JJMS **17**, No. 3, 363-376 (2024)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.3.4

Explicit formulas for flux surfaces and scalar flux functions according to Killing magnetic vectors in $\mathbb{S}L(2,\mathbb{R})$

Noria BEKKOUCHE[1,*], *Fouzi HATHOUT*[2]

[1] Department of Mathematics, University of Saïda, Saïda 20000, Algeria.
[2] Department of Computer Science, University of Saïda, Saïda 20000, Algeria.

Received: May 9, 2023 Accepted : Nov. 2, 2023

Abstract: In this paper, we determine flux surfaces according to Killing magnetic vectors and its associate scalar flux functions in 3-dimensional Riemannian space $\mathbb{S}L(2,\mathbb{R})$. We give for each parametric flux surface an example and its graphical representations in Euclidean 3-space.

Keywords: Flux; Killing magnetic flux surface; magnetic curves; Scalar flux function; $\mathbb{S}L(2,\mathbb{R})$ manifold.

2010 Mathematics Subject Classification. 53A35; 53C21; 37C10.

1 Introduction

Flux surface is a 2-dimensional geometric structure that does not influence the flux, which describes any effect that appears to pass or travel through a surface or substance, in an ambient environment. Flux is a vector calculus given as

$$\phi_B = \int_M g(\overrightarrow{B}, \overrightarrow{\mathbf{n}})ds,$$

where $\overrightarrow{B}$ is a smooth vector corresponding to the flux ϕ_B passes through the surface M in Riemannian manifold (N,g), with the normal vector $\overrightarrow{\mathbf{n}}$ (To simplify, we denote the vectors $\overrightarrow{B}, \overrightarrow{\mathbf{n}}$ by $B, \mathbf{n}$). Therefore, M is called flux surface if B is orthogonal to $\mathbf{n}$ everywhere on M which is part of the constant angle surface set, to be determined explicitly in the sequel in $\mathbb{S}L(2,\mathbb{R})$. The flux surfaces was determined in Euclidean 3-space [15], in Heisenberg three group [7] and $\mathbb{S}ol3$ space in [2].

Moreover, if B is a magnetic vector fields (i.e. B is zero divergences according to Biot and Savart's law see [4] and [15]) then M is called flux surface according to the magnetic vector fields B. Hence, we can define a scalar flux function f according to B, such that its value of f is constant on the surface M, and

$$g(B, \nabla f) = 0,$$

where ∇f is Riemannian gradient on M.

If B is also a Killing i.e. magnetic vector fields satisfying the Killing equation i.e.

$$g(\nabla_X B, Y) + g(\nabla_Y B, X) = 0, \tag{1}$$

where ∇ is a connection and X, Y are a vector fields on N, then M is called flux surface according to the Killing magnetic vector B.

* Corresponding author e-mail: m_bekkouchenoria277@gmail.com

In particular case when (N,g) is three dimensional Riemannian manifold, the topological toroid is the only closed flux surface corresponding to a non-vanishing vector field B which is the basis of the design of magnetic confinement devices. Assuming the flux surfaces have this toroidal topology, the function f defines a set of nested surfaces, so it makes sense to use this function to label the flux surfaces. Each toroidal surface f encloses a volume $\mathscr{V}(f)$. The surface corresponding to an infinitesimal volume $\mathscr{V}$ is essentially a line that corresponds to the toroidal axis (called magnetic axis when B is a magnetic field). When B is a magnetic field with toroidal nested flux surfaces, two magnetic fluxes can be defined from two corresponding surfaces. The poloidal flux toroidal flux are defined by

$$\phi_1 = \int_{M_1} g(B,\mathbf{n})ds \text{ and } \phi_2 = \int_{M_2} g(B,\mathbf{n})ds,$$

where M_1 is a ring-shaped ribbon stretched between the magnetic axis and the flux surface f. (Complementarily, M_1 can be taken to be a surface spanning the central hole of the torus) and M_2 is a poloidal section of the flux surface, respectively. (For more detail see [3,7,9,15])

The famous example of magnetic surface which is a flux surfaces is the plasma which is considered as fourth state of matter. It is a hot ionized gas made up of approximately equal numbers of positively charged ions and negatively charged electrons that it makes it a good electrical conductor. The most of the visible matter in the universe is in the plasma state. (see [3,4,5,9,17]). Flux surfaces, in Minkowski context, appeared as application on the dynamics of solitons and dispersive effects in [10,11,12].

Note that the magnetic surfaces which all magnetic curves lie, whose definition is not yet definitively posed, are flux surfaces. Recall that Killing magnetic curve $\gamma : I \subset \mathbb{R} \to N$ with parametric t, is a trajectories of the charged particles moving under the action of the magnetic fields F which is a solution of Lorentz equation

$$\nabla_{\gamma(t)}\gamma'(t) = V \times \gamma'(t),$$

where V is a Killing vector fields corresponding to Lorentz force F of magnetic flux and "$\times$" is cross product. The magnetic curves have been intensively studied by many authors in different manifolds that we just cite that we will use in the sequel. (See [8,6,16]).

The purpose of paper is to investigate to the determination of flux surfaces in Riemannian space $\mathbb{S}L(2,\mathbb{R})$ which appear in the literature with two models. The first model is called the hyperboloid that appear in [13] and denoted by $\widetilde{\mathbb{S}L}(2,\mathbb{R})$ and the second, which we will use in this paper, is called the right-half space model of $\mathbb{S}L(2,\mathbb{R})$ that its geometry is detailed in [14,18].

This paper is organized as follows. In the second section, we give a brief overview on a geometry of Riemannian space $\mathbb{S}L(2,\mathbb{R})$. We present in the third section the definitions and the determinations of flux surfaces and scalar flux functions in general Riemannian manifold. Next, we determine all parameterizations of flux surface and scalar flux function f according to Killing magnetic vectors $\mathbb{K}$ on $\mathbb{S}L(2,\mathbb{R})$ and we present an examples of each cases. We use the computer software "Wolfram Mathematica" for a graphical representations in Euclidean space $\mathbb{E}^3$.

