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Examination of the geodesic curvatures and geodesic torsions of the transversal intersection curve in $\mathbb{E}^n$

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Abstract: This study aims to compute all geodesic curvatures and geodesic torsions of an implicit curve in Euclidean n-space. Since an implicit curve is determined by the intersection of $(n-1)$ implicit hypersurfaces in $\mathbb{E}^n$, the results obtained in this paper complete the lacking curvatures for a transversal intersection curve. As a result, the approach proposed in the paper can be viewed as a partial solution to the open problem that Goldman identified in 2005. An algorithm has been proposed to handle the difficulty of calculating geodesic curvatures and geodesic torsions in spaces of higher dimensions. Then, an computational algorithms produced by MATLAB are utilized to display the outcomes in the form of illustrative examples.

Keywords: Geodesic torsion; geodesic curvature; transversal intersection; implicit curves; surfaces intersections.

2010 Mathematics Subject Classification. 53A04, 53A07.

1 Introduction

Frenet vectors and curvatures of a parametric curve can be obtained in Euclidean n-space [17,25,15]. However, the computations become more challenging if the curve is an intersection curve. In the last few decades, a variety of approaches have been developed to tackle surface intersection problems, which have played a significant role in CAGD/CAD. Surface intersection problems are classified as tangential, transversal or non-transversal based on the linear dependency or independency of normal vectors.

Willmore [35] constructed the Frenet frame of the curve that represents the intersection of two implicit surfaces in $\mathbb{E}^3$ utilizing the operator $\Delta = \lambda \dfrac{d}{ds} = \sum_{i=1}^{3} h_i \dfrac{\partial}{\partial u_i}$ where $h = \nabla \mathcal{F}_1 \times \nabla \mathcal{F}_2$, $\mathcal{F}_1(u_1,u_2,u_3) = 0$, $\mathcal{F}_2(u_1,u_2,u_3) = 0$, and he computed the curvature and torsion. Düldül and Akbaba [10] provide some new operators for determining the Frenet apparatus of the transversal intersecting curves of two and three (hyper)surfaces in which at least one (hyper)surface is represented parametrically. These operators are based on the concept that was proposed by Willmore. By extending the Willmore approach into $\mathbb{E}^4$, Uyar Düldül and Düldül [29] calculated the differential characteristics of the curves representing the intersections of three implicit hypersurfaces. In addition to this, they introduced two novel approaches in $\mathbb{E}^3$ for two surfaces intersect tangentially [30]. Hartmann [20] offers methods for calculating the curvature of intersection curves for all kinds of intersection problem. Faux and Pratt [16] provide the curvature of an intersection curve formed by two parametric surfaces. Papaioannou and Patrikoussakis [27] introduced reduced curvature formulae for surfaces and curves, covering normal, mean, and Gaussian curvatures, as well as curves on surfaces and intersection curves, where each surface can be parametrically or implicitly defined. In all surface representation modes, formulas were provided for curves on surfaces and surface intersection curves. Additionally, formulas were presented for the normal curvature of offset surfaces and the curvature of intersection curves of offset surfaces. Alessio [4] utilized the implicit function theorem and the approach of Ye and Meakava [36], which computes the Frenet apparatus for tangential and transversal intersection problems in $\mathbb{E}^3$, to determine the Frenet vectors and curvatures of the curve that is obtained by the transversal intersection of three implicit hypersurfaces in $\mathbb{E}^4$. The research produced by Soliman et al.[28] focuses on the curve formed by the intersection of implicit and parametric surfaces. Abdel-All et al.[1] calculate the Frenet apparatus of the intersection curve

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formed by two implicit surfaces. In addition, they analyze the intersection curves of hypersurfaces in $\mathbb{E}^4$ that are defined by parametric-parametric-implicit and parametric-implicit-implicit equations[2].

Uyar Düldül and Çalışkan computed the geodesic curvature and torsion of the two surfaces defined by the parametric-parametric and implicit-implicit equations [31]. In addition, they compute various intersection problems in $\mathbb{E}^3$, including the unit tangent vector and the geodesic torsion of the tangential intersection curve of the two surfaces [32]. Also, Lone et al. [22] developed a new method for calculating geodesic curvature and geodesic torsion by using Rodrigues' rotation formula in $\mathbb{R}^3$. Lewiner et al. [21] presented a novel approach for estimating the curvature and torsion of planar and spatial curves. The proposed method utilizes weighted least-squares fitting and provides an approximation of the local arc-length. Düldül et al. [11] examined the extension of the Darboux frame field into Euclidean 4-space and analyze the relationships between the curvatures of the Frenet and Darboux frames. Furthermore, in Minkowski space-time E_1^4, Uyar Düldül [33] extended the Darboux frame field along a non-null curve that lies on an orientable non-null hypersurface. Düldül [12] and Lone et al. [23] presented calculations for three and four parametric hypersurfaces for their transversal intersection curves, respectively. Moreover, numerous methods for non-transversal [5,6,9], and tangential [3] hypersurface intersection problems have been investigated in $\mathbb{E}^4$. In non-Euclidean spaces, the transversal intersection curve of spacelike (hyper)surfaces is also examined [34,8].

The motivation for this study comes from Goldman's unresolved problems-laden article [18]. Goldman calculated the first curvature of the transversal intersection curve of n-implicit hypersurfaces in $(n+1)$-dimensions and he proposed an open problem about derivating torsion equations in $(n+1)$-dimensions for implicit curves. Zhang and Xu [37] solved an open Goldman problem on implicit surfaces with higher co-dimensions by obtaining explicit formulas for mean curvature vector and Gauss-Kronecker curvature for n-manifolds. For implicit curves in n-dimensions, Alessio [7] obtained the normal curvature, the first geodesic torsion, the first geodesic curvature, and the second and third curvatures. Recently, in Euclidean n-space, Özçetin and Düldül [26] computed Frenet apparatus of the intersection curve of $(n-1)$ transversally intersecting hypersurfaces. In this study, the approach developed for the implicit situation might be regarded as a partial solution to one of the unresolved problems presented by Goldman [18]. Also, they derived Frenet vectors and curvatures of intersection curve of two parametric surfaces [13], two implicit surfaces and implicit-parametric surfaces [14] in $\mathbb{E}^n$.

In this work, we compute all geodesic curvatures and geodesic torsions of the transversal intersection curve of $(n-1)$ hypersurfaces in $\mathbb{E}^n$. To determine the geometric characteristics of an transversal intersection curve of $(n-1)$ implicit hypersurfaces, we first compute Frenet vectors and then by using Gram-Schmidt orthonormalization method, the Darboux frame is constructed. Utilizing the algorithms in [26], we developed a MATLAB code to solve the computational complexity in high-dimensional spaces. Lastly, we provide illustrative examples by applying our code.

2 Fundamental Tools

This section provides the fundamental definitions and theorems needed to construct the Darboux Frame.

