\hat{\theta} = \frac{\frac{n(\bar{y}-\mu)}{k} - \frac{(n-k)\alpha}{k(1+\alpha)}}{1-[\frac{n(\bar{y}-\mu)}{k} - \frac{(n-k)\alpha}{k(1+\alpha)}]}, \tag{20}$$

according to equations (17) and (18) we have:

$$V(D) = s_d^2 = [\frac{k}{n(1+\hat{\theta})} + \frac{n-k}{n(1+\alpha)}][\frac{k\hat{\theta}}{n(1+\hat{\theta})} + \frac{(n-k)\alpha}{n(1+\alpha)}]$$
$$= [1-(\bar{y}-\mu)](\bar{y}-\mu) = (\bar{y}-\mu) - (\bar{y}-\mu)^2,$$

$$Var(Y) = s_y^2 = 2[\frac{k}{n}(\frac{\hat{\theta}}{\hat{\theta}+1})^2 + \frac{n-k}{n}(\frac{\alpha}{\alpha+1})^2] - (\bar{y}-\mu)^2,$$

$$s_y^2 - s_d^2 = 2[\frac{k}{n}(\frac{\hat{\theta}}{\hat{\theta}+1})^2 + \frac{n-k}{n}(\frac{\alpha}{\alpha+1})^2] - (\bar{y}-\mu),$$

according to equation (20) can be write

$$s_y^2 - s_d^2 = 2[\frac{k}{n}(\frac{n(\bar{y}-\mu)}{k} - \frac{(n-k)\alpha}{k(1+\alpha)})^2 + \frac{n-k}{n}(\frac{\alpha}{\alpha+1})^2] - (\bar{y}-\mu),$$

$$f(\alpha) = s_y^2 - s_d^2 + (\bar{y}-\mu) - 2[\frac{k}{n}(\frac{n(\bar{y}-\mu)}{k} - \frac{(n-k)\alpha}{k(1+\alpha)})^2 + \frac{n-k}{n}(\frac{\alpha}{\alpha+1})^2], \tag{21}$$

there is no closed-form solution to this system of equations, so we will solve for θ and α iteratively, using the Newton-Raphson method:

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)},$$

such that,

$$f'(\alpha_i) = -4[\frac{(n-k)\alpha}{n(1+\alpha)^3} - (\frac{n(\bar{y}-\mu)}{k} - \frac{(n-k)\alpha}{k(1+\alpha)})(\frac{(n-k)}{n(1+\alpha)^2})].$$

3.2 Maximum likelihood estimation (MLE)

In this section, we estimate the parameters of the two-parameter exponential distribution with censored data in the presence of outliers using the maximum likelihood method. For this purpose, suppose $(\underset{\sim}{y}, \underset{\sim}{d}) = (y_i, d_i), i = 1, 2, \ldots, n$ is a random sample of size n from the density function defined in (10), then likelihood function (L) is given by

$$L = f(\underset{\sim}{y}, \underset{\sim}{d} | \mu, \theta, \alpha) \tag{22}$$

$$= [\frac{1}{\binom{n}{k}}(\frac{1}{\alpha})^{n-k}(\frac{1}{\theta})^k exp\{-\sum_{i=1}^{n}(y_{(i)} - y_{(1)})(\frac{\alpha+1}{\alpha})\}exp\{-n(y_{(1)}-\mu)(\frac{\alpha+1}{\alpha})\}$$

$$\sum_{\underline{A}} exp\{-\sum_{j=1}^{k}(y_{A_j} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}exp\{-k(y_{(1)}-\mu)(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}]^{\sum_{i=1}^{n} d_i}$$

$$[\frac{1}{\binom{n}{k}} e^{-\sum_{i=1}^{n}(y_{(i)}-y_{(1)})(\frac{\alpha+1}{\alpha})} exp\{-n(y_{(1)}-\mu)(\frac{\alpha+1}{\alpha})\}$$

$$\sum_{\underline{A}} exp\{-\sum_{j=1}^{k}(y_{A_j} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}exp\{-k(y_{(1)}-\mu)(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}]^{(n-\sum_{i=1}^{n} d_i)}$$

$$= [\frac{1}{\binom{n}{k}}]^n [\frac{1}{\alpha}]^{(n-k)\sum_{i=1}^{n} d_i} [\frac{1}{\theta}]^{k\sum_{i=1}^{n} d_i} exp\{-n\sum_{i=1}^{n}(y_{(i)} - y_{(1)})(\frac{\alpha+1}{\alpha})\}exp\{n(y_{(1)}-\mu)(\frac{\alpha+1}{\alpha})\}$$

$$[\sum_{\underline{A}} exp\{-\sum_{j=1}^{k}(y_{A_j} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}exp\{-k(y_{(1)}-\mu)(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}]^n I_{(\mu,\infty)}(y_{(1)}) \qquad \theta, \alpha > 0,$$

such that

$$I_{(\mu,\infty)}(y_{(1)}) = \begin{cases} 1 & y_{(1)} \geq \mu \\ 0 & y_{(1)} < \mu \end{cases}, \tag{23}$$

where, $y_{(1)} = min(y_1, y_2, \ldots, y_n)$. It is well known, that the MLE of μ is $\hat{\mu} = y_{(1)}$. The log likelihood function (l) is given by

$$l(\theta,\alpha) = log(L) = -nlog(\binom{n}{k}) - (\sum_{i=1}^{n} d_i)klog(\theta) - (\sum_{i=1}^{n} d_i)(n-k)log(\alpha)$$
$$-n\sum_{i=1}^{n}(y_{(i)} - y_{(1)})(\tfrac{\alpha+1}{\alpha}) + nlog[\sum_{\underline{A}} exp\{-\sum_{j=1}^{k}(y_{A_j} - y_{(1)})(\tfrac{\theta+1}{\theta} - \tfrac{\alpha+1}{\alpha})\}], \tag{24}$$

take the derivative of the equation (24) with respect to the parameters θ and α, set the partial derivatives equal to zero, and solve for θ and α:

$$l_\theta = \frac{\partial l(\theta,\alpha)}{\partial\theta} = \frac{-(\sum_{i=1}^{n} d_i)k}{\theta} + \frac{n\sum_{\underline{A}} w_\theta}{\sum_{\underline{A}} w(y_A;\theta,\alpha)} = 0,$$
$$l_\alpha = \frac{\partial l(\theta,\alpha)}{\partial\alpha} = \frac{-(\sum_{i=1}^{n} d_i)(n-k)}{\alpha} + \frac{n\sum_{i=1}^{n}(y_{(i)} - y_{(1)})}{\alpha^2} + \frac{n\sum_{\underline{A}} w_\alpha}{\sum_{\underline{A}} w(y_A;\theta,\alpha)} = 0, \tag{25}$$

such that $w(y_A;\theta,\alpha) = exp\{-\sum_{j=1}^{k}(y_{A_j} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}$. Now to obtain the first and second derivatives of $w(y_A;\theta,\alpha)$, we can write:

$$w_\theta = \frac{\sum_{r=1}^{k}(y_{A_r} - y_{(1)})}{\theta^2} exp\{-\sum_{r=1}^{k}(y_{A_r} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}, \tag{26}$$

$$w_\alpha = \frac{-\sum_{r=1}^{k}(y_{A_r} - y_{(1)})}{\alpha^2} exp\{-\sum_{r=1}^{k}(y_{A_r} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}, \tag{27}$$

$$w_{\theta\theta} = \left[\frac{[\sum_{r=1}^{k}(y_{A_r} - y_{(1)})]^2 - 2\theta\sum_{r=1}^{k}(y_{A_r} - y_{(1)})}{\theta^4}\right] exp\{-\sum_{r=1}^{k}(y_{A_r} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}, \tag{28}$$

$$w_{\alpha\alpha} = \left[\frac{[\sum_{r=1}^{k}(y_{A_r} - y_{(1)})]^2 + 2\alpha\sum_{r=1}^{k}(y_{A_r} - y_{(1)})}{\alpha^4}\right] exp\{-\sum_{r=1}^{k}(y_{A_r} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}, \tag{29}$$

$$w_{\theta\alpha} = w_{\alpha\theta} = \frac{-[\sum_{r=1}^{k}(y_{A_r} - y_{(1)})]^2}{\theta^2\alpha^2} exp\{-\sum_{r=1}^{k}(y_{A_r} - y_{(1)})(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha})\}, \tag{30}$$