2 Geometry of Riemannian space $\mathbb{S}L(2,\mathbb{R})$

The $\mathbb{S}L(2,\mathbb{R})$ space with the standard representation as

$$\mathbb{S}L(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

is seen as a space $\mathbb{R}^3$ endowed with the Riemannian metric given by

$$g_{\mathbb{S}L(2,\mathbb{R})} = \frac{1}{4y^2}dx^2 + \frac{1}{4y^2}dy^2 + \left(\frac{1}{2y}dx + dz\right)^2. \tag{2}$$

We define an orthonormal basis $(e_i)_{i=\overline{1,3}}$ as

$$e_1 = 2y\partial x - \partial z, \quad e_2 = 2y\partial y, \quad e_3 = \partial z, \tag{3}$$

where $\left(\partial x = \frac{\partial}{\partial x}, \partial y = \frac{\partial}{\partial y}, \partial z = \frac{\partial}{\partial z}\right)$ and (dx, dy, dz) are local bases of $T_p\mathbb{R}^3$ and $T_p^*\mathbb{R}^3$ respectively. The dual basis $(\omega^i)_{i=\overline{1,3}}$ of $(e_i)_{i=\overline{1,3}}$ is

$$\omega^1 = \frac{1}{2y}dx; \quad \omega^2 = \frac{1}{2y}dy; \quad \omega^3 = \frac{1}{2y}dx + dz.$$

The Levi-Civita connection ∇ of the metric $g_{\mathbb{S}L(2,\mathbb{R})}$ with respect to the orthonormal basis $(e_i)_{i=\overline{1,3}}$ is

$$\begin{cases} \nabla_{e_1}e_1 = 2e_2 & \nabla_{e_2}e_1 = e_3 & \nabla_{e_3}e_1 = e_2 \\ \nabla_{e_1}e_2 = -2e_1 - e_3\,; & \nabla_{e_2}e_2 = 0 \quad ; & \nabla_{e_3}e_2 = -e_1 \\ \nabla_{e_1}e_3 = e_2 & \nabla_{e_2}e_3 = -e_1 & \nabla_{e_3}e_3 = 0 \end{cases}. \tag{4}$$

The algebra of Killing vector field of $\mathbb{S}L(2,\mathbb{R})$ it is generated by the basis $\mathbb{K} = (K_i)_{i=\overline{1,3}}$, which are solutions of Eq.(1), where the Killing vectors $(K_i)_{i=\overline{1,3}}$ are presented in the following

$$K_1 = \partial z, \;\; K_2 = \partial x, \;\; K_3 = x\partial x + y\partial y \text{ and}$$
$$K_4 = \frac{1}{2}\left(x^2 - y^2\right)\partial x + xy\partial y. \tag{5}$$

The Killing vectors, in the base $(e_i)_{i=\overline{1,3}}$ from the Eq.(3), are

$$K_1 = e_3, \quad K_2 = \frac{1}{2y}(e_1 + e_3), \;\; K_3 = \frac{x}{2y}e_1 + \frac{1}{2}e_2 + \frac{x}{2y}e_3, \tag{6}$$
$$K_4 = \frac{x^2 - y^2}{4y}e_1 + \frac{x}{2}e_2 + \frac{x^2 - y^2}{4y}e_3,$$

(for more detail see [1,8,18]).

3 Flux surfaces and scalar flux functions

3.1 Flux surfaces in $\mathbb{S}L(2,\mathbb{R})$

Definition 1. *Let M be a smooth surface in a Riemannian manifold (N,g) and $\mathbf{n}$ be its normal vector field. We call M a flux surface according to the vector field V on (N,g) if*

$$g(V,\mathbf{n}) = 0$$

everywhere on M. The surface M is denoted by V-flux surface.
If V is a Killing field then M is called a flux surface according to the Killing vector V and denoted by Killing V-flux surface.
Moreover, if V is magnetic vector fields, M is denoted by Killing magnetic V-flux surface.

Let M be a surface in $\mathbb{S}L(2,\mathbb{R})$ parameterized by $X(s,t) = (x(s,t), y(s,t), z(s,t))$. Its tangent vectors X_s and X_t are described by

$$\begin{cases} X_s = x_s\partial x + y_s\partial y + z_s\partial z = \frac{x_s}{2y}e_1 + \frac{y_s}{2y}e_2 + \left(\frac{x_s}{2y} + z_s\right)e_3, \\ X_t = x_t\partial x + y_t\partial y + z_t\partial z = \frac{x_t}{2y}e_1 + \frac{y_t}{2y}e_2 + \left(\frac{x_t}{2y} + z_t\right)e_3. \end{cases}$$

The normal vector $\mathbf{n}$ of M, in the base $(e_i)_{i=\overline{1,3}}$, is given by

$$\mathbf{n} = \frac{X_s \times X_t}{\|X_s \times X_t\|} = \frac{1}{2y\|X_s \times X_t\|}\begin{pmatrix} \left(\frac{1}{2y}(y_s x_t - y_t x_s) + y_s z_t - y_t z_s\right)e_1 \\ + \left(\frac{1}{2y}(x_t x_s - x_s x_t) + x_t z_s - x_s z_t\right)e_2 \\ + \frac{1}{2y}(x_s y_t - x_t x_s)e_3 \end{pmatrix}. \tag{7}$$

Now, we have the theorem.

Theorem 1. *Let M be a surface in $\mathbb{S}L(2,\mathbb{R})$ parameterized by $X(s,t) = (x(s,t), y(s,t), z(s,t))$. Then M is a V-flux surface if and only if*

$$(y_s x_t - x_s y_t + 2y(y_s z_t - y_t z_s))v_1 +$$
$$2y(x_t z_s - x_s z_t)v_2 + (x_s y_t - x_t y_s)v_3 = 0,$$

where $V = v_1e_1 + v_2e_2 + v_3e_3$.

Proof. Its a direct consequence by using the metric given by Eq.(2) in the orthonormal base $(e_i)_{i=\overline{1,3}}$, the Definition 1 and the normal vector $\mathbf{n}$ given in the Eq.(7).

3.2 Scalar flux functions in $\mathbb{S}L(2,\mathbb{R})$

Definition 2. *Let f be a function on (N,g) and M be a V-magnetic flux surface in N. The function f is called a scalar flux function on M according to the magnetic vector fields V if the value of f is constant on the surface M and*

$$g(V,\nabla f)=0.$$

We denoted here f, to simplify, a magnetic V-scalar flux function. Moreover, if V is Killing vector then f is denoted by Killing magnetic V-scalar flux function.