Definition 1. *Let standard basis of $\mathbb{R}^n$ represented by $\{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_n\}$. The following formula can be used to define the vector product of the vectors $\mathbf{b}_1=\sum_{j=1}^{n} b_{1j}\mathbf{e}_j$, $\mathbf{b}_2=\sum_{j=1}^{n} b_{2j}\mathbf{e}_j$,..., $\mathbf{b}_{n-1}=\sum_{j=1}^{n} b_{n-1,j}\mathbf{e}_j$:*

$$\mathbf{K}=\mathbf{b}_1\times\mathbf{b}_2\times\cdots\times\mathbf{b}_{n-1}=\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{(n-1)1} & b_{(n-1)2} & \cdots & b_{(n-1)n} \end{vmatrix}.$$

and its norm is provided by the formula [24]

$$||\mathbf{K}||=||\mathbf{b}_1||\cdot||\mathbf{b}_2||\dots||\mathbf{b}_{n-1}||\cdot L, \tag{1}$$

where

$$L=\begin{vmatrix} 1 & \cos\beta_{12} & \cdots & \cos\beta_{1(n-1)} \\ \cos\beta_{21} & 1 & \cdots & \cos\beta_{2(n-1)} \\ \vdots & \vdots & & \vdots \\ \cos\beta_{(n-1)1} & \cos\beta_{(n-1)2} & \cdots & 1 \end{vmatrix}^{1/2}$$

with $\cos\beta_{ij}=\dfrac{\langle\mathbf{b}_i,\mathbf{b}_j\rangle}{||\mathbf{b}_i||\cdot||\mathbf{b}_j||}$.

Definition 2. *Let I be an interval in $\mathbb{R}^1$ and $F: I \to \mathbb{R}^n$ a C^k-parametrization by arc length. Suppose that for each $t \in I$, the vectors $F'(t), F''(t), ..., F^{(r)}(t)$, $r < k$ are linearly independent. Utilizing the Gram-Schmidt orthonormalization process to these vectors, one obtains an orthonormal r-tuple of vectors $\{\mathbf{V}_1(t), \mathbf{V}_2(t), ..., \mathbf{V}_r(t)\}$ called the Frenet r-frame associated with the curve at the point $F(t)$ [17].*

Suppose $\gamma: I \subset \mathbb{R} \to \mathbb{E}^n$ denotes a unit-speed curve with the arc-length t. If the Frenet r-frame along the curve is denoted by $\{\mathbf{V}_1(t), \mathbf{V}_2(t), ..., \mathbf{V}_r(t)\}$, then the Frenet formulas are provided by [17]

$$\begin{aligned}
&\mathbf{V}_1'(t) = k_1(t)\mathbf{V}_2(t),\\
&\mathbf{V}_i'(t) = -k_{i-1}(t)\mathbf{V}_{i-1}(t) + k_i(t)\mathbf{V}_{i+1}(t), \quad 2 \le i \le r-1,\\
&\mathbf{V}_r'(t) = -k_{r-1}(t)\mathbf{V}_{r-1}(t),
\end{aligned}$$

where $k_1(t), k_2(t), ..., k_{r-1}(t)$ indicate the curvatures of the given curve at the point $\gamma(t)$.

Let $\mathcal{S}$ be a hypersurface in $\mathbb{R}^n$, $\{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{n-1}\}$ be γ Frenet frame field that resides on the hypersurface and $\mathbf{W}_n$ be the unit normal vector field to $\mathcal{S}$. Then the system of orthonormals $\{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{n-1}, \mathbf{W}_n\}$ is referred to as natural frame field for curve-hypersurface pair $(\gamma, \mathcal{S})$ [19].

Definition 3. *Suppose that γ is a curve on hypersurface $\mathcal{S} \subset \mathbb{R}^n$. Then the ith geodesic curvature of γ curve at $\gamma(t)$ is defined as*

$$\kappa_{ig}(t) = \left\langle \mathbf{W}_i'(t), \mathbf{W}_{i+1}(t) \right\rangle, \qquad 1 \le i \le n-1 \tag{2}$$

where the function $\kappa_{ig}: I \to \mathbb{R}$ is given for $t \in I$ [19].

Theorem 1. *Suppose $\mathcal{S}$ be a hypersurface in $\mathbb{R}^n$ and let γ represent a curve on $\mathcal{S}$. Then the derivation equations of the natural frame field $\{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{n-1}, \mathbf{W}_n\}$ are*

$$\frac{d}{ds}(\mathbf{W}_i(t)) = -\kappa_{(i-1)g}\mathbf{W}_{i-1}(t) + \kappa_{ig}\mathbf{W}_{i+1}(t) + II(\mathbf{W}_1, \mathbf{W}_i)\mathbf{W}_n(t)$$

where the symbol II shows the second fundamental form of $\mathcal{S}$, $\tau_{ig} = II(\mathbf{W}_1, \mathbf{W}_{i+1})$, $1 \le i \le n-1$ and $\kappa_{0g} = \kappa_{(n-1)g} = 0$ [7].

2.1 Formulas for intersection curve of transversely intersecting implicit hypersurfaces in $\mathbb{E}^n$

In this part, we give Frenet vectors by using [26] to generate the orthonormal system required for the Darboux frame construction.
Let $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_{n-1}$ be the hypersurfaces given by their implicit equations $\mathcal{F}_i(u_1, u_2, ..., u_n) = 0$, $1 \le i \le n-1$ and $\gamma(t)$ denotes their unit speed transversal intersection curve.

Let $\mathcal{H} := \nabla\mathcal{F}_1 \times \nabla\mathcal{F}_2 \times \cdots \times \nabla\mathcal{F}_{n-1} = (h_1, h_2, ..., h_n)$ i.e. $\mathcal{H} = \lambda \mathbf{V}_1$. Let us denote

$$\Delta = \lambda \frac{d}{ds} = h_1 \frac{\partial}{\partial u_1} + h_2 \frac{\partial}{\partial u_2} + \cdots + h_n \frac{\partial}{\partial u_n} = \sum_{i=1}^{n} h_i \frac{\partial}{\partial u_i}.$$

Using the Δ operator, it is possible to calculate the Frenet vectors of intersecting curve as [26]

$$\begin{aligned}
\mathbf{V}_1 &= \frac{\mathcal{H}}{\lambda}, \qquad \mathbf{V}_2 = \frac{1}{\lambda^2\kappa_1}\left(\Delta\mathcal{H} - \lambda\lambda'\mathbf{V}_1\right) \quad \text{with} \quad \lambda\lambda' = \left\langle \Delta\mathcal{H}, \mathbf{V}_1 \right\rangle,\\
\mathbf{V}_n &= \frac{\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\cdots\times\Delta^{n-2}\mathcal{H}}{\left\|\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\cdots\times\Delta^{n-2}\mathcal{H}\right\|},\\
\mathbf{V}_{n-1} &= \frac{\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\cdots\times\Delta^{n-3}\mathcal{H}\times\mathbf{V}_n}{\left\|\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\cdots\times\Delta^{n-3}\mathcal{H}\times\mathbf{V}_n\right\|}\\
\mathbf{V}_{n-2} &= \frac{\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\cdots\times\Delta^{n-4}\mathcal{H}\times\mathbf{V}_n\times\mathbf{V}_{n-1}}{\left\|\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\cdots\times\Delta^{n-4}\mathcal{H}\times\mathbf{V}_n\times\mathbf{V}_{n-1}\right\|}\\
&\vdots\\
\mathbf{V}_4 &= \frac{\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\mathbf{V}_n\times\mathbf{V}_{n-1}\times\cdots\times\mathbf{V}_5}{\left\|\mathcal{H}\times\Delta\mathcal{H}\times\Delta^2\mathcal{H}\times\mathbf{V}_n\times\mathbf{V}_{n-1}\times\cdots\times\mathbf{V}_5\right\|},\\
\mathbf{V}_3 &= \mathbf{V}_n\times\mathbf{V}_{n-1}\times\mathbf{V}_{n-2}\times\cdots\times\mathbf{V}_4\times\mathbf{V}_2\times\mathbf{V}_1.
\end{aligned}$$

3 Geodezic Curvatures and Geodezic Torsions Computations in $\mathbb{E}^n$

3.1 Darboux Frame

Let $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_{n-1}$ be transversally intersection hypersurfaces given by their implicit forms $\mathcal{F}_i\left(u_1, u_2, ..., u_n\right) = 0$, $1 \leq i \leq n-1$, respectively. Using the Frenet vectors $\left\{\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_{n-1}\right\}$ as obtained in Section 2.1, we build the $\left\{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_n\right\}$ Darboux frame of intersection curve of $n-1$ implicit hypersurfaces in n-dimensional Euclidean space.