there is no closed-form solution to this system of equations, so we will solve for θ and α iteratively, using the Newton-Raphson method. In our case we will estimate $\beta = (\theta,\alpha)$ iteratively:

$$\hat{\beta}_{i+1} = \hat{\beta}_i - G^{-1}g, \tag{31}$$

where$g$ is the vector of normal equations for which we want $g = [g_1, g_2]$ with $g_1 = l_\theta$ and $g_2 = l_\alpha$ and G is:

$$G = \begin{bmatrix} \frac{\partial g_1}{\partial\theta} & \frac{\partial g_1}{\partial\alpha} \\ \frac{\partial g_2}{\partial\theta} & \frac{\partial g_2}{\partial\alpha} \end{bmatrix} = \begin{bmatrix} l_{\theta\theta} & l_{\theta\alpha} \\ l_{\alpha\theta} & l_{\alpha\alpha} \end{bmatrix},$$

and

$$\frac{\partial g_1}{\partial\theta} = l_{\theta\theta} = \frac{(\sum_{i=1}^{n} d_i)k}{\theta^2} + \frac{n\sum_{\underline{A}} w_{\theta\theta}}{\sum_{\underline{A}} w(y_A;\theta,\alpha)} - n[\frac{\sum_{\underline{A}} w_\theta}{\sum_{\underline{A}} w(y_A;\theta,\alpha)}]^2, \tag{32}$$

$$\frac{\partial g_2}{\partial\alpha} = l_{\alpha\alpha} = \frac{(\sum_{i=1}^{n} d_i)(n-k)}{\alpha^2} - \frac{2n\sum_{i=1}^{n}(y_{(i)} - y_{(1)})}{\alpha^3} + \frac{n\sum_{\underline{A}} w_{\alpha\alpha}}{\sum_{\underline{A}} w(y_A;\theta,\alpha)} - n[\frac{\sum_{\underline{A}} w_\alpha}{\sum_{\underline{A}} w(y_A;\theta,\alpha)}]^2, \tag{33}$$

$$\frac{\partial g_1}{\partial \alpha} = l_{\theta\alpha} = \frac{n\sum_A w_{\theta\alpha}}{\sum_A w(y_A;\theta,\alpha)} - n\frac{\sum_A w_\theta \sum_A w_\alpha}{(\sum_A w(y_A;\theta,\alpha))^2}, \tag{34}$$

$$\frac{\partial g_2}{\partial \theta} = l_{\alpha\theta} = \frac{n\sum_A w_{\alpha\theta}}{\sum_A w(y_A;\theta,\alpha)} - n\frac{\sum_A w_\theta \sum_A w_\alpha}{(\sum_A w(y_A;\theta,\alpha))^2}, \tag{35}$$

where $w_{\theta\theta}$, $w_{\alpha\alpha}$ and $w_{\alpha\theta}$ are defined in the equations (28)-(30), respectively.

3.3 Asymptotic distance

Based on $\hat{\mu} = y_{(1)}$ and

$$f(y_{(1)}) = n[\frac{k(\theta+1)}{n\theta}exp\{\frac{-(y_{(1)}-\mu)(\theta+1)}{\theta}\} + \frac{n-k(\alpha+1)}{n\alpha}e^{\frac{-(y_{(1)}-\mu)(\alpha+1)}{\alpha}}][\frac{k}{n}exp\{\frac{-(y_{(1)}-\mu)(\theta+1)}{\theta}\} + \frac{n-k}{n}exp\{\frac{-(y_{(1)}-\mu)(\alpha+1)}{\alpha}\}]^{n-1} \tag{36}$$

$$= n\sum_{j=0}^{n-1}\binom{n-1}{j}[(\frac{k}{n})^{j+1}(\frac{n-k}{n})^{n-j-1}\frac{(\theta+1)}{\theta}exp\{-(y_{(1)}-\mu)a\} + (\frac{k}{n})^{j}(\frac{n-k}{n})^{n-j}\frac{(\alpha+1)}{\alpha}exp\{-(y_{(1)}-\mu)b\}],$$

where $a = \frac{(\theta+1)}{\theta}(j+1) + \frac{(\alpha+1)}{\alpha}(n-j-1)$, $b = \frac{(\theta+1)}{\theta}j + \frac{(\alpha+1)}{\alpha}(n-j)$, so

$$E(Y_{(1)}) = n\sum_{j=0}^{n-1}\binom{n-1}{j}[(\frac{k}{n})^{j+1}(\frac{n-k}{n})^{n-j-1}(\frac{\theta+1}{\theta})(\frac{-ay_{(1)}-1}{a^2}) + (\frac{k}{n})^{j}(\frac{n-k}{n})^{n-j}(\frac{\alpha+1}{\alpha})(\frac{-by_{(1)}-1}{b^2})], \tag{37}$$

$$E(Y_{(1)}^2) = n\sum_{j=0}^{n-1}\binom{n-1}{j}[(\frac{k}{n})^{j+1}(\frac{n-k}{n})^{n-j-1}(\frac{\theta+1}{\theta})(-\frac{y_{(1)}^2}{a} - \frac{2y_{(1)}}{a^2} - \frac{2}{a^3}) + (\frac{k}{n})^{j}(\frac{n-k}{n})^{n-j}(\frac{\alpha+1}{\alpha})(-\frac{y_{(1)}^2}{b} - \frac{2y_{(1)}}{b^2} - \frac{2}{b^3})],$$

$$V(Y_{(1)}) = E(Y_{(1)}^{(2)}) - E^2(Y_{(1)}) \tag{38}$$

The $(1-\gamma)\%$ confidence interval for μ is given by

$$[\hat{\mu} - \sqrt{Var(\hat{\mu})}Z_{(1-\frac{\gamma}{2})}, \hat{\mu} + \sqrt{Var(\hat{\mu})}Z_{(1-\frac{\gamma}{2})}].$$

Now to find of confidence interval for $\beta = (\theta,\alpha)$, we derived the asymptotic confidence interval (ACI) forβ by utilizing its asymptotic distribution. With the help of log likelihood function, we use the Fisher information matrix to find the variances of the MLEs of the parameters θ and α. The Fisher information matrix for $\beta = (\theta,\alpha)$ is written as follows:

$$I(\beta) = -E\begin{bmatrix} \frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2}{\partial\theta\partial\alpha} \\ \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\alpha^2} \end{bmatrix} = \begin{bmatrix} I_{11}(\beta) & I_{12}(\beta) \\ I_{21}(\beta) & I_{22}(\beta) \end{bmatrix}.$$

This Fisher's information matrix includes unknown parameters $\beta = (\theta,\alpha)$. Therefore, by replacing the MLE estimation of the parameters $\hat{\beta} = (\hat{\theta},\hat{\alpha})$, the estimated values of the Fisher information matrix are:

$$I(\hat{\beta}) = \begin{bmatrix} I_{11}(\hat{\theta},\hat{\alpha}) & I_{12}(\hat{\theta},\hat{\alpha}) \\ I_{21}(\hat{\theta},\hat{\alpha}) & I_{22}(\hat{\theta},\hat{\alpha}) \end{bmatrix},$$

hence, $(1-\gamma)\%$ confidence interval for parameters β_i, $i = 1,2$ is given by

$$[\hat{\beta}_i - \sqrt{Var(\hat{\beta}_i)}Z_{(1-\frac{\gamma}{2})}, \hat{\beta}_i + \sqrt{Var(\hat{\beta}_i)}Z_{(1-\frac{\gamma}{2})}], \tag{39}$$

where, $Var(\hat{\beta}_i)$ is the $(i,i)^{th}$ diagonal element of $I^{-1}(\hat{\beta})$ and $Z_{(1-\frac{\gamma}{2})}$ is the $(1-\frac{\gamma}{2})^{th}$ quantile of the standard normal distribution.