Lemma 1. *Let f be a scalar function in $\mathbb{S}L(2,\mathbb{R})$, the Riemannian gradient of f is*

$$\nabla f=f_x\partial x+f_y\partial y+f_z\partial z=\frac{f_x}{2y}e_1+\frac{f_y}{2y}e_2+\left(\frac{f_x}{2y}+f_z\right)e_3.$$

Proposition 1. *Let f be a function on (N,g) and $M\subset N$ be a magnetic V-flux surface. f is scalar flux function to M if*

$$\frac{f_x}{2y}v_1+\frac{f_y}{2y}v_2+\left(\frac{f_x}{2y}+f_z\right)v_3=0 \text{ and } f_{/M} \text{ is constant,}$$

where $V=v_1e_1+v_2e_2+v_3e_3$ and "$f_{/M}$" is the restriction of f to M.

Proof. We get the proof by a simple computation using the Definition 2 and the Lemma 1.

4 Flux surfaces and scalar flux functions according to Killing vectors in $\mathbb{S}L(2,\mathbb{R})$

The determination of Killing flux surfaces needs the resolution methods of a homogeneous and nonhomogeneous partial differential equations (PDE). Therefore from [19], we present the resolution methods of PDE in the following;

Proposition 2. *Let P and Q two functions in real parameters s and t. The general solutions of PDE*

$$P(s,t)h_s+Q(s,t)h_t=R(s,t), \tag{8}$$

with the unknown function $h(s,t)$ are in the form:
1. *When the PDE (8) is homogeneous (i.e. $R\equiv 0$).*
i. *If $P\equiv 0$ (resp. $Q\equiv 0$) then $h(s,t)=h(s)$ (resp. $h(s,t)=h(t)$).*
ii. *If P and Q are non null functions, then*

$$h(s,t)=\varphi\left(\psi(s,t)\right),$$

where $\psi(s,t)=c$ (constant), is the solution of ODE

$$\frac{ds}{P}=\frac{dt}{Q}$$

and φ is arbitrary real function.
2. *When the PDE (8) is nonhomogeneous (i.e. $R\neq 0$):*
i. *If $P\equiv 0$ (resp. $Q\equiv 0$) then $h(s,t)=\int\frac{R}{Q}dt$ (resp. $h(s,t)=\int\frac{R}{P}ds$),*
ii. *If P and Q are non null functions, then solution h is given implicitly from*

$$\overline{\Psi}_1(s,t,h)=\varphi\left(\overline{\Psi}_2(s,t,h)\right),$$

where $\overline{\Psi}_{1,2}(s,t)=c_{1,2}$ (constants) are the choice of two functions among three functions solutions of three ODEs

$$\frac{ds}{P}=\frac{dt}{Q}=\frac{dh}{R}$$

and φ is arbitrary function in $\mathbb{R}$.

The Killing vectors K_1 and K_2 has a same study, in the following section we determine only flux surfaces and scalar flux function corresponding to K_1.

4.1 Killing K_1-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$

We characterise all Killing K_1-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$ given in the Eq.(6) in the following theorem.

Theorem 2. *Let M_1 be a surface in $\mathbb{S}L(2,\mathbb{R})$ parametrized by $X(s,t) = (x(s,t), y(s,t), z(s,t))$. M_1 is a Killing K_1-flux surface if and only if*

$$x_s y_t - x_t y_s = 0.$$

Proof. Using the Theorem 1 and the value of Killing vector K_1 given in Eq.(6), we get the the proof.

Proposition 3. *The Killing K_1-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$ are parameterized by*

1. $X(s,t) = (\varphi_1(s), \varphi_2(s), z(s,t))$,
2. $X(s,t) = (\varphi_1(t), \varphi_2(t), z(s,t))$,
3. $X(s,t) = (x(s,t), \varphi(\psi_1(s,t)), z(s,t))$,
4. $X(s,t) = (\varphi(\psi_2(s,t)), y(s,t), z(s,t))$,

where x, z y and φ, $\varphi_{1,2}$ are arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively, and $\psi_{1,2}$ are solution functions given in Proposition 2(1-ii).

Proof. From the Proposition 2(1-i) (resp. (1-ii)), the parameterizations $X(s,t)$ is a general solution of the first order linear PDE given in the Theorem 2 for arbitrary functions x, z and y, z for the assertions 1 and 2 (resp. for the assertions 3 and 4), respectively.

Example 1. We consider in this example the fourth assertion of Proposition 3. Let $y(s,t) = st$, from the assertion 1 of Proposition 2, we have

$$\frac{ds}{x_t} = -\frac{dt}{x_s}$$

its solution is

$$\psi_2(s,t) = st = c \text{ constant},$$

then the surface M_1 parameterized by

$$X(s,t) = (\varphi(st), st, z(s,t))$$

is Killing K_1-flux surface in $\mathbb{S}L(2,\mathbb{R})$, where φ and z are arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. We present in Figure 1, the Killing K_1-flux surface M_1 in $\mathbb{S}L(2,\mathbb{R})$ parameterized by $X(s,t) = \left((st)^2, st, \sin(s+t)\right)$ where $(s,t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4.1.1 Killing magnetic K_1-scalar flux functions in $\mathbb{S}L(2,\mathbb{R})$

Theorem 3. *Let M_1 be a Killing magnetic K_1-flux surface in $\mathbb{S}L(2,\mathbb{R})$. Then the function f given by*

$$f(x,y,z) = \overline{\varphi}(2yx - z)$$

and f constant on M_1, is Killing magnetic K_1-scalar flux function to M, where $\overline{\varphi}$ is a real arbitrary function.

Proof. Using the Proposition 1 and the value of K_1 given in the Eq.(6), we have

$$g(K_1, \nabla f) = \frac{f_x}{2y} + f_z = 0.$$

By solving the above first order PDE and using the Proposition 2(1-ii), we get

$$f(x,y,z) = \overline{\varphi}(2yx - z)$$

where $\overline{\varphi}$ is a real arbitrary function. Then the function f is Killing magnetic K_1-scalar flux function and f must be also constant on M.