Let $\mathbf{W}_1^{\mathcal{F}_i} = \mathbf{V}_1$ and $\mathbf{W}_n^{\mathcal{F}_i} = \dfrac{\nabla\mathcal{F}_i}{\|\nabla\mathcal{F}_i\|}$. Then, the Gram-Schmidt orthonormalization method yields

$$\mathbf{W}_2^{\mathcal{F}_i} = \frac{\mathbf{V}_2 - \langle\mathbf{V}_2, \mathbf{W}_1^{\mathcal{F}_i}\rangle\mathbf{W}_1^{\mathcal{F}_i} - \langle\mathbf{V}_2, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}}{\left\|\mathbf{V}_2 - \langle\mathbf{V}_2, \mathbf{W}_1^{\mathcal{F}_i}\rangle\mathbf{W}_1^{\mathcal{F}_i} - \langle\mathbf{V}_2, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}\right\|}, \tag{3}$$

$$\mathbf{W}_3^{\mathcal{F}_i} = \frac{\mathbf{V}_3 - \langle\mathbf{V}_3, \mathbf{W}_1^{\mathcal{F}_i}\rangle\mathbf{W}_1^{\mathcal{F}_i} - \langle\mathbf{V}_3, \mathbf{W}_2^{\mathcal{F}_i}\rangle\mathbf{W}_2^{\mathcal{F}_i} - \langle\mathbf{V}_3, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}}{\left\|\mathbf{V}_3 - \langle\mathbf{V}_3, \mathbf{W}_1^{\mathcal{F}_i}\rangle\mathbf{W}_1^{\mathcal{F}_i} - \langle\mathbf{V}_3, \mathbf{W}_2^{\mathcal{F}_i}\rangle\mathbf{W}_2^{\mathcal{F}_i} - \langle\mathbf{V}_3, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}\right\|}, \tag{4}$$

$$\mathbf{W}_4^{\mathcal{F}_i} = \frac{\mathbf{V}_4 - \langle\mathbf{V}_4, \mathbf{W}_1^{\mathcal{F}_i}\rangle\mathbf{W}_1^{\mathcal{F}_i} - \langle\mathbf{V}_4, \mathbf{W}_2^{\mathcal{F}_i}\rangle\mathbf{W}_2^{\mathcal{F}_i} - \langle\mathbf{V}_4, \mathbf{W}_3^{\mathcal{F}_i}\rangle\mathbf{W}_3^{\mathcal{F}_i} - \langle\mathbf{V}_4, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}}{\left\|\mathbf{V}_4 - \langle\mathbf{V}_4, \mathbf{W}_1^{\mathcal{F}_i}\rangle\mathbf{W}_1^{\mathcal{F}_i} - \langle\mathbf{V}_4, \mathbf{W}_2^{\mathcal{F}_i}\rangle\mathbf{W}_2^{\mathcal{F}_i} - \langle\mathbf{V}_4, \mathbf{W}_3^{\mathcal{F}_i}\rangle\mathbf{W}_3^{\mathcal{F}_i} - \langle\mathbf{V}_4, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}\right\|}, \tag{5}$$

and in general form

$$\mathbf{W}_j^{\mathcal{F}_i} = \frac{\mathbf{V}_j - \sum_{t=1}^{j-1}\langle\mathbf{V}_j, \mathbf{W}_t^{\mathcal{F}_i}\rangle\mathbf{W}_t^{\mathcal{F}_i} - \langle\mathbf{V}_j, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}}{\left\|\mathbf{V}_j - \sum_{t=1}^{j-1}\langle\mathbf{V}_j, \mathbf{W}_t^{\mathcal{F}_i}\rangle\mathbf{W}_t^{\mathcal{F}_i} - \langle\mathbf{V}_j, \mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i}\right\|} \quad \text{with} \quad 2 \leq j \leq n-1. \tag{6}$$

3.2 Calculations of Geodesic Curvatures and Geodesic Torsions

In this section, we provide all geodesic curvatures and geodesic torsions of the intersection curve of $n-1$ hypersurfaces that are given by their implicit equations.

Theorem 2. Let $\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_{n-1}$ be transversally intersection hypersurfaces given by their implicit forms $\mathcal{F}_i\big(u_1,u_2,...,u_n\big)=0$, $1\leq i\leq n-1$, respectively. Then, geodesic curvatures of the intersection curve is given by

$$\kappa_{jg}^{\mathcal{F}_i}=\frac{\det \mathtt{T}}{\left|\left|P_j^{\mathcal{F}_i}\right|\right|^3\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|},\quad 1\leq j\leq n-1 \tag{7}$$

where

$$\mathtt{T}=\begin{bmatrix}\left\langle\left(P_j^{\mathcal{F}_i}\right)',P_{j+1}^{\mathcal{F}_i}\right\rangle & \left\langle P_j^{\mathcal{F}_i},P_{j+1}^{\mathcal{F}_i}\right\rangle\\ \left\langle\left(P_j^{\mathcal{F}_i}\right)',P_j^{\mathcal{F}_i}\right\rangle & \left\langle P_j^{\mathcal{F}_i},P_j^{\mathcal{F}_i}\right\rangle\end{bmatrix},\quad P_j^{\mathcal{F}_i}=\mathbf{V}_j-\sum_{t=1}^{j-1}\langle\mathbf{V}_j,\mathbf{W}_t^{\mathcal{F}_i}\rangle\mathbf{W}_t^{\mathcal{F}_i}-\langle\mathbf{V}_j,\mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i},\quad P_1^{\mathcal{F}_i}=\mathbf{V}_1,\quad P_n^{\mathcal{F}_i}=\mathbf{W}_n^{\mathcal{F}_i}.$$

Proof. We can express the jth geodesic curvature by using (2) in the form

$$\kappa_{jg}^{\mathcal{F}_i}=\left\langle\left(\frac{P_j^{\mathcal{F}_i}}{\left|\left|P_j^{\mathcal{F}_i}\right|\right|}\right)',\frac{P_{j+1}^{\mathcal{F}_i}}{\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|}\right\rangle\quad\text{with}\quad\mathbf{W}_j^{\mathcal{F}_i}=\frac{P_j^{\mathcal{F}_i}}{\left|\left|P_j^{\mathscr{F}_i}\right|\right|}.$$

Then, we have

$$\begin{aligned}\kappa_{jg}^{\mathcal{F}_i}&=\frac{\left\langle\left(P_j^{\mathcal{F}_i}\right)',P_{j+1}^{\mathcal{F}_i}\right\rangle}{\left|\left|P_j^{\mathcal{F}_i}\right|\right|.\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|}-\frac{\left\langle P_j^{\mathcal{F}_i},P_{j+1}^{\mathcal{F}_i}\right\rangle\left\langle\left(P_j^{\mathcal{F}_i}\right)',P_j^{\mathcal{F}_i}\right\rangle}{\left|\left|P_j^{\mathcal{F}_i}\right|\right|^3\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|}\\&=\frac{\det\mathtt{T}}{\left|\left|P_j^{\mathcal{F}_i}\right|\right|^3\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|},\quad 1\leq j\leq n-1.\end{aligned}$$