4 Bayesian estimation

In this section, we now derive the Bayesian estimation of the parameters μ,θ and α of Two-parameter exponential distribution with censored data in the presence of outliers when the parameters μ,θ and α is known. We consider two different loss functions, square error loss (SEL) function and LINEX loss function. Let θ and α follow conjugate inverted gamma priors with parameters (a_1,b_1) and (a_2,b_2), respectively, with pdfs:

$$\Pi_1(\theta,a_1,b_1) \propto \frac{1}{\theta^{a_1+1}}\exp\big(\frac{-b_1}{\theta}\big) \qquad \theta>0,a_1,b_1>0, \tag{40}$$

$$\Pi_2(\theta,a_2,b_2) \propto \frac{1}{\alpha^{a_2+1}}\exp\big(\frac{-b_2}{\alpha}\big) \qquad \alpha>0,a_2,b_2>0, \tag{41}$$

we consider the following improper uniform prior of μ as follows:

$$\Pi_3(\mu) \propto \frac{1}{c} \qquad -\infty<\mu<\infty,\;\; c>0, \tag{42}$$

moreover, all the three parameters are assumed to be independent. Now, the joint prior distribution of the random vector (μ,θ,α) is thus given by

$$\Pi(\mu,\theta,\alpha) = \Pi_1(\theta)\Pi_2(\alpha)\Pi_3(\mu) = \frac{1}{c}\frac{1}{\theta^{a_1+1}}\frac{1}{\alpha^{a_2+1}}\exp\big(-\frac{b_1}{\theta}-\frac{b_2}{\alpha}\big). \tag{43}$$

The posterior density function of (μ,θ,α) given $(\underset{\sim}{y},\underset{\sim}{d}) = (y_i,d_i), i=1,2,\ldots,n$ is obtained by combining equations (22) and (43), and be written as

$$\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d}) = \frac{f(\underset{\sim}{y},\underset{\sim}{d}|\mu,\theta,\alpha)\Pi(\mu,\theta,\alpha)}{\int_0^\infty\int_0^\infty\int_0^{y_1} f(\underset{\sim}{y},\underset{\sim}{d}|\mu,\theta,\alpha)\Pi(\mu,\theta,\alpha)d\mu d\theta d\alpha} \tag{44}$$

$$= \frac{\big[\frac{1}{\alpha}\big]^{(n-k)\sum d_i+a_2+1}\big[\frac{1}{\theta}\big]^{k\sum d_i+a_1+1} exp\big\{-n\sum_{i=1}^{n}(y_i-\mu)\big(\frac{\alpha+1}{\alpha}\big)\big\}\big[B_{A_r}\big]^n\exp\big(\frac{-b_1}{\theta}\big)\exp\big(\frac{-b_2}{\alpha}\big)}{\int_0^\infty\int_0^\infty\int_0^{y_1}\big[\frac{1}{\alpha}\big]^{(n-k)\sum d_i+a_2+1}\big[\frac{1}{\theta}\big]^{k\sum d_i+a_1+1} exp\big\{-n\sum_{i=1}^{n}(y_i-\mu)\big(\frac{\alpha+1}{\alpha}\big)\big\}\big[B_{A_r}\big]^n\exp\big(\frac{-b_1}{\theta}\big)\exp\big(\frac{-b_2}{\alpha}\big)d\mu d\theta d\alpha},$$

such that

$$B_{A_r} = \sum_{\underline{A}} exp\big\{-\sum_{r=1}^{k}(y_{A_r}-\mu)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}.$$

It is well known that under squared error loss function, the Bayes estimator of a function is the posterior mean of the function and is given by a ratio of two integrals.

$$\hat{U}(\mu,\theta,\alpha) = E\big[U(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})\big] = \frac{\int U(\mu,\theta,\alpha)e^{l(\mu,\theta,\alpha)+G(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{l(\mu,\theta,\alpha)+G(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}, \tag{45}$$

such that, $U(\mu,\theta,\alpha)$ function of μ,θ and α, $l(\mu,\theta,\alpha)$= log of likelihood function, $G(\mu,\theta,\alpha)$= log of joint prior of μ,θ and α. Both integrals cannot be obtained in a simple closed form and we must use numerical integration technique, which can be computationally intensive, especially in high dimensional parameter space. Instead, we use Lindley's method [2] approximation to obtain the Bayes estimator of the function $U(\mu,\theta,\alpha)$. Under squared error loss function, Lindley showed that the approximate Bayes estimate of the function Ψ about the posterior mode $\hat{\Psi}$ is of the form

$$\widehat{U(\Psi)} = U(\hat{\Psi}) + \tfrac{1}{2}\sum_{i,j=1}^{N}U_{ij}(\hat{\Psi})\tau_{ij} + \tfrac{1}{2}\sum_{i,j,k,l=1}^{N}\Delta_{ijk}(\hat{\Psi})\tau_{ij}\tau_{kl}U_l(\hat{\Psi}) \;\; i,j,k=1,2,\ldots,N, \tag{46}$$

$$U_i(\hat{\Psi}) = \frac{\partial U(\Psi)}{\partial\theta_i}\;,\; U_{ij}(\hat{\Psi}) = \frac{\partial^2 U(\Psi)}{\partial\psi_i\partial\psi_j}\;,\; \Delta_{ij}(\hat{\Psi}) = \frac{\partial^2\Delta(\Psi)}{\partial\psi_i\partial\psi_j}, \Delta_{ijk}(\hat{\Psi}) = \frac{\partial^3\Delta(\Psi)}{\partial\theta_i\partial\psi_j\partial\psi_k}\;,\; \Sigma_{N\times N} = \tau_{ij} = (-\Delta_{ij})^{-1}_{N\times N},$$

for the three-parameter case, Lindleys approximation (46) reduces to the following

$$\hat{U}_B(\mu,\theta,\alpha) \equiv U(\hat{\mu},\hat{\theta},\hat{\alpha}) + \frac{1}{2}\sum_{i,j,k=1}^{3}U_{i,j,k}(\hat{\mu},\hat{\theta},\hat{\alpha})\tau_{ijk} + \frac{1}{2}\sum_{i,j,k,l=1}^{3}\Delta_{ijk}(\hat{\mu},\hat{\theta},\hat{\alpha})\tau_{ij}\tau_{kl}U_l(\hat{\mu},\hat{\theta},\hat{\alpha}), \tag{47}$$

such that,

$$\Delta(\mu,\theta,\alpha) \propto l(\mu,\theta,\alpha)+G(\mu,\theta,\alpha) = -\big(\sum_{l}^{n} d_i(n-k)+a_2+1\big)\log(\alpha) - \big(\sum_{l}^{n} d_i(k)+a_1+1\big)\log(\theta)$$
$$-n\sum_{ji=1}^{n}\big(y_{(i)}-\mu\big)\big(\frac{\alpha+1}{\alpha}\big)+n\log\Big[\sum_{A}\Big(exp\Big\{-\sum_{r=1}^{k}(y_{A_r}-\mu)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\Big\}\Big)\Big]-\big[\frac{b_1}{\theta}+\frac{b_2}{\alpha}\big], \quad (48)$$

for simplicity, we put $w(y_A;\mu,\theta,\alpha)=exp\Big\{-\sum_{r=1}^{k}(y_{A_r}-\mu)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\Big\}$. To obtain the first, second and third derivatives of $w(y_A;\mu,\theta,\alpha)$, we can write:

$$w_\mu = k\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (49)$$

$$w_\theta = \frac{\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)}{\theta^2}exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (50)$$

$$w_\alpha = \frac{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)}{\alpha^2}exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (51)$$

$$w_{\mu\mu} = k^2\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)^2exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (52)$$

$$w_{\theta\theta} = \left[\frac{\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^2-2\theta\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)}{\theta^4}\right]exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (53)$$

$$w_{\alpha\alpha} = \left[\frac{\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^2+2\alpha\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)}{\alpha^4}\right]exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (54)$$

$$w_{\theta\alpha} = w_{\alpha\theta} = \frac{-\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^2}{\theta^2\alpha^2}exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (55)$$

$$w_{\theta\mu} = w_{\mu\theta} = \frac{k}{\theta^2}(-1+\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big))exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (56)$$

$$w_{\alpha\mu} = w_{\mu\alpha} = \frac{k}{\alpha^2}(1-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big))exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (57)$$

$$w_{\alpha\theta\theta} = w_{\theta\alpha\theta} = w_{\theta\theta\alpha} = \frac{2\theta\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^2-\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^3}{\alpha^2\theta^4}exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (58)$$