Example 2. Using the Example 1, we have Killing K_1-magnetic flux surface M_{1-1} parameterized by $X(s,t) = (\varphi(st), st, z(s,t))$ where $z(s,t) = 2st\cos st - a \mid a \in \mathbb{R}$ and $\varphi(u) = \cos u \mid u \in \mathbb{R}$. The Killing K_1-magnetic scalar flux function f to M_{1-1}, from the Theorem 3, is in the form $f(x,y,z) = \overline{\varphi}(2xy - z)$ and it must be constant on M_{1-1}, (i.e. $f(X(s,t)) \equiv C$ a constant). We have

$$f(x,y,z)_{/M_1} = \overline{\varphi}(2xy - z) = \overline{\varphi}(a) \text{ a constant},$$

then f is Killing K_1-magnetic scalar flux function to the Killing K_1-magnetic flux surface M_{1-1} parameterized by $X(s,t) = (\cos st, st, 2st\cos st - a)$.

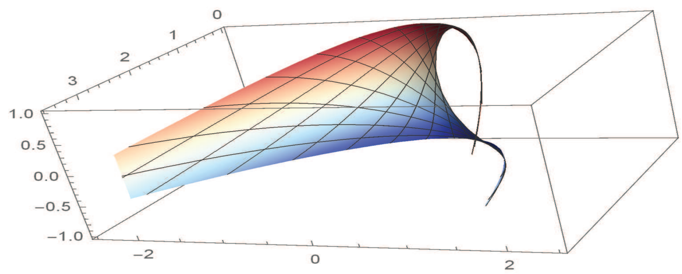

Fig. 1: Killing K_1-flux surface M_1

4.2 Killing K_3-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$

Theorem 4. *Let M_3 be a surface in $\mathbb{S}L(2,\mathbb{R})$ parametrized by $X(s,t) = (x(s,t), y(s,t), z(s,t))$. M_3 is a Killing K_3-flux surface if and only if*

$$(x_t z_s - x_s z_t) + \frac{x}{y}(y_s z_t - y_t z_s) = 0. \tag{9}$$

Proposition 4. *The Killing K_3-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$ are parameterized by*

$$\begin{cases} 1.\ X(s,t) = \left(x(s,t), e^{\varphi_1(s)}x(s,t), \varphi_2(s)\right) \\ 2.\ X(s,t) = \left(x(s,t), e^{\varphi_1(t)}x(s,t), \varphi_2(t)\right) \\ 3.\ X(s,t) = \left(e^{\varphi_1(s)}x(s,t), x(s,t), \varphi_2(s)\right) \\ 4.\ X(s,t) = \left(e^{\varphi_1(t)}x(s,t), x(s,t), \varphi_2(t)\right) \end{cases}, \quad \begin{matrix} 5.\ X(s,t) = \left(x(s,t), e^{\varphi_1(s)}, \varphi_2(s)\right) \\ 6.\ X(s,t) = \left(x(s,t), e^{\varphi_1(t)}, \varphi_2(t)\right) \\ 7.\ X(s,t) = \left(e^{\varphi_1(s)}, x(s,t), \varphi_2(s)\right) \\ 8.\ X(s,t) = \left(e^{\varphi_1(t)}, x(s,t), \varphi_2(t)\right) \end{matrix}$$

and

$$\begin{cases} 9.\ X(s,t) = (x(s,t), y(s,t), \varphi(\psi(s,t))) \\ 10.\ X(s,t) = (x(s,t), y(s,t), \varphi(\overline{\psi}(s,t))) \\ 11.\ X(s,t) = (x(s,t), \exp(\mathscr{Y}(s,t)), z(s,t)) \\ 12.\ X(s,t) = (\exp(\mathscr{X}(s,t)), y(s,t), z(s,t)) \end{cases},$$

where x, z y and φ, $\varphi_{1,2}$ are arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively, $\psi, \overline{\psi}$ are a functions given in cases (1-ii and 2-ii) and $\mathscr{X}, \mathscr{Y}$ a solutions given in Eq.(11).

Proof. The Killing K_3-flux surface M_3 in $\mathbb{S}L(2,\mathbb{R})$ is a solution of PDE given the Eq.(9) with three unknown functions x, y and z.

1. Let x and y be a given functions, then the PDE given the Eq.(9) is homogenous PDE

$$P(s,t)z_s + Q(s,t)z_t = 0$$

where

$$\begin{cases} P(s,t) = x_t - \frac{x}{y}y_t \\ Q(s,t) = \frac{x}{y}y_s - x_s \end{cases}$$

Using the Proposition 2(1), we have:

i. If $P \equiv 0$ (resp. $Q \equiv 0$), after an integration, we obtain $y = e^{\varphi_1(s)}x$ and $z(s,t) = \varphi_2(s)$ (resp. $y = e^{\varphi_1(t)}x$ and $z(s,t) = \varphi_2(t)$. Then the parametrization of M_3 is

$$X(s,t) = \left(x(s,t), e^{\varphi_1(s)}x(s,t), \varphi_2(s)\right), \left(\text{resp. } X(s,t) = \left(x(s,t), e^{\varphi_1(t)}x(s,t), \varphi_2(t)\right)\right).$$

ii. If $P,Q \neq 0$, we have $z(s,t) = \varphi_3(\overline{\psi}(s,t))$ and M_3 is parameterized by

$$X(s,t) = (x(s,t), y(s,t), \varphi_3(\overline{\psi}(s,t))),$$

where $\varphi_{1,2,3}$ are arbitrary functions in $\mathbb{R}$.
2. Let x and z be a given functions, then the PDE given the Eq.(9) turns to nonhomogeneous PDE.