Theorem 3. Let $\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_{n-1}$ be transversally intersection hypersurfaces given by their implicit forms $\mathcal{F}_i\big(u_1,u_2,...,u_n\big)=0$, $1\leq i\leq n-1$, respectively. Then, geodesic torsions of the intersection curve is given by

$$\tau_{jg}^{\mathcal{F}_i}=\frac{\det\mathcal{D}}{\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|^3\left|\left|\nabla\mathcal{F}_i\right|\right|},\quad 1\leq j\leq n-1 \tag{8}$$

where

$$\mathcal{D}=\begin{bmatrix}\left\langle\left(P_{j+1}^{\mathcal{F}_i}\right)',\nabla\mathcal{F}_i\right\rangle & \left\langle P_{j+1}^{\mathcal{F}_i},\nabla\mathcal{F}_i\right\rangle\\ \left\langle\left(P_{j+1}^{\mathcal{F}_i}\right)',P_{j+1}^{\mathcal{F}_i}\right\rangle & \left\langle P_{j+1}^{\mathcal{F}_i},P_{j+1}^{\mathcal{F}_i}\right\rangle\end{bmatrix},\quad P_j^{\mathcal{F}_i}=\mathbf{V}_j-\sum_{t=1}^{j-1}\langle\mathbf{V}_j,\mathbf{W}_t^{\mathcal{F}_i}\rangle\mathbf{W}_t^{\mathcal{F}_i}-\langle\mathbf{V}_j,\mathbf{W}_n^{\mathcal{F}_i}\rangle\mathbf{W}_n^{\mathcal{F}_i},\quad P_1^{\mathcal{F}_i}=\mathbf{V}_1,\quad P_n^{\mathcal{F}_i}=\mathbf{W}_n^{\mathcal{F}_i}.$$

Proof. We can represent the jth geodesic torsion by using Theorem 1 in the form

$$\tau_{jg}^{\mathcal{F}_i}=\left\langle\left(\frac{P_{j+1}^{\mathcal{F}_i}}{\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|}\right)',\frac{\nabla\mathcal{F}_i}{\left|\left|\nabla\mathcal{F}_i\right|\right|}\right\rangle\quad\text{with}\quad\mathbf{W}_j^{\mathcal{F}_i}=\frac{P_j^{\mathcal{F}_i}}{\left|\left|P_j^{\mathcal{F}_i}\right|\right|}.$$

Then, we have

$$\begin{aligned}\tau_{jg}^{\mathcal{F}_i}&=\frac{\left\langle\left(P_{j+1}^{\mathcal{F}_i}\right)',\nabla\mathcal{F}_i\right\rangle}{\left|\left|\nabla\mathcal{F}_i\right|\right|.\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|}-\frac{\left\langle\nabla\mathcal{F}_i,P_{j+1}^{\mathcal{F}_i}\right\rangle\left\langle\left(P_{j+1}^{\mathcal{F}_i}\right)',P_{j+1}^{\mathcal{F}_i}\right\rangle}{\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|^3\left|\left|\nabla\mathcal{F}_i\right|\right|}\\&=\frac{\det\mathcal{D}}{\left|\left|P_{j+1}^{\mathcal{F}_i}\right|\right|^3\left|\left|\nabla\mathcal{F}_i\right|\right|},\quad 1\leq j\leq n-1.\end{aligned}$$

In geodesic curvature and geodesic torsion calculations, u_i' derivatives can be simply determined by substituting $\dfrac{h_i}{\lambda}$ for u_i',

$1 \le i \le n$ [26]. However, in higher dimensions, computing the geodesic curvatures and geodesic torsions of the intersection curve via the formulas (7) and (8) will become incredibly difficult. To address this issue, we produce a MATLAB R2018a code (see Appendix).

As shown in Figure 1, the algorithmic structure may be summed up as follows:

i.Input dimension (n), (hyper)surfaces $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n-1}\}$, intersection point $p_0 = (p_1, p_2, ..., p_n)$.
ii. Compute $\{\nabla\mathcal{F}_1, \nabla\mathcal{F}_2, ..., \nabla\mathcal{F}_{n-1}\}$.
iii. Compute $\{\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_{n-1}\}$ using the algorithm defined in [26] and $i = 0$.
iv.$i = i + 1$
v.Define $\mathbf{W}_1^{\mathcal{F}_i} = P_1^{\mathcal{F}_i} = \mathbf{V}_1$, $\mathbf{W}_n^{\mathcal{F}_i} = P_n^{\mathcal{F}_i} = \dfrac{\nabla\mathcal{F}_i}{\|\nabla\mathcal{F}_i\|}$ and $j = 2$.
vi. Compute $\mathbf{W}_j^{\mathcal{F}_i} = \dfrac{P_j^{\mathcal{F}_i}}{\left\|P_j^{\mathcal{F}_i}\right\|}$ to construct the Darboux frame by using equations (6) until the vaule of j reaches $n-1$.
vii. Compute derivative of $P_j^{\mathcal{F}_i}$ from $j = 1$ to $j = n-1$.
viii. Calculate $\kappa_{ag}^{\mathcal{F}_i}$ geodesic curvatures by using equation (7) from $a = 1$ to $a = n-1$.
ix. Calculate $\tau_{ag}^{\mathcal{F}_i}$ geodesic torsions by using equation (8) from $a = 1$ to $a = n-1$.
x.Repeat steps from step **iv** until the value of i reaches $n-1$.

4 Examples

4.1 Example 1

Consider the surfaces that are produced by the subsequent equations in $\mathbb{E}^3$ (Figure 2):

$$\mathcal{F}_1(u_1, u_2, u_3) = u_3 - u_1 u_2 = 0,$$
$$\mathcal{F}_2(u_1, u_2, u_3) = u_1^2 + u_2^2 + u_3 - 3 = 0.$$

The geodesic curvatures and geodesic torsions at $p_0 = (1, -2, -2)$ of intersection curve of these surfaces were given in the work of Uyar Düldül and Çalışkan [31]. The normal vector fields of these surfaces are given by

$$\nabla\mathcal{F}_1 = (-u_2, -u_1, 1), \qquad \nabla\mathcal{F}_2 = (2u_1, 2u_2, 1),$$

respectively. It is evident and simple to confirm that these surfaces meet transversely at $p_0 = (1, -2, -2)$. Then, we have

$$\mathcal{H} = \nabla\mathcal{F}_1 \times \nabla\mathcal{F}_2 = (-u_1 - 2u_2, 2u_1 + u_2, 2u_1^2 - 2u_2^2) \quad \text{and} \quad \lambda = \sqrt{(u_1 + 2u_2)^2 + (2u_1 + u_2)^2 + (2u_1^2 - 2u_2^2)^2}.$$

By using the Δ operator, the Frenet vectors of the intersection curve $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ at point $p_0 = (1, -2, -2)$ are determined.