$$w_{\alpha\theta\alpha} = w_{\theta\alpha\alpha} = w_{\alpha\alpha\theta} = \frac{2\alpha\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^2+\big[\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big]^3}{\alpha^4\theta^2}exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (59)$$

$$w_{\mu\mu\alpha} = w_{\mu\alpha\mu} = w_{\alpha\mu\mu} = \frac{k^2}{\alpha^2}\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big(2-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big)$$
$$exp\big\{-\sum_{r=1}^{k}\big(y_{A_r}-\mu\big)\big(\frac{\theta+1}{\theta}-\frac{\alpha+1}{\alpha}\big)\big\}, \quad (60)$$

$$w_{\mu\mu\theta} = w_{\mu\theta\mu} = w_{\theta\mu\mu} = \frac{k^2}{\theta^2}\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)\big(-2 + \sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)\big)$$
$$exp\big\{-\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)\big\}, \quad (61)$$

$$w_{\mu\alpha\alpha} = w_{\alpha\mu\alpha} = w_{\alpha\alpha\mu} \quad (62)$$
$$= (k\Bigg[\frac{\big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^2 + 2\alpha\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)}{\alpha^4}\Bigg]\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big) - 2\frac{k}{\alpha^4}\big(\alpha + \sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big)\big)$$
$$exp\big\{-\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)\big\}, \quad (63)$$

$$w_{\mu\theta\theta} = w_{\theta\mu\theta} = w_{\theta\theta\mu}$$
$$= (k\Bigg[\frac{\big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^2 - 2\theta\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)}{\theta^4}\Bigg]\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big) + 2\frac{k}{\theta^4}\big(\theta - \sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big)\big)$$
$$exp\big\{-\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)\big\}, \quad (64)$$

$$w_{\mu\mu\mu} = k^3\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)^3 exp\big\{-\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big)\big\}, \quad (65)$$

$$w_{\theta\theta\theta} = \Bigg[\frac{6\theta^4\sum_{r=1}^{k}\big(y_{A_r} - \mu\big) - 4\theta^3\big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^2}{\theta^8} - \frac{2\theta\big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^2 - \big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^3}{\theta^6}\Bigg]$$
$$exp\big\{-\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big]\big\}, \quad (66)$$

$$w_{\alpha\alpha\alpha} = \Bigg[\frac{-6\alpha^4\sum_{r=1}^{k}\big(y_{A_r} - \mu\big) - 4\alpha^3\big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^2}{\alpha^8} - \frac{2\alpha\big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^2 + \big[\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big]^3}{\alpha^6}\Bigg]$$
$$exp\big\{-\sum_{r=1}^{k}\big(y_{A_r} - \mu\big)\big(\frac{\theta+1}{\theta} - \frac{\alpha+1}{\alpha}\big]\big\}. \quad (67)$$

Now, in order to use Lindley's approximation, successive derivatives are taken from the equation (48) with respect to the parameters, We can write:

$$\Delta_{11} = \frac{n\sum_{\underline{A}} w_{\mu\mu}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)} - n\Bigg(\frac{\sum_{\underline{A}} w_{\mu}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}\Bigg)^2,$$
$$\Delta_{12} = \Delta_{21} = \frac{n\sum_{\underline{A}} w_{\theta\mu}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)} - n\frac{\sum_{\underline{A}} w_{\theta}\sum_{\underline{A}} w_{\mu}}{\big(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\big)^2},$$
$$\Delta_{13} = \Delta_{31} = -\frac{n^2}{\alpha^2} + \frac{n\sum_{\underline{A}} w_{\alpha\mu}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)} - n\frac{\sum_{\underline{A}} w_{\alpha}\sum_{\underline{A}} w_{\mu}}{\big(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\big)^2},$$
$$\Delta_{32} = \Delta_{23} = \frac{n\sum_{\underline{A}} w_{\theta\alpha}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)} - n\frac{\sum_{\underline{A}} w_{\theta}\sum_{\underline{A}} w_{\alpha}}{\big(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\big)^2}, \quad (68)$$

$$\Delta_{22}=\frac{a_1+1+k\left(\sum d_i\right)}{\theta^2}-2\frac{b_1}{\theta^3}+\frac{n\sum_{\underline{A}} w_{\theta\theta}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-n\left(\frac{\sum_{\underline{A}} w_\theta}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}\right)^2,$$

$$\Delta_{33}=\frac{a_2+1+\sum_{i=1}^{n} d_i(n-k)}{\alpha^2}-2\frac{\left(b_2+n\sum_{i=1}^{n}\left(y_i-\mu\right)\right)}{\alpha^3}+n\frac{\sum_{\underline{A}} w_{\alpha\alpha}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-n\left(\frac{\sum_{\underline{A}} w_\alpha}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}\right)^2,$$

$$\Delta_{222}=-2\frac{a_1+1+k\left(\sum d_i\right)}{\theta^3}+6\frac{b_1}{\theta^4}+\frac{n\sum_{\underline{A}} w_{\theta\theta\theta}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-n\frac{3\sum_{\underline{A}} w_\theta\sum_{\underline{A}} w_{\theta\theta}}{(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha))^2}+2n\left(\frac{\sum_{\underline{A}} w_\theta}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}\right)^3,$$

$$\Delta_{333}=-2\frac{a_2+1+(n-k)\left(\sum d_i\right)}{\alpha^3}$$
$$+6\frac{\left(b_2+n\sum_{i=1}^{n}\left(y_i-\mu\right)\right)}{\alpha^4}+\frac{n\sum_{\underline{A}} w_{\alpha\alpha\alpha}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-n\frac{3\sum_{\underline{A}} w_\alpha\sum_{\underline{A}} w_{\alpha\alpha}}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^2}+2n\left(\frac{\sum_{\underline{A}} w_\alpha}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}\right)^3,$$

$$\Delta_{112}=\Delta_{121}=\Delta_{211}=n\left[\frac{\sum_{\underline{A}} w_{\mu\mu\theta}}{\sum_{\underline{A}} W(y_A;\mu,\theta,\alpha)}-\frac{\sum_{\underline{A}} w_\theta\cdot\sum_{\underline{A}} w_{\mu\mu}+2\sum_{\underline{A}} w_\mu\cdot\sum_{\underline{A}} w_{\mu\theta}}{\left(\sum_{\underline{A}} W(y_A;\mu,\theta,\alpha)\right)^2}+\frac{2\sum_{\underline{A}} w_\theta\cdot\left(\sum_{\underline{A}} w_\mu\right)^2}{\left(\sum_{\underline{A}} W(y_A;\mu,\theta,\alpha)\right)^3}\right],$$

$$\Delta_{113}=\Delta_{131}=\Delta_{311}=n\left[\frac{\sum_{\underline{A}} w_{\mu\mu\alpha}}{\sum_{\underline{A}} W(y_A;\mu,\theta,\alpha)}-\frac{\sum_{\underline{A}} w_\alpha\cdot\sum_{\underline{A}} w_{\mu\mu}+2\sum_{\underline{A}} w_\mu\cdot\sum_{\underline{A}} w_{\alpha\mu}}{\left(\sum_{\underline{A}} W(y_A;\mu,\theta,\alpha)\right)^2}+\frac{2\sum_{\underline{A}} w_\alpha\cdot\left(\sum_{\underline{A}} w_\mu\right)^2}{\left(\sum_{\underline{A}} W(y_A;\mu,\theta,\alpha)\right)^3}\right],$$

$$\Delta_{122}=\Delta_{212}=\Delta_{221}=n\left[\frac{\sum_{\underline{A}} w_{\theta\theta\mu}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-\frac{\sum_{\underline{A}} w_\mu\cdot\sum_{\underline{A}} w_{\theta\theta}+2\sum_{\underline{A}} w_\theta\cdot\sum_{\underline{A}} w_{\mu\theta}}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^2}+\frac{2\sum_{\underline{A}} w_\mu\cdot\left(\sum_{\underline{A}} w_\theta\right)^2}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^3}\right],$$

