JJMS **17**, No. 3, 377-389 (2024)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.3.5

Frames as Operator Orbits for Quaternionic Hilbert spaces

Ruchi Bhardwaj[1], *S. K. Sharma*[2,*] *and S. K. Kaushik*[3]

[1] Department of Mathematics, University of Delhi
[2] Department of Mathematics, Kirori Mal College, University of Delhi
[3] Department of Mathematics, Kirori Mal College, University of Delhi

Received: May 19, 2023 Accepted : Feb. 4, 2024

Abstract: In this paper, we study frames which can be expressed as operator orbits $\{\mathscr{T}^n(\phi)\}_{n\in\mathbb{Z}}$ under a single generator ϕ and an operator $\mathscr{T}$ on a right quaternionic Hilbert space $\mathfrak{H}$ and prove a necessary and sufficient condition under which the sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is expressible as orbit of some operator $\mathscr{T}$. Also, a necessary condition for a frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ to have an operator orbit representation $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ using a bounded operator $\mathscr{T}$ is given. Further, a characterization for the boundedness of the operator $\mathscr{T}$, given that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ forms a frame is obtained. Moreover, it is proved that a redundant frame with finite excess can never be expressed as an orbit of a bounded operator whereas for a Riesz sequence an operator orbit representation with a bounded operator is always possible. Furthermore, we discuss the stability of frames that can be expressed as an orbit of some operator and prove that it remains undisturbed under some perturbation conditions. Finally as an application, we approximate frames that cannot be expressed as operator orbit using the sub-orbit representation of hypercyclic operators.

Keywords: Frames; Operator orbit; Quaternions.

2010 Mathematics Subject Classification. 94A20; 42C15.

1 Introduction and Preliminaries

In recent years, many researchers have studied the concept of dynamical sampling with the help of frames. Aldroubi et al. [1,2,3] presented the notion of dynamical sampling that analyses the properties of the sequences expressible as orbit of some operator under single generator. Also, Christensen et al. [5,6] gave some crucial results concerning the boundedness of the operator and stability of operator representation of frames. One of the main concerns of dynamical sampling is to examine the frames expressible as $\{\mathscr{T}^n(\psi)\}_{n\in\mathbb{Z}}$ where $\mathscr{T}:\mathfrak{H}\to\mathfrak{H}$ is a bounded operator and $\psi\in\mathfrak{H}$ is a fixed element. But, this is quite restrictive and not very easy to achieve expression using a bounded operator. Various necessary and sufficient conditions required for these orbit representations have been already studied in Hilbert spaces [8,9]. Along with that, there comes some limitations as well that are also considered and rectified by many researchers in context of dynamical sampling [7]. But, can this very significant topic of dynamical sampling be extended to quaternionic Hilbert spaces? This question motivates us to prove some of the important results of Hilbert spaces concerning orbit representation of frames sequences, for quaternionic Hilbert spaces.

Moreover, the concept of frame sequences and their generalizations have been already introduced to quaternionic Hilbert spaces in [11,13]. The generalization of this concept of frame sequences as operator orbits, to quaternionic Hilbert spaces has become an interesting problem because of its significant applications in dynamical sampling concerning quaternionic Hilbert spaces. In this article, our main concern is to deal with the frame properties of the sequences $\{\mathscr{T}^n(\psi)\}_{n\in\mathbb{Z}}$, which are expressible as an orbit of some operator $\mathscr{T}$ in quaternionic Hilbert spaces. Once we get the desired representation with the help of some operator, the very next crucial part is to check for the conditions under which such an expression is feasible with some bounded operator $\mathscr{T}$. It gives rise to many questions, as if every frame could have this representation? If not, can we approximate these frames with orbit or sub-orbit of any operator?

* Corresponding author e-mail: sumitkumarsharma@gmail.com

Can this operator $\mathscr{T}$ be bounded? and many more. In this paper, the answers of all these natural arising questions in quaternionic Hilbert spaces are given, extending the important results of Hilbert spaces.