Calculations regarding the hypersurface $\mathcal{F}_1$

We derive Darboux frame with using the formulas

$$\mathbf{W}_1^{\mathcal{F}_1}(p_0) = \frac{\mathcal{H}(p_0)}{\lambda(p_0)},$$
$$\mathbf{W}_2^{\mathcal{F}_1}(p_0) = \frac{\mathbf{V}_2(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_1^{\mathcal{F}_1}(p_0) \right\rangle \mathbf{W}_1^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_3^{\mathcal{F}_1}(p_0) \right\rangle \mathbf{W}_3^{\mathcal{F}_1}(p_0)}{\left\| \mathbf{V}_2(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_1^{\mathcal{F}_1}(p_0) \right\rangle \mathbf{W}_1^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_3^{\mathcal{F}_1}(p_0) \right\rangle \mathbf{W}_3^{\mathcal{F}_1}(p_0) \right\|},$$
$$\mathbf{W}_3^{\mathcal{F}_1}(p_0) = \frac{\nabla\mathcal{F}_1(p_0)}{\|\nabla\mathcal{F}_1(p_0)\|}.$$

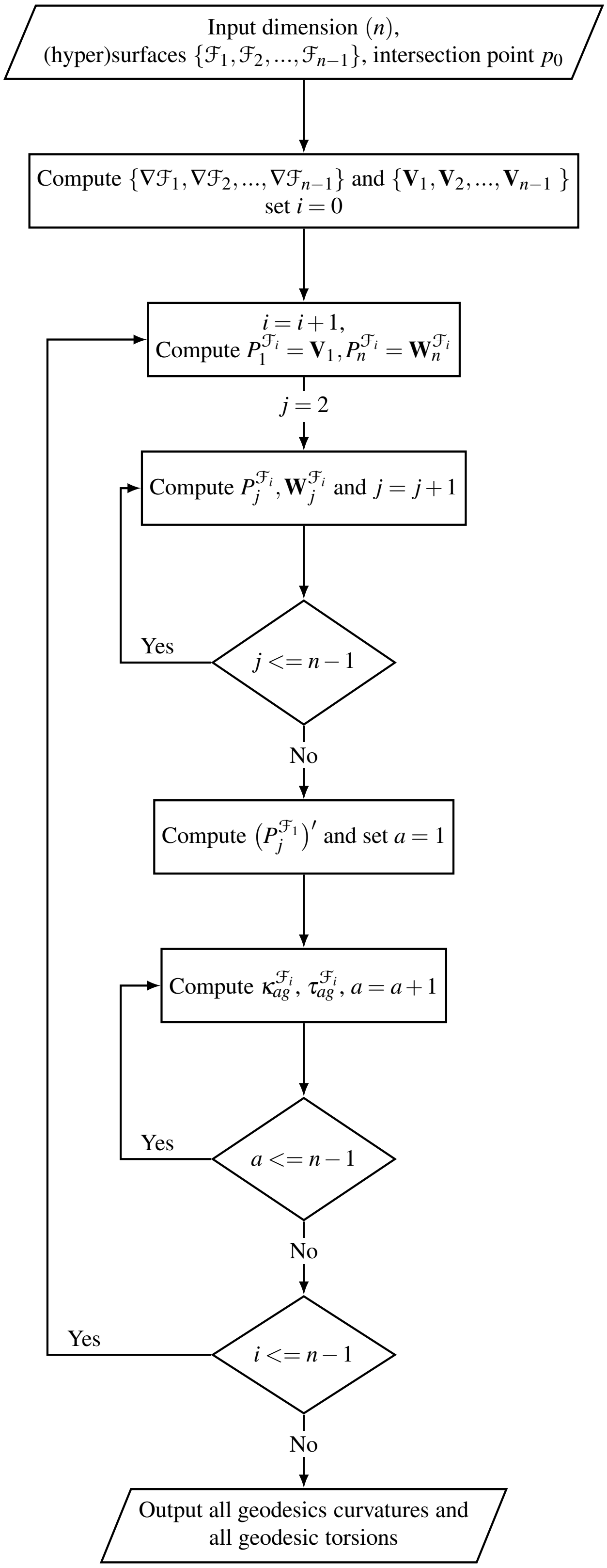

Fig. 1: Flowchart of computations

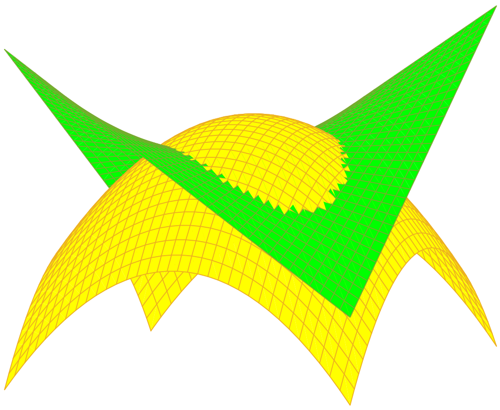

Fig. 2: Intersection of surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$

We get

$$P_1^{\mathcal{F}_1}(p_0)=\left(\frac{\sqrt{5}}{5},0,-\frac{2\sqrt{5}}{5}\right),\quad P_2^{\mathcal{F}_1}(p_0)=\left(\frac{\sqrt{30}}{15},\frac{\sqrt{30}}{6},\frac{\sqrt{30}}{30}\right),\quad P_3^{\mathcal{F}_1}(p_0)=\left(\frac{\sqrt{6}}{3},-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6}\right)$$

at intersection point p_0. By taking derivative of $P_j^{\mathcal{F}_1}$, $1\leq j\leq 3$, we obtain

$$(P_1^{\mathcal{F}_1})'(p_0)=\left(\frac{4}{75},\frac{2}{15},\frac{2}{75}\right)\quad (P_2^{\mathcal{F}_1})'(p_0)=\left(\frac{13\sqrt{6}}{450},-\frac{\sqrt{6}}{36},\frac{73\sqrt{6}}{900}\right)\quad (P_3^{\mathcal{F}_1})'(p_0)=\left(-\frac{\sqrt{30}}{90},-\frac{\sqrt{30}}{36},-\frac{\sqrt{30}}{180}\right)$$

at intersection point p_0. Thus, the geodesic curvature and geodesic torsion of the intersection curve with respect to $\mathcal{F}_1$ are $\kappa_{1g}^{\mathcal{F}_1}(p_0)=\dfrac{2\sqrt{30}}{75}$ and $\tau_{1g}^{\mathcal{F}_1}(p_0)=\dfrac{1}{6}$ as provided by Uyar Düldül and Çalışkan [31].

Calculations regarding the hypersurface $\mathcal{F}_2$

Similarly we produce the Darboux frame by using

$$\mathbf{W}_1^{\mathcal{F}_2}(p_0)=\frac{\mathcal{H}(p_0)}{\lambda(p_0)},$$

$$\mathbf{W}_2^{\mathcal{F}_2}(p_0)=\frac{\mathbf{V}_2(p_0)-\left\langle \mathbf{V}_2(p_0),\mathbf{W}_1^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_2}(p_0)-\left\langle \mathbf{V}_2(p_0),\mathbf{W}_3^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_3^{\mathcal{F}_2}(p_0)}{\left\|\mathbf{V}_2(p_0)-\left\langle \mathbf{V}_2(p_0),\mathbf{W}_1^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_2}(p_0)-\left\langle \mathbf{V}_2(p_0),\mathbf{W}_n^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_n^{\mathcal{F}_2}(p_0)\right\|},$$

$$\mathbf{W}_4^{\mathcal{F}_2}(p_0)=\frac{\nabla\mathcal{F}_2(p_0)}{\|\nabla\mathcal{F}_2(p_0)\|}$$