$$\Delta_{133}=\Delta_{313}=\Delta_{331}=2\frac{n^2}{\alpha^3}+n\left[\frac{\sum_{\underline{A}} w_{\alpha\alpha\mu}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-\frac{\sum_{\underline{A}} w_\mu\cdot\sum_{\underline{A}} w_{\alpha\alpha}+2\sum_{\underline{A}} w_\alpha\cdot\sum_{\underline{A}} w_{\alpha\mu}}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^2}+\frac{2\sum_{\underline{A}} w_\mu\cdot\left(\sum_{\underline{A}} w_\alpha\right)^2}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^3}\right],$$

$$\Delta_{223}=\Delta_{232}=\Delta_{322}=n\left[\frac{\sum_{\underline{A}} w_{\theta\theta\alpha}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-\frac{\sum_{\underline{A}} w_\alpha\cdot\sum_{\underline{A}} w_{\theta\theta}+2\sum_{\underline{A}} w_\theta\cdot\sum_{\underline{A}} w_{\alpha\theta}}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^2}+\frac{2\sum_{\underline{A}} w_\alpha\cdot\left(\sum_{\underline{A}} w_\theta\right)^2}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^3}\right],$$

$$\Delta_{332}=\Delta_{323}=\Delta_{233}=n\left[\frac{\sum_{\underline{A}} w_{\alpha\alpha\theta}}{\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)}-\frac{\sum_{\underline{A}} w_\theta\cdot\sum_{\underline{A}} w_{\alpha\alpha}+2\sum_{\underline{A}} w_\alpha\cdot\sum_{\underline{A}} w_{\alpha\theta}}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^2}+\frac{2\sum_{\underline{A}} w_\theta\cdot\left(\sum_{\underline{A}} w_\alpha\right)^2}{\left(\sum_{\underline{A}} w(y_A;\mu,\theta,\alpha)\right)^3}\right].$$

4.1 Bayesian estimation under squared error loss function

In this subsection, we have obtained the Bayes estimator of μ,θ and α under the most common symmetric loss function, that is, the squared error loss function(SELF). If $\hat{\mu},\hat{\theta},\hat{\alpha}$ is an estimator of (μ,θ,α), then we can define the SELF as

$$L(\hat{\alpha},\alpha)=(\hat{\alpha}-\alpha)^2. \tag{69}$$

We know that the Bayes estimator of under the SEL function is

$$\hat{\alpha}_s=E_\alpha\left[(\alpha|\underset{\sim}{y},\underset{\sim}{d})\right]=\int_{(\mu,\theta,\alpha)}\alpha\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})d(\mu,\theta,\alpha),$$

after substituting the value of $\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})$ from (44), can write:

$$E_\alpha\left[(\alpha|\underset{\sim}{y},\underset{\sim}{d})\right]=\frac{\int U(\mu,\theta,\alpha)e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}, \tag{70}$$

where

$$U(\mu,\theta,\alpha)=\alpha,\ U_l=U_\alpha=1,\ U_{ij}=U_{\alpha\theta}=U_{\theta\alpha}=U_{\alpha\alpha}=U_\theta=U_{\theta\theta}=U_{\alpha\mu}=U_{\mu\alpha}=U_{\theta\mu}=U_{\mu\theta}=U_\mu=0.$$

Also, the Bayes estimator of μ under the SEL function is

$$\hat{\mu}_s=E_\mu\left[(\mu|\underset{\sim}{y},\underset{\sim}{d})\right]=\int_{(\mu,\theta,\alpha)}\mu\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})d(\mu,\theta,\alpha), \tag{71}$$

after substituting the value of $\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})$ from (44), can write:

$$E_\mu\left[(\mu|\underset{\sim}{y},\underset{\sim}{d})\right]=\frac{\int U(\mu,\theta,\alpha)e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)},$$

where

$$U(\mu,\theta,\alpha)=\mu,\ U_\mu=1,\ U_{\alpha\theta}=U_{\theta\alpha}=U_{\alpha\alpha}=U_\alpha=U_{\theta\theta}=U_{\alpha\mu}=U_{\mu\alpha}=U_{\theta\mu}=U_{\mu\theta}=U_\theta=0.$$

Also, the Bayes estimator of θ under the SEL function is

$$\hat{\theta}_s=E_\theta\left[(\theta|\underset{\sim}{y},\underset{\sim}{d})\right]=\int_{(\mu,\theta,\alpha)}\theta\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})d(\mu,\theta,\alpha), \tag{72}$$

after substituting the value of $\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})$ from (44), can write:

$$E_\theta\left[(\theta|\underset{\sim}{y},\underset{\sim}{d})\right]=\frac{\int U(\mu,\theta,\alpha)e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)},$$

where

$$U(\mu,\theta,\alpha)=\theta,\ U_\theta=1,\ U_{\alpha\theta}=U_{\theta\alpha}=U_{\alpha\alpha}=U_\alpha=U_{\theta\theta}=U_{\alpha\mu}=U_{\mu\alpha}=U_{\theta\mu}=U_{\mu\theta}=U_\mu=0.$$

We can write the approximate Bayes estimators for μ,θ and α, as follows:

$$\hat{\mu}_s=\hat{\mu}+\frac{1}{2}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k1},\ \hat{\theta}_s=\hat{\theta}+\frac{1}{2}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k2},\ \hat{\alpha}_s=\hat{\alpha}+\frac{1}{2}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k3}.$$

4.2 Bayesian estimation under LINEX loss function

In this subsection, we have obtained the Bayes estimator of parameters under the LINEX loss function, The LINEX loss function with parameter α given by

$$L(\hat{\alpha},\alpha)=[e^{a(\hat{\alpha}-\alpha)}-a(\hat{\alpha}-\alpha)-1], \tag{73}$$

the Bayes estimator of α under LINEX loss function is given by

$$\hat{\alpha}=\frac{-1}{a}\ln\left\{E_\alpha\left[e^{-a\alpha}|\underset{\sim}{y},\underset{\sim}{d}\right]\right\}, \tag{74}$$

such that,

$$E_\alpha\left[(e^{-a\alpha}|\underset{\sim}{y},\underset{\sim}{d})\right]=\int_{(\mu,\theta,\alpha)}e^{-a\alpha}\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})d(\mu,\theta,\alpha),$$

after substituting the value of $\Pi(\mu,\theta,\alpha|\underset{\sim}{y},\underset{\sim}{d})$ from (44), can write:

$$E_\alpha\left[(e^{-a\alpha}|\underset{\sim}{y},\underset{\sim}{d})\right]=\frac{\int U(\mu,\theta,\alpha)e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}, \tag{75}$$

where

$$U(\mu,\theta,\alpha)=e^{-a\alpha},\ U_\alpha=-ae^{-a\alpha},\ U_{\alpha\alpha}=a^2e^{-a\alpha},U_{\alpha\theta}=U_{\theta\alpha}=U_{\theta\theta}=U_{\mu\mu}=U_{\alpha\mu}=U_{\mu\alpha}=U_{\theta\mu}=U_{\mu\theta}=U_\mu=0.$$

The Bayes estimator of μ under the LINEX loss function is given by

$$\hat{\mu}_l=-\frac{1}{a}\ln E_\mu(e^{-a\mu}|\underset{\sim}{y},\underset{\sim}{d}),$$

where

$$E_\mu(e^{-a\mu}|\underset{\sim}{y},\underset{\sim}{d})=\frac{\int U(\mu,\theta,\alpha)e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}, \tag{76}$$

$$U(\mu,\theta,\alpha)=e^{-a\mu},\ U_\mu=-ae^{-a\mu},\ U_{\mu\mu}=a^2e^{-a\mu},U_{\alpha\theta}=U_{\theta\alpha}=U_{\alpha\alpha}=U_\theta=U_{\alpha\mu}=U_{\mu\alpha}=U_{\theta\mu}=U_{\mu\theta}=U_\alpha=U_{\theta\theta}=0.$$

The Bayes estimator of θ under the LINEX loss function is given by

$$\hat{\theta}_l=-\frac{1}{a}\ln E_\theta(e^{-a\theta}|\underset{\sim}{y},\underset{\sim}{d}),$$

where

$$E_\theta(e^{-a\theta}|\underset{\sim}{y},\underset{\sim}{d})=\frac{\int U(\mu,\theta,\alpha)e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}{\int e^{\Delta(\mu,\theta,\alpha)}d(\mu,\theta,\alpha)}, \tag{77}$$

$$U=e^{-a\theta},\ U_\theta=-ae^{-a\theta},\ U_{\theta\theta}=a^2e^{-a\theta},U_{\alpha\theta}=U_{\theta\alpha}=U_{\alpha\alpha}=U_{\mu\mu}=U_{\alpha\mu}=U_{\mu\alpha}=U_{\theta\mu}=U_{\mu\theta}=U_\mu=0.$$