$$P(s,t)\mathscr{Y}_s + Q(s,t)\mathscr{Y}_t = R(s,t), \ (R \neq 0), \tag{10}$$

where

$$P(s,t) = z_t; \ \ Q(s,t) = -z_s; \ \ R(s,t) = \frac{1}{x}(x_s z_t - x_t z_s) \text{ and}$$
$$\mathscr{Y} = \ln|y|. \tag{11}$$

a. If $R \equiv 0$ then the solution of the Eq.(10), using the Proposition 2(1), is
i. If $P \equiv 0$ (resp. $Q \equiv 0$), we obtain $\mathscr{Y} = \varphi_1(s)$ and $z(s,t) = \varphi_2(s)$ (resp. $\mathscr{Y} = \varphi_1(t)$ and $z(s,t) = \varphi_2(t)$). Then the parametrization of M_3 is

$$X(s,t) = \left(x(s,t), e^{\varphi_1(s)}, \varphi_2(s)\right) \left(\text{resp. } X(s,t) = \left(x(s,t), e^{\varphi_1(t)}, \varphi_2(t)\right)\right).$$

ii. If P and Q are non null functions, then solution of the Eq.(10) is

$$X(s,t) = (x(s,t), \exp(\varphi(\psi(s,t))), z(s,t)),$$

where $\psi(s,t) = c$, a constant, is a solution of ode

$$\frac{ds}{P} = \frac{dt}{Q}.$$

b. Using the Proposition 2(2), we have:
i. If $P \equiv 0$ (resp. $Q \equiv 0$), we obtain $\mathscr{Y} = \int \frac{R}{Q} dt = \ln|x| + \varphi_1(s)$ and $z(s,t) = \varphi_2(s)$ (resp. $\mathscr{Y} = \int \frac{R}{P} ds = \ln|x| + \varphi_1(t)$ and $z(s,t) = \varphi_2(t)$). Then the parametrization of M_3 is

$$X(s,t) = \left(x(s,t), e^{\varphi_1(s)} x(s,t), \varphi_2(s)\right) \left(\text{resp. } X(s,t) = \left(x(s,t), e^{\varphi_1(t)} x(s,t), \varphi_2(t)\right)\right).$$

ii. f $P,Q \neq 0$, using Proposition 2(2-ii), then $\mathscr{Y}$ is given implicitly from $\overline{\varphi}(\overline{\psi}_1(s,t,\mathscr{Y})) = \overline{\psi}_2(s,t,\mathscr{Y})$.
Then the parametrization of M_3 is

$$X(s,t) = (x(s,t), \exp(\mathscr{Y}(s,t)), z(s,t))$$

Similarly as in (2) for arbitrary functions y and z, we have

$$\begin{cases} X(s,t) = \left(e^{\varphi_1(s)} y(s,t), y(s,t), \varphi_2(s)\right) \\ X(s,t) = \left(e^{\varphi_1(t)} y(s,t), y(s,t), \varphi_2(t)\right) \\ X(s,t) = (\exp(\mathscr{X}(t)), y(s,t), z(s,t)) \end{cases}$$

*Example 3.***1.** We consider in following the assertion 2 of the Proposition 4. Let $x(s,t) = (a + b\cos s)\cos t$ and $y(s,t) = (a + b\cos s)\sin t$, from assertion 1 of the Proposition proof 4, we have to solve homogenous PDE

$$P(s,t)z_s + Q(s,t)z_t = 0,$$

where

$$\begin{cases} P(s,t) = -(a + b\cos s)\sin t - \frac{\cos t}{\sin t}(a + b\cos s)\cos s \\ Q(s,t) = 0. \end{cases},$$

Then we get $z(s,t) = \varphi(t)$ and $y = e^{\varphi_1(t)} x$ where $\varphi_1(t) = \log\tan t$. Hence, the surface $M_{3,1}$ parameterized by

$$X(s,t) = ((a + b\cos s)\cos t, (a + b\cos s)\sin t, \varphi_2(t)),$$

is Killing K_3-flux surface in $\mathbb{S}L(2,\mathbb{R})$, where φ_2 is arbitrary smooth functions in $\mathbb{R}$. (See Figures 2 and 3)

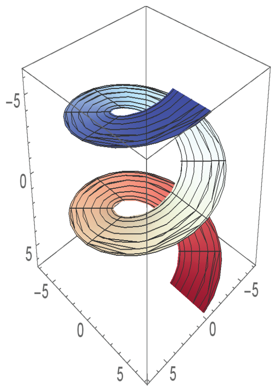

Fig. 2: Killing K_3-flux surface $M_{3,1}$ for $\varphi(t)=t$

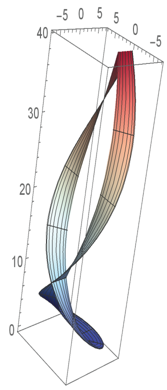

Fig. 3: Killing K_3-flux surface $M_{3,1}$ for $\varphi(t)=t^2$

2. Now, we consider assertion 10 of the Proposition 4. Let $x(s,t)=\sin s\sin t$ and $y(s,t)=\sin s\cos t$, from assertion 1 of the Proposition proof 4, we have to solve homogenous PDE

$$P(s,t)z_s+Q(s,t)z_t=0,$$

where

$$\begin{cases} P(s,t)=\cos s\cos t+\frac{\sin t}{\cos t}\cos s\sin t \\ Q(s,t)=\frac{\cos s}{\sin s}\cos s\sin t+\sin s\sin t \end{cases},$$

using the Proposition 2, then we must solve the ODE

$$\frac{ds}{\cos s\cos t+\frac{\sin t}{\cos t}\cos s\sin t}=\frac{dt}{\frac{\cos s}{\sin s}\cos s\sin t+\sin s\sin t}$$

which give a solution

$$\psi(s,t)=\frac{\tan s}{\tan t}=c \text{ a constant}$$

and its general solution is

$$z(s,t)=\varphi\left(\frac{\tan s}{\tan t}\right),$$

where φ is arbitrary real smooth function. Hence, the surface $M_{3,2}$ parameterized by

$$X(s,t)=\left(\sin s\sin t,\sin s\cos t,\varphi\left(\frac{\tan s}{\tan t}\right)\right)$$

is Killing K_3-flux surface in $\mathbb{S}L(2,\mathbb{R})$. (See Figure 4)