and we get

$$P_1^{\mathcal{F}_2}(p_0)=\left(\frac{\sqrt{5}}{5},0,-\frac{2\sqrt{5}}{5}\right),\quad P_2^{\mathcal{F}_2}(p_0)=\left(\frac{4\sqrt{30}}{35},\frac{\sqrt{30}}{14},\frac{2\sqrt{30}}{35}\right),\quad P_3^{\mathcal{F}_2}(p_0)=\left(\frac{2\sqrt{21}}{21},-\frac{4\sqrt{21}}{21},\frac{\sqrt{21}}{21}\right)$$

at intersection point p_0. By taking derivative of $P_j^{\mathcal{F}_2}$, $1 \leq j \leq 3$, we obtain

$$(P_1^{\mathcal{F}_2})'(p_0) = \left(\frac{4}{75}, \frac{2}{15}, \frac{2}{75}\right)$$
$$(P_2^{\mathcal{F}_2})'(p_0) = \left(\frac{941\sqrt{6}}{11025}, -\frac{1069\sqrt{6}}{8820}, \frac{943\sqrt{6}}{11025}\right)$$
$$(P_3^{\mathcal{F}_2})'(p_0) = \left(\frac{34\sqrt{105}}{2205}, \frac{16\sqrt{105}}{2205}, -\frac{4\sqrt{105}}{2205}\right)$$

at intersection point p_0. Hence, the geodesic curvature and geodesic torsion of the intersection curve with respect to $\mathcal{F}_2$ are $\kappa_{1g}^{\mathcal{F}_2}(p_0) = \dfrac{2\sqrt{105}}{175}$ and $\tau_{1g}^{\mathcal{F}_2}(p_0) = -\dfrac{16}{105}$ as provided by Uyar Düldül and Çalışkan [31].

4.2 Example 2

Consider the surfaces that are produced by the subsequent equations in $\mathbb{E}^3$ (Figure 3):

$$\mathcal{F}_1(u_1,u_2,u_3) = u_2 - u_3^2 - 1 = 0,$$
$$\mathcal{F}_2(u_1,u_2,u_3) = u_3(u_1^2+u_2^2+u_3^2-1) - 2u_2(u_1^2+u_3^2+u_1) = 0.$$

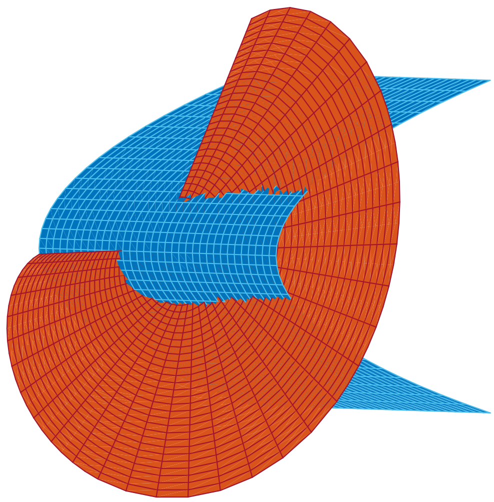

Fig. 3: Intersection of surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$

Then, we have

$$\nabla\mathcal{F}_1 = \big(0,1,-2u_3\big), \qquad \nabla\mathcal{F}_2 = \big(2u_1u_3 - 2u_2(2u_1+1), -2u_1^2 - 2u_1 - 2u_3^2 + 2u_2u_3, u_1^2 + u_2^2 - 4u_2u_3 + 3u_3^2 - 1\big),$$

respectively. It is easy to see that we have a transversal intersection at $p_0 = \big(0,0,1\big)$. Then, we have

$$\mathcal{H} = \big(-4u_1^2u_3 + u_1^2 - 4u_1u_3 + u_2^2 + 4u_2u_3^2 - 4u_2u_3 - 4u_3^3 + 3u_3^2 - 1, 4u_3(u_2 + 2u_1u_2 - u_1u_3), 2u_2 + 4u_1u_2 - 2u_1u_3\big)$$

and

$$\lambda = \sqrt{16u_3^2(u_2+2u_1u_2-u_1u_3)^2 + (4u_1^2u_3 - u_1^2 + 4u_1u_3 - u_2^2 - 4u_2u_3^2 + 4u_2u_3 + 4u_3^3 - 3u_3^2 + 1)^2 + 4(u_2+2u_1u_2-u_1u_3)^2}$$

If Δ and Darboux frame formulas are used similarly to the previous example, then we obtain

$$P_1^{\mathcal{F}_1}(p_0) = \big(-1,0,0\big), \quad P_2^{\mathcal{F}_1}(p_0) = \left(0, \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right), \quad P_3^{\mathcal{F}_1}(p_0) = \left(0, \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}\right)$$

and

$$\big(P_1^{\mathcal{F}_1}\big)'(p_0) = \big(0,2,1\big) \quad \big(P_2^{\mathcal{F}_1}\big)'(p_0) = \big(\sqrt{5},0,0\big) \quad \big(P_3^{\mathcal{F}_1}\big)'(p_0) = \big(0,0,0\big)$$

at intersection point p_0. Thus, the geodesic curvature and geodesic torsion of the intersection curve with respect to $\mathcal{F}_1$ are $\kappa_{1g}^{\mathcal{F}_1}(p_0) = \sqrt{5}$ and $\tau_{1g}^{\mathcal{F}_1}(p_0) = 0$. If similar operations are applied for the surface $\mathcal{F}_2$, then we compute

$$P_1^{\mathcal{F}_2}(p_0) = \big(-1,0,0\big), \quad P_2^{\mathcal{F}_2}(p_0) = \left(0, \frac{3\sqrt{5}}{10}, \frac{3\sqrt{5}}{10}\right), \quad P_3^{\mathcal{F}_2}(p_0) = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

and

$$\big(P_1^{\mathcal{F}_2}\big)'(p_0) = \big(0,2,1\big) \quad \big(P_2^{\mathcal{F}_2}\big)'(p_0) = \left(\frac{9\sqrt{5}}{10}, \frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{10}\right) \quad \big(P_3^{\mathcal{F}_2}\big)'(p_0) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$$

at intersection point p_0. Thus, the geodesic curvature and geodesic torsion of the intersection curve with respect to $\mathcal{F}_2$ are $\kappa_{1g}^{\mathcal{F}_2}(p_0) = \dfrac{3\sqrt{2}}{2}$ and $\tau_{1g}^{\mathcal{F}_2}(p_0) = -\dfrac{1}{2}$.

4.3 Example 3

Consider the hypersurfaces that are produced by the subsequent equations in $\mathbb{E}^4$:

$$\mathcal{F}_1\big(u_1,u_2,u_3,u_4\big) = u_1 - u_2^4 = 0,$$
$$\mathcal{F}_2\big(u_1,u_2,u_3,u_4\big) = 2u_2 - u_3 - u_1 = 0,$$
$$\mathcal{F}_3\big(u_1,u_2,u_3,u_4\big) = u_3 + u_4 - 2u_1^2 = 0.$$

We have

$$\nabla\mathcal{F}_1 = \big(1,-4u_2^3,0,0\big), \quad \nabla\mathcal{F}_2 = \big(-1,2,-1,0\big), \quad \nabla\mathcal{F}_3 = \big(-4u_1 1,0,1,1\big).$$

It is clear and simple to confirm that these hypersurfaces meet transversely at $p_0 = \big(1,1,1,1\big)$. Then, we have

$$\mathcal{H} = \nabla\mathcal{F}_1 \times \nabla\mathcal{F}_2 \times \nabla\mathcal{F}_3 = \big(4u_2^3, 1, 2 - 4u_2^3, 16u_1u_2^3 + 4u_2^3 - 2\big)$$

and

$$\lambda = \sqrt{(16u_1u_2^3 + 4u_2^3 - 2)^2 + 16u_2^6 + (4u_2^3 - 2)^2 + 1}.$$

Using the Δ operator and Frenet calculations provided Section 2.1 gives $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$.