We can write the approximate Bayes estimators for $e^{-a\mu}, e^{-a\theta}$ and $e^{-a\alpha}$, as follows:

$$\begin{aligned}
E_\mu(e^{-a\mu}|\underset{\sim}{y},\underset{\sim}{d})&=e^{-a\hat{\mu}}+\frac{1}{2}a^2e^{-a\hat{\mu}}\tau_{11}-\frac{1}{2}ae^{-a\hat{\mu}}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k1},\\
E_\theta(e^{-a\theta}|\underset{\sim}{y},\underset{\sim}{d})&=e^{-a\hat{\theta}}+\frac{1}{2}a^2e^{-a\hat{\theta}}\tau_{22}-\frac{1}{2}ae^{-a\hat{\theta}}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k2},\\
E_\alpha(e^{-a\alpha}|\underset{\sim}{y},\underset{\sim}{d})&=e^{-a\hat{\alpha}}+\frac{1}{2}a^2e^{-a\hat{\alpha}}\tau_{33}-\frac{1}{2}ae^{-a\hat{\alpha}}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k3},
\end{aligned} \tag{78}$$

thus, the Bayes estimator of μ,θ and α, under the LINEX loss function given by

$$\hat{\mu}_l=-\frac{1}{a}\ln\left(e^{-a\hat{\mu}}+\frac{1}{2}a^2e^{-a\hat{\mu}}\tau_{11}-\frac{1}{2}ae^{-a\hat{\mu}}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k1}\right),$$

$$\hat{\theta}_l=-\frac{1}{a}\ln\left(e^{-a\hat{\theta}}+\frac{1}{2}a^2e^{-a\hat{\theta}}\tau_{22}-\frac{1}{2}ae^{-a\hat{\theta}}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k2}\right),$$

$$\hat{\alpha}_l=-\frac{1}{a}\ln\left(e^{-a\hat{\alpha}}+\frac{1}{2}a^2e^{-a\hat{\alpha}}\tau_{33}-\frac{1}{2}ae^{-a\hat{\alpha}}\sum_{i,j,k=1}^{3}\Delta_{ijk}(\mu,\theta,\alpha)\tau_{ij}\tau_{k3}\right).$$

5 Simulation

In this section our main aim is to compare the estimators in terms of their MSE's. Changes in the estimators and their MSE's have been determined when changing the shape parameter of the LINEX loss function for $a = \pm 0.5, \pm 1$. The simulation study was carried out for $\theta = 3,2$ and $\alpha = 1,2$ for $k = 1,2$ with sample size $n = 10(5)40$. Obtain different values of the parameters equal to the means of prior distributions so that $\theta = \dfrac{b_1}{a_1 - 1}$ and $\alpha = \dfrac{b_2}{a_2 - 1}$, respectively. The results are summarized in tables 1 - 6 for k = 1,2. We also computed confidence interval of the parameters and the same is presented in Tables 7 and 8.

n	parameter	Moment	MLE	Bayes	a			
					1	0.5	-0.5	-1
10	μ	0.0354654	0.0041961	0.0027114	0.0030696	0.0028877	0.0026424	0.0026082
	θ	2.9795067	0.6664383	0.1496310	0.1943296	0.1785521	0.1078956	0.0610692
	α	0.2166915	0.0283981	0.0118961	0.0192078	0.0153635	0.0088338	0.0062011
15	μ	0.0298834	0.0023403	0.0004208	0.0005687	0.0004920	0.0003939	0.0003808
	θ	2.7709235	0.6199781	0.0558355	0.0833406	0.0712425	0.0385236	0.0219165
	α	0.1449901	0.0219890	0.0016856	0.0028105	0.0022138	0.0012275	0.0008408
20	μ	0.0293554	0.0013598	0.0002375	0.0003513	0.0002916	0.0002174	0.0002076
	θ	2.4212515	0.6175838	0.0288033	0.0455608	0.0376766	0.0197111	0.0114418
	α	0.1042527	0.0196167	0.0004527	0.0007619	0.0005975	0.0003277	0.0002227
25	μ	0.0227985	0.0008881	0.0001752	0.0002746	0.0002221	0.0001580	0.0001497
	θ	2.2937153	0.6128282	0.0177978	0.0289957	0.0235576	0.0121588	0.0071638
	α	0.0775364	0.0195482	0.0001522	0.0002586	0.0002019	0.0001095	0.0000737
30	μ	0.0188690	0.0008628	0.0001714	0.0002698	0.0002178	0.0001544	0.0001462
	θ	2.1166801	0.5719994	0.0123446	0.0205649	0.0164967	0.0083949	0.0049606
	α	0.0607984	0.0186510	0.0000688	0.0001174	0.0000915	0.0000493	0.0000331
35	μ	0.0187930	0.0008597	0.0000876	0.0001612	0.0001216	0.0000756	0.0000699
	θ	2.0684202	0.5659042	0.0086157	0.0145569	0.0115741	0.0058609	0.0034942
	α	0.0549938	0.0154480	0.0000367	0.0000626	0.0000488	0.0000263	0.0000176
40	μ	0.0175697	0.0003457	0.0000117	0.0000456	0.0000258	0.0000076	0.0000058
	θ	1.8399674	0.5243080	0.0066857	0.0114450	0.0090339	0.0045306	0.0026978
	α	0.0501709	0.0099215	0.0000201	0.0000345	0.0000268	0.0000144	0.0000096

Table 1: Mean squared errors and risk of estimators when $k = 1, \mu = 1, \theta = 2, \alpha = 1$

n	parameter	Moment	MLE	Bayes	a			
					1	0.5	-0.5	-1
10	μ	0.0034696	0.0006407	0.0005483	0.0006013	0.0005632	0.0005409	0.0003760
	θ	2.8811222	1.6160872	1.1940453	1.4323331	1.2936883	0.9953392	0.6020656
	α	0.1998201	0.0253330	0.0163926	0.0251037	0.0206599	0.0125656	0.0092094
15	μ	0.0031032	0.0001751	0.0000898	0.0001222	0.0001054	0.0000839	0.0000810
	θ	2.6196614	0.5972067	0.4573851	0.4739443	0.4581095	0.3818436	0.2376179
	α	0.1674512	0.0199156	0.0021256	0.0034511	0.0027503	0.0015787	0.0011115
20	μ	0.0027083	0.0001490	0.0000360	0.0000575	0.0000461	0.0000323	0.0000306
	θ	2.4890385	0.5825261	0.2465495	0.2880852	0.2782227	0.1873665	0.1093755
	α	0.1191478	0.0189533	0.0004188	0.0006977	0.0005496	0.0003056	0.0002100
25	μ	0.0023100	0.0000458	0.0000269	0.0000357	0.0000297	0.0000237	0.0000222
	θ	2.3610041	0.5202164	0.1346054	0.1765426	0.1611705	0.0974906	0.0562765
	α	0.0858601	0.0186828	0.0001780	0.0002984	0.0002344	0.0001294	0.0000884
30	μ	0.0021982	0.0000260	0.0000097	0.0000220	0.0000152	0.0000078	0.0000070
	θ	2.1310193	0.4807318	0.0954844	0.1331662	0.1179806	0.0672235	0.0380707
	α	0.0621373	0.0167609	0.0000723	0.0001223	0.0000957	0.0000521	0.0000353
35	μ	0.0018236	0.0000251	0.0000085	0.0000202	0.0000137	0.0000068	0.0000060
	θ	2.0337450	0.4219897	0.0649706	0.0955880	0.0823674	0.0449405	0.0254170
	α	0.0582723	0.0158709	0.0000374	0.0000634	0.0000495	0.0000269	0.0000182
40	μ	0.0011514	0.0000222	0.0000079	0.0000192	0.0000129	0.0000062	0.0000054
	θ	1.6885341	0.3371218	0.0491487	0.0750126	0.0634271	0.0335273	0.0188325
	α	0.0479331	0.0156061	0.0000213	0.0000362	0.0000282	0.0000153	0.0000103