3. We present an example following 11^{th} assertion of the Proposition 4. Let $x(s,t)=s\cos t$ and $z(s,t)=s\sin t$, using Proposition 2(2-ii), we have

$$P(s,t)=z_t;\quad Q(s,t)=-z_s;\quad R(s,t)=\frac{1}{x}\left(x_s z_t-x_t z_s\right) \text{ and }$$
$$\mathscr{Y}=\ln|y|.$$

$$P=s\cos t,\ Q=-\sin t \text{ and } R=\frac{1}{s\cos t}\left(s\cos^2 t+s\sin^2 t\right)=\frac{1}{\cos t}.$$

and the EDOs

$$\frac{ds}{s\cos t}=\frac{dt}{\sin t}=\cos t\ d\mathscr{Y}.$$

We chose two ODEs to solve it

$$\begin{cases} \frac{ds}{s\cos t}=\frac{dt}{\sin t} \\ \frac{ds}{s\cos t}=\cos t d\mathscr{Y} \end{cases},$$

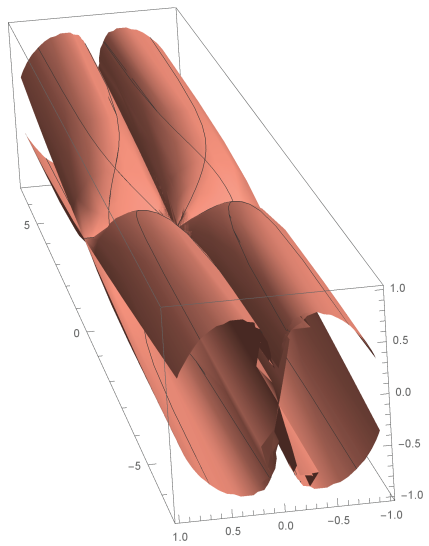

Fig. 4: Killing K_3-flux surface $M_{3,2}$ where $\varphi = Id_{\mathbb{R}}$

we get

$$\overline{\Psi}_1(s,t) = \frac{s}{\sin t} = c_1 \text{ and } \overline{\Psi}_1(s,t) = \mathscr{Y} - \ln s = c_2;\ c_{1,2} \in \mathbb{R}$$

and

$$\mathscr{Y}(s,t) = \varphi\left(\frac{s}{\sin t}\right) + \ln s,$$

which give

$$y = s\exp\left(\varphi\left(\frac{s}{\sin t}\right)\right),$$

where φ is arbitrary real function. Then the surface M_3 parameterized by

$$M_3 = \left(s\cos t, s\exp\left(\varphi\left(\frac{s}{\sin t}\right)\right), s\sin t\right)$$

is Killing K_3-flux surface in $\mathbb{S}L(2,\mathbb{R})$. See the Figure 5 for $\varphi(u) = \ln(u)$.

*Remark.*According to the Theorem 3 (page 18) given in [8], the Killing magnetic curve γ corresponding to K_3

$$\gamma(t) = \left(e^{-\frac{1}{4}t}, -\frac{1}{4}e^{-\frac{1}{4}t}, t\right),$$

lie to the K_3-flux surface $M_{3.4}$, using assertion 8 of the Proposition 2, parameterized by

$$M_{3.4} = \left(e^{-\frac{1}{4}t}, -\frac{\cos s}{4}e^{-\frac{1}{4}t}, t\right).$$

We present K_3-flux surface $M_{3.4}$ in Fig.6 with magnetic curve γ, in Euclidean space, where $(s,t) \in [-2,2] \times [-6,6]$.

4.2.1 Killing magnetic K_3-scalar flux functions in $\mathbb{S}L(2,\mathbb{R})$

Theorem 5.*Let M_3 be a Killing magnetic K_3-flux surface in $\mathbb{S}L(2,\mathbb{R})$. Then the function f given by*

$$f(x,y,z) = \varphi(xy, xy - z)$$

and constant on M_3 is Killing magnetic K_3-scalar flux function to M_3.

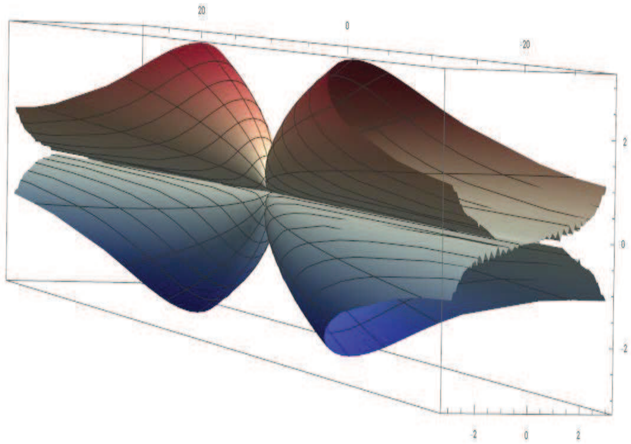

Fig. 5: Killing K_3-flux surface $M_{3,3}$

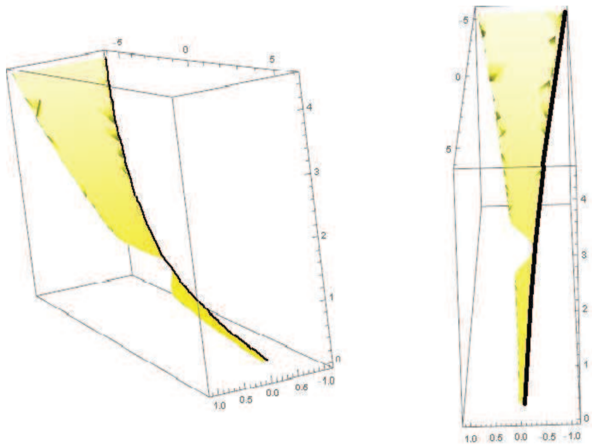

Fig. 6: K_3-Flux surface $M_{3,4}$ with magnetic curve γ (in black color)

*Proof.*Using the Proposition 1 and the value of K_3 given in the Eq.(6), we have

$$g(K_1, \nabla f) = \frac{1}{4y} f_y + \frac{1}{2}\frac{x}{y^2} f_x + \frac{1}{2}\frac{x}{y} f_z = 0.$$

By solving the above first order PDE (here we have a function f of three variable x,y,z), we get

$$f(x,y,z) = \varphi(xy, xy - z),$$

where φ is arbitrary function in $\mathbb{R}^2$, then the Killing magnetic K_3-scalar flux function f must be also constant on M_3.