Calculations regarding the hypersurface $\mathcal{F}_1$

We derive Darboux frame with using the formulas

$$\mathbf{W}_1^{\mathcal{F}_1}(p_0) = \frac{\mathcal{H}(p_0)}{\lambda(p_0)},$$
$$\mathbf{W}_2^{\mathcal{F}_1}(p_0) = \frac{\mathbf{V}_2(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_1^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_4^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_1}(p_0)}{\left\| \mathbf{V}_2(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_1^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_4^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_1}(p_0)\right\|},$$
$$\mathbf{W}_3^{\mathcal{F}_1}(p_0) = \frac{\mathbf{V}_3(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_1^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_2^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_2^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_4^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_1}(p_0)}{\left\| \mathbf{V}_3(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_1^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_2^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_2^{\mathcal{F}_1}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_4^{\mathcal{F}_1}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_1}(p_0)\right\|},$$
$$\mathbf{W}_4^{\mathcal{F}_1}(p_0) = \frac{\nabla \mathcal{F}_1(p_0)}{\|\nabla \mathcal{F}_1(p_0)\|}$$

and we get

$$P_1^{\mathcal{F}_1}(p_0) = \big(0.215352, 0.053838, -0.107676, 0.969087\big)$$
$$P_2^{\mathcal{F}_1}(p_0) = \big(-0.925953, -0.231488, 0.0814633, 0.227679\big)$$
$$P_3^{\mathcal{F}_1}(p_0) = \big(0.092262, -0.023065, -0.896680, -0.077846\big)$$
$$P_4^{\mathcal{F}_1}(p_0) = \big(0.242535, -0.970142, 0, 0\big)$$

at intersection point p_0. By taking derivative of $P_j^{\mathcal{F}_1}$, $1 \leq j \leq 4$, we obtain

$$\big(P_1^{\mathcal{F}_1}\big)'(p_0) = \big(-0.042646, -0.019357, 0.003931, 0.010989\big)$$
$$\big(P_2^{\mathcal{F}_1}\big)'(p_0) = \big(-0.009669, 0.034971, 0.129474, -0.033692\big)$$
$$\big(P_3^{\mathcal{F}_1}\big)'(p_0) = \big(-0.108043, -0.023285, 0.020554, 0.027587\big)$$
$$\big(P_4^{\mathcal{F}_1}\big)'(p_0) = \big(-0.036868, -0.009217, 0, 0\big)$$

at intersection point p_0. Then, by using (7) and (8), we compute

$$\kappa_{1g}^{\mathcal{F}_1}(p_0) = 0.047523 \qquad \tau_{1g}^{\mathcal{F}_1}(p_0) = -0.036839$$

$$\kappa_{2g}^{\mathcal{F}_1}(p_0) = -0.127241 \qquad \tau_{2g}^{\mathcal{F}_1}(p_0) = -0.003993$$

Calculations regarding the hypersurface $\mathcal{F}_2$

Similarly we produce the Darboux frame by using

$$\mathbf{W}_1^{\mathcal{F}_2}(p_0) = \frac{\mathcal{H}(p_0)}{\lambda(p_0)},$$
$$\mathbf{W}_2^{\mathcal{F}_2}(p_0) = \frac{\mathbf{V}_2(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_1^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_2}(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_4^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_2}(p_0)}{\left\| \mathbf{V}_2(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_1^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_2}(p_0) - \left\langle \mathbf{V}_2(p_0), \mathbf{W}_4^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_2}(p_0)\right\|},$$
$$\mathbf{W}_3^{\mathcal{F}_2}(p_0) = \frac{\mathbf{V}_3(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_1^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_2}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_2^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_2^{\mathcal{F}_2}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_4^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_2}(p_0)}{\left\| \mathbf{V}_3(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_1^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_1^{\mathcal{F}_2}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_2^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_2^{\mathcal{F}_2}(p_0) - \left\langle \mathbf{V}_3(p_0), \mathbf{W}_4^{\mathcal{F}_2}(p_0)\right\rangle \mathbf{W}_4^{\mathcal{F}_2}(p_0)\right\|},$$
$$\mathbf{W}_4^{\mathcal{F}_2}(p_0) = \frac{\nabla \mathcal{F}_2(p_0)}{\|\nabla \mathcal{F}_2(p_0)\|}$$

and we obtain

$$P_1^{\mathcal{F}_2}(p_0) = (0.215352, 0.053838, -0.107676, 0.969087)$$
$$P_2^{\mathcal{F}_2}(p_0) = (-0.883563, -0.401050, 0.081463, 0.227679)$$
$$P_3^{\mathcal{F}_2}(p_0) = (0.079195, -0.411817, -0.902830, -0.095034)$$
$$P_4^{\mathcal{F}_2}(p_0) = (-0.408248, 0.816496, -0.408248, 0)$$

at intersection point p_0. By taking derivative of $P_j^{\mathcal{F}_2}$, $1 \leq j \leq 4$, we have

$$\left(P_1^{\mathcal{F}_2}\right)'(p_0) = (-0.042646, -0.019357, 0.003931, 0.010989)$$
$$\left(P_2^{\mathcal{F}_2}\right)'(p_0) = (0.021295, 0.054089, 0.129474, -0.033692)$$
$$\left(P_3^{\mathcal{F}_2}\right)'(p_0) = (-0.121625, -0.055205, 0.011213, 0.031340)$$
$$\left(P_4^{\mathcal{F}_2}\right)'(p_0) = (0, 0, 0, 0)$$

at intersection point p_0. Then, utilizing (7) and (8), we get

$$\kappa_{1g}^{\mathcal{F}_2}(p_0) = 0.048266 \qquad \tau_{1g}^{\mathcal{F}_2}(p_0) = 0$$

$$\kappa_{2g}^{\mathcal{F}_2}(p_0) = -0.137653 \qquad \tau_{2g}^{\mathcal{F}_2}(p_0) = 0$$

Calculations regarding the hypersurface $\mathcal{F}_3$

In a similar manner, if the Darboux frame is determined for the $\mathcal{F}_3$ hypersurface with $\mathbf{W}_1^{\mathcal{F}_3} = \mathbf{V}_1$ and $\mathbf{W}_4^{\mathcal{F}_3} = \dfrac{\nabla \mathcal{F}_3}{\|\nabla \mathcal{F}_3\|}$ at intersection point p_0, then equations (7) and (8) will provide us with the following:

$$\kappa_{1g}^{\mathcal{F}_3}(p_0) = 0.020440 \qquad \tau_{1g}^{\mathcal{F}_3}(p_0) = -0.014131$$

$$\kappa_{2g}^{\mathcal{F}_3}(p_0) = -0.327838 \qquad \tau_{2g}^{\mathcal{F}_3}(p_0) = -0.049688$$

5 Conclusions

We determined the formulas for all geodesic curvatures and geodesic torsions of the implicit curve defined by the transversal intersection curve of $(n-1)$ implicit hypersurfaces in Euclidean n-space. $(n-1)$ implicit hypersurfaces establish an implicit curve in the n-dimensional Eucliedean space. Therefore, the provided results might be viewed as a partial solution to the unresolved problem stated by Goldman [18]. In addition, to apply our strategy, we created a MATLAB algorithm that works effectively in any dimension.