Table 2: Mean squared errors and risk of estimators when $k = 1, \mu = 1, \theta = 3, \alpha = 1$

n	parameter	Moment	MLE	Bayes	a			
					1	0.5	-0.5	-1
10	μ	0.100738	0.533334	0.155385	1.067346	0.099449	0.016678	0.033638
	θ	3.245432	1.947860	0.496780	0.616123	0.509719	0.434571	0.326527
	α	1.652906	1.543708	0.429367	0.735023	0.535173	0.241388	0.184788
15	μ	0.383906	0.088666	0.03687	0.684775	0.062496	0.007358	0.008363
	θ	3.167014	1.739497	0.484110	0.506123	0.485972	0.404257	0.247727
	α	1.547602	1.401110	0.350738	0.700023	0.352573	0.171788	0.084788
20	μ	0.229659	0.047207	0.020388	0.366349	0.033407	0.003872	0.00426
	θ	2.945020	0.409890	0.244053	0.286233	0.275646	0.185613	0.108566
	α	1.107138	0.974187	0.300814	0.138775	0.133063	0.070338	0.043021
25	μ	0.205508	0.016992	0.011661	0.089594	0.008781	0.002839	0.010602
	θ	1.466138	0.343417	0.151140	0.195179	0.179710	0.109624	0.062595
	α	1.103641	0.866726	0.003423	0.003753	0.003571	0.003319	0.003274
30	μ	0.170546	0.01554	0.00895	0.100432	0.009463	0.001874	0.004597
	θ	1.395049	0.372543	0.098564	0.136759	0.121479	0.069521	0.039390
	α	1.071603	0.798827	0.000883	0.001157	0.001015	0.000762	0.000654
35	μ	0.121434	0.013631	0.006417	0.100483	0.009245	0.00126	0.001877
	θ	0.799634	0.384229	0.069532	0.101556	0.087876	0.048123	0.027102
	α	1.028991	0.795777	0.000355	0.000525	0.000436	0.000281	0.000216
40	μ	0.109144	0.010137	0.002768	0.073762	0.006802	0.000965	0.001531
	θ	0.574606	0.386391	0.052611	0.079747	0.067691	0.035932	0.020140
	α	0.977091	0.704616	0.000100	0.000168	0.000132	0.000073	0.000050

Table 3: Mean squared errors and risk of estimators when $k = 1, \mu = 1, \theta = 3, \alpha = 2$

n	parameter	Moment	MLE	Bayes	a			
					1	0.5	-0.5	-1
10	μ	0.0493294	0.0113471	0.0104291	0.0109599	0.0105794	0.0103543	0.0072988
	θ	2.7817550	0.6215728	0.0828180	0.1107717	0.0993793	0.0618833	0.0393810
	α	0.2430123	0.0643586	0.0309029	0.0521799	0.0409652	0.0308721	0.0220434
15	μ	0.0299823	0.0101658	0.0092979	0.0097995	0.0094399	0.0092273	0.0025562
	θ	2.6407878	0.5747529	0.0299747	0.0424766	0.0366878	0.0227346	0.0155784
	α	0.2181708	0.0295076	0.0033383	0.0054604	0.0043384	0.0024643	0.0017195
20	μ	0.0299497	0.0074788	0.0067372	0.0071651	0.0068581	0.0066771	0.0007412
	θ	2.2834697	0.5745496	0.0166916	0.0240095	0.0204928	0.0127977	0.0090561
	α	0.1841664	0.0224089	0.0007254	0.0012078	0.0009517	0.0005293	0.0003637
25	μ	0.0214984	0.0011136	0.0008393	0.0009946	0.0008823	0.0008182	0.0007211
	θ	2.1647008	0.5449814	0.0101750	0.0146881	0.0124728	0.0078902	0.0057291
	α	0.1571600	0.0223556	0.0002301	0.0003859	0.0003031	0.0001671	0.0001142
30	μ	0.0199443	0.0006959	0.0000441	0.0001063	0.0000719	0.0000307	0.0000260
	θ	1.7151691	0.5299187	0.0075731	0.0109854	0.0092933	0.0058887	0.0043105
	α	0.1532507	0.0211999	0.0000955	0.0001615	0.0001264	0.0000689	0.0000467
35	μ	0.0182262	0.0004086	0.0000349	0.0000902	0.0000588	0.0000340	0.0000224
	θ	1.6983753	0.5198236	0.0051465	0.0074937	0.0063172	0.0040184	0.0029717
	α	0.1396413	0.0178924	0.0000501	0.0000846	0.0000663	0.0000362	0.0000246
40	μ	0.0157614	0.0002498	0.0000149	0.0000567	0.0000325	0.0000098	0.0000076
	θ	1.6862910	0.3597495	0.0040633	0.0059312	0.0049899	0.0031779	0.0023608
	α	0.1264531	0.0132382	0.0000272	0.0000461	0.0000360	0.0000196	0.0000132

Table 4: Mean squared errors and risk of estimators when $k = 2, \mu = 1, \theta = 2, \alpha = 1$

n	parameter	Moment	MLE	Bayes	a			
					1	0.5	-0.5	-1
10	μ	0.0023720	0.0003951	0.0003165	0.0003613	0.0003290	0.0003104	0.0002927
	θ	2.8098240	0.6991896	0.5444805	0.5809883	0.5757578	0.5383800	0.3798773
	α	0.2542098	0.0597072	0.0415444	0.0572235	0.0454419	0.0347436	0.0252860
15	μ	0.0020464	0.0003873	0.0003095	0.0003538	0.0003218	0.0003034	0.0002279
	θ	2.5390217	0.5453236	0.1925543	0.2251680	0.2159367	0.1519921	0.0984046
	α	0.2114026	0.0351986	0.0032764	0.0053493	0.0042534	0.0024220	0.0016934
20	μ	0.0018285	0.0001233	0.0000538	0.0000823	0.0000673	0.0000488	0.0000464
	θ	2.4731183	0.5421827	0.0927681	0.1218603	0.1102568	0.0700304	0.0449700
	α	0.1933246	0.0264478	0.0007326	0.0012125	0.0009579	0.0005370	0.0003714
25	μ	0.0016133	0.0000999	0.0000269	0.0000479	0.0000366	0.0000234	0.0000217
	θ	2.3155201	0.4842251	0.0544837	0.0762838	0.0667458	0.0403021	0.0258616
	α	0.1718360	0.0234545	0.0002379	0.0003969	0.0003124	0.0001734	0.0001191
30	μ	0.0015001	0.0000679	0.0000142	0.0000304	0.0000216	0.0000117	0.0000106
	θ	2.0548902	0.4734207	0.0375135	0.0546150	0.0468097	0.0273908	0.0175152
	α	0.1636477	0.0205947	0.0001032	0.0001732	0.0001360	0.0000749	0.0000511
35	μ	0.0014616	0.0000516	0.0000103	0.0000245	0.0000167	0.0000082	0.0000073
	θ	1.9907460	0.3095870	0.0258482	0.0386682	0.0326322	0.0187854	0.0120939
	α	0.1619473	0.0178938	0.0000495	0.0000833	0.0000653	0.0000359	0.0000244
40	μ	0.0012213	0.0000191	0.0000066	0.0000185	0.0000118	0.0000049	0.0000042
	θ	1.4128690	0.3013983	0.0193319	0.0295625	0.0246491	0.0139624	0.0089842
	α	0.1542396	0.0157656	0.0000267	0.0000450	0.0000353	0.0000193	0.0000131