Throughout this paper, we consider the orbits generated by single elements, i.e., the orbit of $\mathscr{T}$ under some $\mathfrak{h}_0 \in \mathfrak{H}$ is given by the sequence of the form $\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$. Also, we denote by $\mathfrak{H}$, a right quaternionic Hilbert space, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{Z}$ the set of all integers, $\mathscr{I}$ a countable index set and $\mathfrak{Q}$ denotes the set of all real quaternions. The kernel and range of an operator $\mathscr{T}$ are denoted by $\ker(\mathscr{T})$ and $\operatorname{ran}(\mathscr{T})$, respectively. Also the terms 'right-subspace' and 'rspan' refers to the 'subspace' and 'span' respectively, with scalars on the right side of the vectors only.

The paper is structured as follows. In the remaining part of this section, we recall some basic definitions and results concerning quaternionic Hilbert spaces. In Section 2, we firstly give a classification of sequences $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ having the representation as an orbit of some operator i.e., $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$. A necessary condition for a frame to have a representation as an orbit of some bounded operator is given. Moreover, for a frame of the form $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$, a crucial characterization for the boundedness of the operator $\mathscr{T}$ is given with the help of the synthesis operator and some important consequences of this result are derived. Further, it is proved that if a redundant frame with finite excess is expressible as an operator orbit, the operator must be unbounded. Section 3 deals mainly with the stability of the frames expressible as an orbit of some operator under some perturbation conditions. Lastly in Section 4, we define approximation of a frame in $\mathfrak{H}$ and approximate any arbitrary frame in $\mathfrak{H}$ with a frame as a sub-orbit of some hypercyclic operator.

Quaternions are basically a four dimensional non-commutative extension of the set of complex numbers over $\mathbb{R}$. The quaternionic algebra $\mathfrak{Q}$ contain elements of the form

$$\mathfrak{q} = \mathfrak{q}_0 + i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3,\ \mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3 \in \mathbb{R}$$

where, $i^2 = j^2 = k^2 = ijk = -1$. For any $\mathfrak{q} = \mathfrak{q}_0 + i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3 \in \mathfrak{Q}$, $\mathfrak{q}_0$ is called the scalar (or real part) and $i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3$ is called the imaginary (or vector part) of $\mathfrak{q}$. Further, it can be expressed as $\mathfrak{q} = a + v$, where a is the scalar part and v is the imaginary part of $\mathfrak{q}$ and its conjugate $\bar{\mathfrak{q}}$ is given by

$$\bar{\mathfrak{q}} = \mathfrak{q}_0 - i\mathfrak{q}_1 - j\mathfrak{q}_2 - k\mathfrak{q}_3.$$

This directs a norm for $\mathfrak{q} \in \mathfrak{Q}$ defined by

$$|\mathfrak{q}| = \sqrt{\mathfrak{q}\bar{\mathfrak{q}}} = \sqrt{\mathfrak{q}_0^2 + \mathfrak{q}_1^2 + \mathfrak{q}_2^2 + \mathfrak{q}_3^2}.$$

The quaternion's exponential function for $\mathfrak{q} = a + v \in \mathfrak{Q}$ is given by

$$e^{\mathfrak{q}} = e^a\Big(\cos(|v|) + \frac{v}{|v|}\sin(|v|)\Big).$$

Definition 1([13]). *A right quaternionic vector space $\mathfrak{H}$ is said to be a right quaternionic pre-Hilbert space (or a right quaternionic inner product space) over $\mathfrak{Q}$ if it is endowed with the inner product $\langle .|.\rangle : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{Q}$ satisfying the following properties:*

(a) $\overline{\langle \mathfrak{h}_1|\mathfrak{h}_2\rangle} = \langle \mathfrak{h}_2|\mathfrak{h}_1\rangle,\ \mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{H}$.
(b) $\langle \mathfrak{h}|\mathfrak{h}\rangle > 0$ *if* $\mathfrak{h} \neq 0$.
(c) $\langle \mathfrak{h}|\mathfrak{h}_1 + \mathfrak{h}_2\rangle = \langle \mathfrak{h}|\mathfrak{h}_1\rangle + \langle \mathfrak{h}|\mathfrak{h}_2\rangle,\ \mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{H}$.
(d) $\langle \mathfrak{h}_1|\mathfrak{h}_2\mathfrak{q}\rangle = \langle \mathfrak{h}_1|\mathfrak{h}_2\rangle\mathfrak{q},\ \mathfrak{h}_1, \mathfrak{h}_2 \in \mathfrak{H}, \mathfrak{q} \in \mathfrak{Q}$.