*Example 4.*Using the Killing magnetic K_3-flux surface M_3

$$X(s,t) = (x(s,t), y(s,t), \varphi(\psi(s,t))),$$

given in the assertion 9 of the Proposition 4, we have

$$f(x,y,z)_{/M_3} = \overline{\varphi}(xy, xy - z)_{/M_3}.$$

By taking $\overline{\varphi}(u,v)=\overline{\overline{\varphi}}(u)$ and $x(s,t)=a/y(s,t)$ where a is a constant, then the function

$$f(x,y,z)=\overline{\overline{\varphi}}(xy) \text{ and } f_{/M_3}=a\in\mathbb{R},$$

is Killing magnetic K_3-scalar flux function to M_3.

4.3 Killing K_4-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$

Theorem 6. *Let M_4 be a surface in $\mathbb{S}L(2,\mathbb{R})$ parameterized by $X(s,t)=(x(s,t),y(s,t),z(s,t))$. M_4 is a Killing K_4-flux surface if and only if*

$$\frac{x^2-y^2}{2y}(y_sz_t-y_tz_s)+x(x_tz_s-x_sz_t)=0. \tag{12}$$

Proposition 5. *The Killing K_4-flux surfaces in $\mathbb{S}L(2,\mathbb{R})$ are parameterized by*

$$X(s,t)=(x(s,t),y(s,t),\varphi(\psi(s,t))),$$

where φ is arbitrary real function and ψ a solution given in Proposition 2(1-ii).

Proof. The PDE given the Eq.(12), give one possible homogenous PDE with respect to z that we can solve it, for a given the functions x and y

$$\underbrace{\left(xx_t-\frac{x^2-y^2}{2y}y_t\right)}_{P}z_s+\underbrace{\left(\frac{x^2-y^2}{2y}y_s-xx_s\right)}_{Q}z_t=0.$$

Using the Proposition 2(1), we get the solution

$$X(s,t)=(x(s,t),y(s,t),\varphi(\psi(s,t))),$$

where φ is arbitrary real function.

Example 5.1. Let $x(s,t)=\cosh s$ and $y(s,t)=\sinh t$, using the Proposition 2(1), we have

$$P=-\frac{\cosh^2 s-\sinh^2 t}{2\sinh t}\cosh t \text{ and } Q=-\cosh s\sinh s$$

and the solution of the ordinary differential equation (ODE)

$$\frac{ds}{P}=\frac{dt}{Q},$$

is

$$\psi_1(s,t)=\frac{\cosh^2 t}{4}-\frac{1}{2}\cosh^2 s(\ln(\sinh t)+1)=c \text{ constant.}$$

Then the surface $\mathbb{M}_{4,1}$ parameterized by

$$X(s,t)=\left(\cosh s,\sinh t,\varphi(\frac{\cosh^2 t}{4}-\frac{1}{2}\cosh^2 s(\ln(\sinh t)+1))\right)$$

is Killing K_4-flux surface in $\mathbb{S}L(2,\mathbb{R})$, where φ is arbitrary real smooth function.
2. Let $x(s,t)=s+2t$ and $y(s,t)=\sqrt{\frac{3}{2}}(s+2t)$, we have

$$P=\frac{5}{2}(s+2t) \text{ and } Q=-\frac{5}{2}(s+2t)$$

and the solution of the ODE

$$\frac{ds}{P}=\frac{dt}{Q}$$

is

$$\psi_1(s,t) = s+t = c \text{ constant}.$$

Then the surface $\mathbb{M}_{4,2}$ parameterized by

$$X(s,t) = \left(s+2t, \sqrt{\tfrac{3}{2}}(s+2t), \varphi(s+t)\right),$$

is Killing K_4-flux surface in $\mathbb{S}L(2,\mathbb{R})$, where φ is arbitrary real smooth function. We present, in $(\mathbb{R}^3, g_{euc})$, the Killing K_4-flux surface $M_{4,1}$ and $M_{4,2}$ for φ is cosine and identity functions in Figures 7 and 8, respectively.

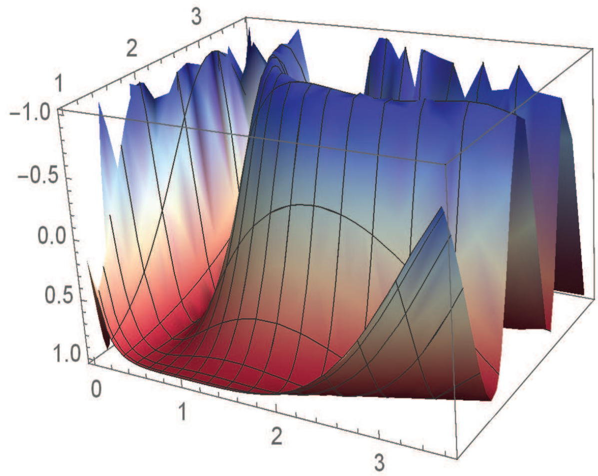

Fig. 7: Killing K_4-flux surface $M_{4,1}$

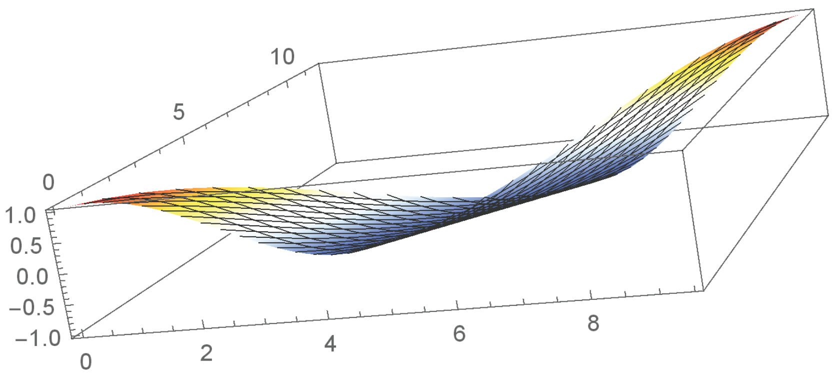

Fig. 8: Killing K_4-flux surface $M_{4,2}$

4.3.1 Killing magnetic K_4-scalar flux functions in $\mathbb{S}L(2,\mathbb{R})$

Theorem 7. *Let M_4 be a Killing magnetic K_4-flux surface in $\mathbb{S}L(2,\mathbb{R})$. Then the function f given by*

$$f(x,y,z) = \overline{\varphi}\left(3x^2y - 2y^3, xy - z\right),$$

and constant on M_4 is Killing magnetic K_4-scalar flux function to M_4, where $\overline{\varphi}$ is arbitrary function in $\mathbb{R}^2$.