Table 5: Mean squared errors and risk of estimators when $k = 2, \mu = 1, \theta = 3, \alpha = 1$

n	parameter	Moment	MLE	Bayes	a			
					1	0.5	-0.5	-1
10	μ	0.43150	0.05729	0.04931	0.27599	0.02673	0.00756	0.02560
	θ	1.55000	0.58900	0.62800	0.60200	0.58400	0.57100	0.40900
	α	2.25000	1.12000	0.81400	2.87000	1.01000	0.52100	0.38800
15	μ	0.40687	0.02879	0.02114	0.12209	0.01175	0.00309	0.00986
	θ	1.31000	0.41900	0.21300	0.24800	0.23600	0.17100	0.11200
	α	1.20000	1.11000	0.38000	1.35000	0.56000	0.25400	0.19700
20	μ	0.36816	0.03702	0.01723	0.25546	0.02378	0.00394	0.00781
	θ	1.25000	0.40200	0.10500	0.13500	0.12300	0.08030	0.05220
	α	1.25000	1.09000	0.10400	0.23600	0.12600	0.07450	0.06080
25	μ	0.23567	0.02375	0.01645	0.16348	0.01523	0.00254	0.00508
	θ	1.20000	0.32700	0.06370	0.08690	0.07700	0.04780	0.03110
	α	1.23000	1.01000	0.01810	0.04740	0.02340	0.01570	0.01440
30	μ	0.14718	0.01475	0.01391	0.09821	0.00920	0.00168	0.00378
	θ	1.05000	0.27500	0.04290	0.06100	0.05290	0.03180	0.02070
	α	1.23000	0.91300	0.00256	0.00287	0.00270	0.00246	0.00241
35	μ	0.13303	0.01199	0.00936	0.08152	0.00761	0.00131	0.00275
	θ	0.91000	0.21600	0.03000	0.04390	0.03740	0.02210	0.01440
	α	1.10000	0.81000	0.00088	0.00115	0.00102	0.00076	0.00064
40	μ	0.10392	0.01098	0.00191	0.08044	0.00741	0.00103	0.00159
	θ	0.60000	0.13100	0.02230	0.03340	0.02810	0.01630	0.01070
	α	1.02000	0.74000	0.00035	0.00049	0.00042	0.00029	0.00023

Table 6: Mean squared errors and risk of estimators when $k = 2, \mu = 1, \theta = 3, \alpha = 2$

(θ,α,μ)	n	θ		α		μ	
		Lower	Lower	Upper	Upper	Lower	Upper
(2,1,1)	10	0.998235	2.281502	0.998917	1.025177	0.948026	1.149298
	15	1.462447	2.095438	0.999787	1.008912	0.963544	1.096498
	20	1.587380	2.089192	0.999611	1.004737	0.975574	1.074845
	25	1.660742	2.084971	0.999763	1.002844	0.97779	1.056998
	30	1.747947	2.040828	0.999922	1.001802	0.984233	1.050125
	35	1.771255	2.050147	0.999770	1.001478	0.986423	1.042831
	40	1.818001	2.026584	0.999869	1.001069	0.986793	1.036105
(3,1,1)	10	0.044169	3.865796	0.998475	1.027224	0.947652	1.150866
	15	1.183366	3.540998	0.999230	1.009819	0.963261	1.097030
	20	1.816321	3.273759	0.999709	1.004750	0.975567	1.075284
	25	2.077302	3.218321	0.999920	1.002714	0.98103	1.060518
	30	2.206872	3.214486	0.999704	1.002048	0.983862	1.049945
	35	2.305604	3.207666	0.999958	1.001294	0.984185	1.040734
	40	2.452594	3.125495	0.999841	1.001089	0.986572	1.035991
(3,2,1)	10	1.843878	3.772835	1.217007	2.819133	0.928378	1.192709
	15	2.028713	3.679881	1.295845	2.735048	0.955526	1.131067
	20	2.047086	3.554594	1.647320	2.248641	0.970496	1.1019
	25	2.411368	3.477614	1.715737	2.057721	0.972131	1.077135
	30	2.242828	3.297097	1.974661	2.048293	0.979926	1.067363
	35	2.285317	3.048793	1.996099	2.055034	0.980276	1.055182
	40	2.265899	3.027624	1.998137	2.019937	0.985833	1.05135

Table 7: Various interval estimates for different combination of $n,\mu,\theta and \alpha$ when $k=1$

(θ,α,μ)	n	θ		α		μ	
		Lower	Lower	Upper	Upper	Lower	Upper
(2,1,1)	10	1.318526	2.167051	0.544312	1.046362	0.944354	1.151141
	15	1.611501	2.067141	0.997314	1.015686	0.967973	1.103288
	20	1.708152	2.052489	0.999135	1.006496	0.975115	1.075688
	25	1.776397	2.031661	0.998753	1.004452	0.980496	1.060527
	30	1.827264	2.013809	0.999718	1.002364	0.986168	1.052626
	35	1.854924	2.007314	0.999731	1.001695	0.984454	1.041277
	40	1.876162	2.003066	0.999846	1.001202	0.988185	1.037813
(3,1,1)	10	1.514659	3.121386	0.978756	1.070055	0.950544	1.161438
	15	1.867451	3.330611	0.999409	1.011947	0.966059	1.103067
	20	2.117307	3.312687	0.999018	1.006733	0.977909	1.079421
	25	2.447464	3.109901	0.999660	1.003621	0.978329	1.058933
	30	2.560464	3.078514	0.999677	1.002387	0.985256	1.052107
	35	2.646281	3.049168	0.999909	1.001545	0.984955	1.042063
	40	2.658078	3.077561	0.999930	1.001141	0.98764	1.037485
(3,2,1)	10	0.366759	4.073541	1.364999	2.799788	0.93475	1.20213
	15	1.536471	3.545819	1.392909	2.598922	0.95513	1.13200
	20	2.036665	3.319306	1.740515	2.271959	0.96763	1.09978
	25	2.379895	3.117336	1.991338	2.158753	0.97208	1.07756
	30	2.435033	3.151772	1.997599	2.070325	0.97787	1.06564
	35	2.568094	3.086660	1.961039	2.083904	0.98168	1.05683
	40	2.630504	3.071392	1.993964	2.035888	0.98327	1.04897

Table 8: Various interval estimates for different combination of $n,\mu,\theta and \alpha$ when $k=2$

According to the obtained results, it can be said that:

- As the sample size increases, MSE estimates of the parameters decrease.
- With the increase of outlier data, the MSE estimate of (distribution parameter of most data) increases and the estimate of parameter (distribution parameter of outlier data) decreases.

•As the value of the risk parameters increase, the estimators under the LINEX loss function decrease.
•In all cases, Bayesian estimator performs better than Maximum Likelihood and Moment.
•As the sample size increases, the length of the confidence intervals of the estimators decrease.
•Also, when the number of outlier data increase, the length of the confidence intervals of the estimators increases.

6 Real lifetime data analysis

To evaluate the methods presented in this paper, we consider a real data set that is the breakdown time (in hours) of 59 conductors in the circuit presented in [26];
6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.941, 5.923.
To goodness of fitting, by using the methods of estimation presented in this paper, the estimators, Akaike information criterion (AIC) and Bayesian information criterion (BIC) are given in Tables 9 and 10 for k=1 and k=2 respectively. According to the results of Tables 9 and 10, goodness of fit based on Bayesian estimation is better than Maximum Likelihood Estimation.

Method	$\hat{\mu}$	$\hat{\theta}$	$\hat{\alpha}$	-Log(L)	AIC	BIC
MLE	2.997000	6.55467	7.292760	161.3627	328.7254	334.9580
Bayes	2.6651107	6.25959	7.0190598	140.9304	287.8609	294.0935

Table 9: Summary statistics for model fitting when $k = 1$.

Method	$\hat{\mu}$	$\hat{\theta}$	$\hat{\alpha}$	-Log(L)	AIC	BIC
MLE	2.997000	6.41929	7.468359	168.3898	342.7795	349.0121
Bayes	3.3470179	6.02053	6.977835	151.3149	308.6299	314.8625

Table 10: Summary statistics for model fitting when $k = 2$.

7 Conclusion

In this article, the parameters of the two-parameter exponential distribution with censored data with the presence of outliers have been estimated by Moment and Maximum Likelihood and Bayesian Methods. In section 3, Moment Estimation, Maximum Likelihood and distance estimation, and in section 4, parameters were estimated by Bayesian estimator under two symmetric (Squared Error) and asymmetric (LINEX) loss functions. According to the Mean Square Error criterion, the Maximum Likelihood Estimator performs better compared to the Moment Estimator, while the Bayesian Estimators perform better than the Maximum Likelihood Estimator. By increasing the value of the risk parameter under the LINEX loss function, the Mean Squared Error decreases. Also, with the increase of the sample size, this criterion decreases. With the increase number of outlier data, the Mean Square Error of the α parameter estimate increases and the Mean Square Error of the θ parameter estimate decreases.

Declarations

Competing interests: We do not have competitive interests.
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