A right quaternionic pre-Hilbert space is said to be a right quaternionic Hilbert space if it is complete with respect to the norm induced by the above defined inner product. For more information on quaternionic Hilbert spaces one may refer [10].

The right quaternionic vector space of all the square integrable quaternionic valued functions $L^2(\mathbb{R}, \mathfrak{Q})$ which is given by

$$L^2(\mathbb{R}, \mathfrak{Q}) = \Big\{\mathfrak{h} : \mathbb{R} \to \mathfrak{Q}\ \Big|\ \int_{\mathbb{R}} |\mathfrak{h}(x)|^2 dx < \infty\Big\}$$

forms a right quaternionic Hilbert space under the quaternionic valued inner product

$$\langle \mathfrak{h}|\mathfrak{g}\rangle = \int_{\mathbb{R}} \overline{\mathfrak{h}(x)}\mathfrak{g}(x)dx,\ \mathfrak{h}, \mathfrak{g} \in L^2(\mathbb{R}, \mathfrak{Q})$$

where, dx denotes the usual Lebesgue measure on $\mathbb{R}$ and the right scalar multiplication $\mathfrak{h}\mathfrak{q} : \mathbb{R} \to \mathfrak{Q}$ with $(\mathfrak{h}\mathfrak{q})(x) = \mathfrak{h}(x)\mathfrak{q}$, $\mathfrak{q} \in \mathfrak{Q}, x \in \mathbb{R}$.

For $b, w \in \mathbb{R}$, we define the quaternionic translation operator $\mathscr{T}_b$ and the quaternionic modulation operator E_w on $L^2(\mathbb{R}, \mathfrak{Q})$ as follows:

$$\mathscr{T}_b \mathfrak{h}(x) = \mathfrak{h}(x-b) \text{ and } E_w \mathfrak{h}(x) = \mathfrak{h}(x) e^{-2\pi jwx}, \ \mathfrak{h} \in L^2(\mathbb{R}, \mathfrak{Q}), \ x \in \mathbb{R},$$

where E_w also denotes the mapping $x \mapsto e^{-2\pi jwx}$ on $\mathbb{R}$.

Definition 2. *Let $\mathfrak{g} \in L^2(\mathbb{R}, \mathfrak{Q})$ be a non-zero function and $\alpha, \beta > 0$. Then, define the Gabor system generated by $\mathfrak{g}, \alpha, \beta$ as the set of time-frequency shifts given by*

$$\mathscr{G}(\mathfrak{g}, \alpha, \beta) = \{\mathfrak{g}_{m,n} := E_{\beta m} \mathscr{T}_{\alpha n} \mathfrak{g} \ : \ m, n \in \mathbb{Z}\},$$

where $\mathfrak{g}$ is called the window function for the system $\mathscr{G}(\mathfrak{g}, \alpha, \beta)$.

Also the space $\ell^2_{\mathbb{Z}}(\mathfrak{Q})$ which is defined by

$$\ell^2_{\mathbb{Z}}(\mathfrak{Q}) = \Big\{ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \subset \mathfrak{Q} \ \Big| \ \sum_{n\in\mathbb{Z}} |\mathfrak{q}_n|^2 < \infty \Big\},$$

forms a right quaternionic Hilbert space under the right multiplication by quaternions together with the inner product given by

$$\langle \{\mathfrak{p}_n\}_{n\in\mathbb{Z}} | \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \rangle = \sum_{n\in\mathbb{Z}} \overline{\mathfrak{p}_n} \mathfrak{q}_n, \ \{\mathfrak{p}_n\}_{n\in\mathbb{Z}}, \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q}).$$

We define the analogue of the quaternionic translation operator, the right-shift operator $\mathscr{R} : \ell^2_{\mathbb{Z}}(\mathfrak{Q}) \to \ell^2_{\mathbb{Z}}(\mathfrak{Q})$ by

$$\mathscr{R}(\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}) = \{\mathfrak{q}_{n-1}\}_{n\in\mathbb{Z}}, \ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q}).$$

Similarly, the left-shift operator on $\ell^2_{\mathbb{Z}}(\mathfrak{Q})$ can be defined.