Declarations

Competing interests: The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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Appendix

```
n=input('enter the dimension of space: n=')
T = sprintfc('x%d', 1:n);
syms(T:)
syms s
for i=1:n-1
    fprintf('enter the %d-th (hyper)surface:',i)
    f(i)=input('');
end
for i=1:n-1
    fprintf('f(%d)=%s n',i,f(i))
end
x = sym('x',[1 n])
assume(x,'real')
for i=1:n-1
    for j=1:n
        m(j)=diff(f(i),x(j));
    end
    fprintf('Gradf(%d)=',i)
    disp(m);
    G(i,:)=m;
end
for i=1:n-1
    A(i,:)=[G(i,:)];
end
if(mod(n,2)==0)
    h=-Product(A)
        else
    h=Product(A)
end
lambda= sqrt(dot(h.',h.'))
for i=1:n
    fprintf('enter the %d-th component of point p:',i)
    p(i)=input('');
end
for i=1:n
    fprintf('p(%d)=%d n',i,p(i))
end
r = sprintfc('x%d(s)', 1:n);
z=str2sym(r);
assume(z,'real')
y(1,:)=subs(h, x(1),z(1));
b(1,:)=subs(lambda, x(1),z(1));
for i=2:n
    y(i,:)=subs(y(i-1,:), x(i),z(i));
    b(i,:)=subs(b(i-1,:), x(i),z(i));
end
hs=y(n,:);
lambdas=b(n,:);
assume(lambdas,'real')
V(1,:)=hs./lambdas
v=str2sym(r);
H=diff(v,s);
A=diff(hs,s);
deltah(1,:)=A;
M(1,:)=subs(deltah(1,:),H,hs)
deltah(2,:)=diff(M(1,:),s);
```

```
M(2,:)=subs(deltah(2,:),H,hs);
for i=3:n-1
    deltah(i,:)=diff(M(i-1,:),s);
    M(i,:)=subs(deltah(i,:),H(1,:),hs);
end
sum=0;
for i=1:n
    t=M(1,i).*V(1,i);
    sum=sum+t;
end
g=sum;
lambdaprime=g/lambdas
Q=0;
for i=1:n
    y=M(1,i).*M(1,i);
    Q=Q+y;
end
R=Q
assume(lambdaprime,'real')
lambdaussu=sym(lambdaprime);
K=(sqrt((R)-(lambdas*lambdaussu)^2))/(lambdas)^2;
k(1,:)=sym(K);
v2=(M(1,:)-(lambdas*lambdaussu*V(1,:)))/(((lambdas)^2)*k(1,:));
V(2,:)=sym(v2)
deltah(2,:)=diff(M(1,:),s)
M(2,:)=subs(deltah(2,:),H,hs)
for i=3:n-1
    deltah(i,:)=diff(M(i-1,:),s)
    M(i,:)=subs(deltah(i,:),H(1,:),hs)
end
for a=1:n-1
    fprintf('Delta%dh=',a)
    disp(M(a,:))
end
u(1,:)=hs;
for i=2:n-1
    u(i,:)=M(i-1,:)
end
if(mod(n,2)==0)
    v4=-Product(u)/norm(Product(u));
        else
    v4=Product(u)/norm(Product(u));
end
V(n,:)=v4
t=n;
if n>4
    L(1,:)=hs
    for i=n-1:-1:4
        j=2;
        while i-j>=1
            L(j,:)=M(j-1,:)
            j=j+1
        end
        s=n
        while j<=n-1
            L(j,:)=V(s,:)
            j=j+1;
            s=s-1;
```

```
        end
        V(i,:)=-Product(L)/norm(Product(L));
    end
    E(n-2,:)=V(1,:)
    E(n-1,:)=V(2,:)
    for p=1:n-3
        E(p,:)=V(t,:)
        t=t-1
    end
    if(mod(n,2)==0)
        c=-Product(E)/norm(Product(E));
            else
        c=Product(E)/norm(Product(E));
    end
    V(3,:)=c;
end
if n>3
    j=1
    for i=n:-1:4
        q(j,:)=V(i,:)
        j=j+1
    end
    q(n-2,:)=V(1,:)
    q(n-1,:)=V(2,:)
    if(mod(n,2)==0)
        v3=-Product(q)/norm(Product(q));
            else
        v3=Product(q)/norm(Product(q));
    end
    V(3,:)=v3;
end
for a=1:n
    fprintf('V%d=',a)
    disp(V(a,:))
end
for i=1:n-1
    GG(i,:)=subs(G(i,:),x,z);
end
for i=1:n-1
    P(1,:)=V(1,:);
    W(1,:)=P(1,:)/norm(P(1,:));
    P(n,:)=GG(i,:)/norm(GG(i,:));
    W(n,:)=P(n,:);
    summation=0;
    for j=2:n-1
        for t=1:j-1
            eta=dot(V(j,:).',W(t,:))*W(t,:);
            summation=eta+summation;
        end
    b=dot(V(j,:).',W(n,:).').*W(n,:);
    P(j,:)=V(j,:)-summation-b;
    W(j,:)=P(j,:)/norm(P(j,:));
    end
    for m=1:n
        DP(m,:)=diff(P(m,:),s)
    end
    P=subs(P,H,V(1,:))
    P=subs(P,z,x)
```

```
    P=subs(P,x,p)
    DP=subs(DP,H,V(1,:))
    DP=subs(DP,z,x)
    DP=subs(DP,x,p)
    for a=1:n-2
        ro=[dot(DP(a,:).',P(a+1,:)) dot(P(a,:).',P(a+1,:)) ;dot(DP(a,:).',P(a,:)) dot(P(a,:).',P(a,:))];
        GC=det(ro)/((norm(P(a,:))^3)*(norm(P(a+1,:))));
        GC=subs(GC,z,x);
        GC=subs(GC,x,p)
        printf('%d-th (hyper)surface %d-th geodesic curvature=',i,a)
        disp(GC)
        beta=[dot(DP(a+1,:).',GG(i,:))
        dot(P(a+1,:).',GG(i,:));dot(DP(a+1,:).',P(a+1,:)) dot(P(a+1,:).',P(a+1,:))];
        GT=det(beta)/((norm(P(a+1,:))^3)*(norm(GG(i,:))));
        GT=subs(GT,z,x);
        GT=subs(GT,x,p)
        fprintf('%d-th (hyper)surface %d-th geodesic torsion=',i,a)
        disp(GT)
    end
end
```

The function representing vector product in n-dimensional Euclidean space

```
function [P,A]=Product(B)
C=sym('e',[1,length(B)]);
D=vertcat(B,C);
A=det(D);
E=coeffs(A,C);
if length(E)==length(D)
    P=fliplr(E);
        elseif size(E,2)==0
    P=sym(zeros(1,length(D)));
        else
    P=sym(zeros(1,length(D)));
    for i=1:length(D)
        if length(coeffs(A,strcat('e',int2str(i))))==1
        P(i)=0;
            else
        F=coeffs(A,strcat('e',int2str(i)));
        P(i)=F(2);
        end
    end
end
if P==zeros([length(B)])
    k=length(B)
    M=eye(length(B))
    y(1,:)=subs(A,E(1),M(1,:))
    for i=2:k
        y(i,:)=subs(A,E(i),M(i,:))
    end
    top=0;
    for i=1:k
        gg(i,:)=y(i,:)
        top=top+gg(i,:)
    end
    P=top-(k-1)*A
end
    rr=sqrt(dot(P.',P.'));
end
```