Proof. Using the Proposition 1 and the value of K_4 given in the Eq.(6), we have

$$g(K_1, \nabla f) = \frac{f_x}{2y}\frac{x^2-y^2}{4y} + \frac{f_y}{2y}\frac{x}{2} + \left(\frac{f_x}{2y} + f_z\right)\frac{x^2-y^2}{4y} = 0,$$

which give

$$\left(x^2-y^2\right)f_x + yxf_y + y\left(x^2-y^2\right)f_z = 0.$$

By solving the above first order PDE of three variable, we get

$$f(x,y,z) = \overline{\varphi}\left(3x^2y - 2y^3, xy - z\right),$$

where $\overline{\varphi}$ is arbitrary function in $\mathbb{R}^2$, when f is also constant on M_4 then f is Killing magnetic K_4-scalar flux function to M_4.

Example 6. Using the Example 5(2), we have

$$f(x,y,z)_{/M_{4,2}} = \overline{\varphi}\left(3x^2y - 2y^3, xy - z\right)_{/M_{4,2}} = \overline{\varphi}\left(0, \sqrt{\tfrac{3}{2}}(s+2t)^2 - \varphi(s+t)\right).$$

By taking $\overline{\varphi}(u,v) = \overline{\overline{\varphi}}(u)$, the function

$$f(x,y,z) = \overline{\overline{\varphi}}\left(3x^2y - 2y^3\right)$$

is Killing magnetic K_4-scalar flux function to $M_{4,2}$, where $\overline{\overline{\varphi}}$ is arbitrary real function.

Declarations

Competing interests: The authors declare that they have no competing interests.
Authors' contributions: All authors contributed equally to the writing of this paper and approved the final manuscript.
Funding: There is no funding for this work.
Availability of data and materials: Not applicable.
Acknowledgments: The authors thank the referees for valuable comments and suggestions which improved the presentation of this paper.

References

[1] M. Belkhalfa, F. Dillen and J. Inoguchi, Parallel surfaces in the real special linear group $\mathbb{S}L(2,\mathbb{R})$, *Bull. Austral. Math. Soc.*, **65**(2002), 183.
[2] N. Benmensour and F. Hathout, Flux surfaces according to Killing magnetic vectors in Riemannian space $\mathbb{S}ol3$, *Fundamental J. of Maths and App.*, **6(2)**(2023), 89-100.
[3] A. H. Boozer, Physics of magnetically confined plasmas, *Rev. Mod. Phys.*, **76**(2004) 1071-1141.
[4] M. Barros, A. Romero: Magnetic vortices, *EPL*, **77**(2007), 34002.
[5] R. B. Bird, W. E. Stewart, E. N. Lightfoot, Transport Phenomena, *Wiley.* ISBN 0-471-07392-X, (1960).
[6] K. Derkaoui, F. Hathout, Explicit formulas for Killing magnetic curves in three-dimensional Heisenberg group, *Int. J. of Geo. Meth. in Modern Phy.*, **18(09)**(2021), 2150135. doi/abs/10.1142/S0219887821501358
[7] M. H. Dida and F. Hathout, Killing magnetic flux surfaces in the Heisenberg three group, *Facta Universitatis (NIS) Ser. Math. Inform.*, **37(5)**(2022), 975-991. doi.org/10.22190/FUMI220226067D
[8] Z. Erjavec, On Killing magnetic curves in $\mathbb{S}L(2,\mathbb{R})$ geometry. *Reports on Math. Phys.*, **84**(2019), 333-350.
[9] R. D. Hazeltine, J. D. Meiss, Plasma Confinement, *Dover publications, inc. Mineola, New York,* (2003).

[10] T. Körpinar, R. C. Demirkol and Z. Körpinar, Approximate solutions for the inextensible Heisenberg antiferromagnetic flow and solitonic magnetic flux surfaces in the normal direction in Minkowski space, *Optik j.*, **238**(2021),166403. doi.org/10.1016/j.ijleo.2021.166403.
[11] T. Körpinar et al., New analytical solutions for the inextensible Heisenberg ferromagnetic flow and solitonic magnetic flux surfaces in the binormal direction, *J. Phys. Scr.*, **96**(2021), 085219. doi 10.1088/1402-4896/ac0274
[12] Talat Körpinar et al., Magnetic flux surfaces by the fractional Heisenberg antiferromagnetic flow of magnetic b-lines in binormal direction in Minkowski space, *J. of Magnetism and Magnetic Materials*, **549**(2022),168952. doi.org/10.1016/j.jmmm.2021.168952
[13] E. Molnàr, The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beiträge Algebra Geom.*, **38**(1997), 261.
[14] E. Molnàr, J. Szirmai and A. Vesnin: The optimal packings by translation balls in $\widetilde{\mathbb{S}L}(2,\mathbb{R})$, *J. of Geometry*, **105**(2014), 287.
[15] Z. Ozdemir , I. Gok, Y. Yayli, F.N. Ekmekci, Killing magnetic flux surfaces in Euclidean 3-space, *Honam Math. J.*, **41(2)**(2019), 329-342. doi.org/10.5831/HMJ.2019.41.2.329
[16] Z. Ozdemir, I. Gok and F. N. Ekmekci, A geometric view of magnetic surfaces and magnetic curves, *Int. ele. J. of Geo.*, **12(1)**(2019), 126-134.
[17] T.S. Pedersen, A. H. Boozer, Confinement of nonneutral plasmas on magnetic surfaces. *Phys. Rev. Lett.*, **88**(2002), 205002.
[18] P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* **15**(1983), 401.
[19] Walter. A. Strauss, Partial differential equations: An introduction, ISBN 0-471-57364-7 (Wiley). *The Mathematical Gazette Lord, N.*, **77(479)**(1993), 286-287. doi:10.2307/3619758by