Definition 3([13]). *A sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ is said to be a frame for $\mathfrak{H}$ if there exist two positive constants $0 < r_1 \leq r_2 < \infty$ such that*

$$r_1 \|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle \mathfrak{h}_n | \mathfrak{h} \rangle|^2 \leq r_2 \|\mathfrak{h}\|^2, \ \mathfrak{h} \in \mathfrak{H}.$$

The real constants r_1, r_2 are known as lower and upper frame bounds for the frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$, respectively. The sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is said to be a Bessel sequence for $\mathfrak{H}$ with Bessel bound r_2, if it satisfies the upper frame condition.

For a Bessel sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$, the operator $\mathfrak{T}_{\mathfrak{h}} : \ell^2_{\mathbb{Z}}(\mathfrak{Q}) \to \mathfrak{H}$ defined by

$$\mathfrak{T}_{\mathfrak{h}}(\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \mathfrak{h}_n \mathfrak{q}_n, \ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$$

is known as the *synthesis operator* and its adjoint $\mathfrak{T}^*_{\mathfrak{h}} : \mathfrak{H} \to \ell^2_{\mathbb{Z}}(\mathfrak{Q})$ given by

$$\mathfrak{T}^*_{\mathfrak{h}}(\mathfrak{h}) = \{\langle \mathfrak{h}_n | \mathfrak{h} \rangle\}_{n\in\mathbb{Z}}, \ \mathfrak{h} \in \mathfrak{H}$$

is known as the *analysis operator* for the frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$. Also, the kernel of the synthesis operator $\mathfrak{T}_{\mathfrak{h}}$ for a frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is given by

$$\ker(\mathfrak{T}_{\mathfrak{h}}) = \Big\{ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q}) \ \Big| \ \sum_{n\in\mathbb{Z}} \mathfrak{h}_n \mathfrak{q}_n = 0 \Big\}.$$

Composing $\mathfrak{T}_{\mathfrak{h}}$ and $\mathfrak{T}^*_{\mathfrak{h}}$, we obtain the *frame operator* $\mathfrak{S}_{\mathfrak{h}} = \mathfrak{T}_{\mathfrak{h}} \mathfrak{T}^*_{\mathfrak{h}} : \mathfrak{H} \to \mathfrak{H}$ such that

$$\mathfrak{S}_{\mathfrak{h}}(\mathfrak{h}) = \sum_{n\in\mathbb{Z}} \mathfrak{h}_n \langle \mathfrak{h}_n | \mathfrak{h} \rangle, \ \mathfrak{h} \in \mathfrak{H}.$$

The frame operator $\mathfrak{S}_{\mathfrak{h}}$ for the frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is a positive, self-adjoint and invertible operator on $\mathfrak{H}$.

Also, a sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ is said to be a *Riesz sequence*, if there exist two positive constants $r_1 \leq r_2$ such that for all finite sequences $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$,

$$r_1 \sum_{n\in\mathbb{Z}} |\mathfrak{q}_n|^2 \leq \Big\| \sum_{n\in\mathbb{Z}} \mathfrak{h}_n \mathfrak{q}_n \Big\|^2 \leq r_2 \sum_{n\in\mathbb{Z}} |\mathfrak{q}_n|^2.$$

A Riesz sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is said to be a *Riesz basis* for $\mathfrak{H}$ if $\overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \mathfrak{H}$. A *redundant frame* is a frame which properly contains a Riesz basis and the *excess* for a frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is the number of elements that can be eliminated from the sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ so that the remaining sequence forms a Riesz basis.

2 Properties of Frames as Operator Orbits

In this section, we consider frames $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ that are expressible as an orbit of some operator $\mathscr{T}$ on $\mathfrak{H}$ and investigate various frame properties and conditions under which such an expression is feasible with the help of some bounded operator. An important point to be mentioned here is that when we write $\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$, then one should not get confused for $n<0$. We take $\mathscr{T}$ as an invertible operator and hence $\mathscr{T}^{-n}=(\mathscr{T}^{-1})^n$ is well-defined for all $n\in\mathbb{N}$. Also, by right-linearly independent sequence we mean linear independence of the sequence with scalars on the right side. First of all, we give a characterization for the sequences $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ that are expressible as $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}=\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ for an operator $\mathscr{T}$(may or may not be bounded) on $\mathfrak{H}$.

Theorem 1.*Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ be a sequence such that rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is infinite-dimensional. Then, $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is right-linearly independent if and only if there exists a bijective right-linear operator $\mathscr{T}$: rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}\to$ rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ such that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}=\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$.*

*Proof.*Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ be a right-linearly independent sequence in $\mathfrak{H}$. Define $\mathscr{T}(\mathfrak{h}_n):=\mathfrak{h}_{n+1}$, $n\in\mathbb{Z}$ and by right-linearity, this $\mathscr{T}$ extends to a well-defined operator on rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$.

Conversely, let $\mathscr{T}$: rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}\to$ rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ be a bijective right-linear operator such that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}=\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$. On the contrary, assume that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is right-linearly dependent. This implies that there exist some $M,N\in\mathbb{Z}$ and a finite sequence $0\neq\{\mathfrak{q}_n\}_{n=M}^N\subset\mathfrak{Q}$, such that $\sum\limits_{n=M}^{N}\mathfrak{h}_n\mathfrak{q}_n=0$. Since the sequence $\{\mathfrak{q}_n\}_{n=M}^N\neq 0$, there must exists some $M\leq m\leq N$ such that $\mathfrak{q}_m\neq 0$. If $m\neq N$, we may re-index the sequences $\{\mathfrak{q}_n\}_{n=M}^N$ and $\{\mathfrak{h}_n\}_{n=M}^N$ to get $\mathfrak{q}_N\neq 0$. So, we may assume $\mathfrak{q}_N\neq 0$ which in turn implies that $\mathfrak{h}_N=\sum\limits_{n=M}^{N-1}\mathfrak{h}_n\mathfrak{p}_n$ for some $\{\mathfrak{p}_n\}_{n=M}^{N-1}\subset\mathfrak{Q}$. Now, let $U:=$ rspan$\{\mathfrak{h}_n\}_{n=M}^{N-1}$. Clearly $\mathfrak{h}_N\in U$. Also, one may observe that $\mathscr{T}(U)\subseteq U$. Indeed, for $\{\mathfrak{q}_n\}_{n=M}^{N-1}\subset\mathfrak{Q}$, we have

$$\mathscr{T}\Big(\sum_{n=M}^{N-1}\mathfrak{h}_n\mathfrak{q}_n\Big)=\sum_{n=M}^{N-2}\mathfrak{h}_{n+1}\mathfrak{q}_n+\mathfrak{h}_N\mathfrak{q}_{N-1}\in U.$$

Therefore, rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}=U$, which is a contradiction since rspan$\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is infinite-dimensional. Hence, $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is a right-linearly independent sequence.

In support of the above theorem, let us give a basic example of a frame that is expressible as an orbit of some operator on $\mathfrak{H}$.

*Example 1.*Let $\{\mathfrak{e}_n\}_{n\in\mathbb{Z}}$ be any orthonormal basis of $\mathfrak{H}$. Define $\mathscr{T}:\mathfrak{H}\to\mathfrak{H}$ such that $\mathscr{T}(\mathfrak{e}_n)=\mathfrak{e}_{n+1}, n\in\mathbb{Z}$. One may observe that $\mathscr{T}$ can extend right-linearly to a bounded operator on $\mathfrak{H}$ and $\{\mathfrak{e}_n\}_{n\in\mathbb{Z}}=\{\mathscr{T}^n(\mathfrak{e}_0)\}_{n\in\mathbb{Z}}$.

The above example has direct implications in Gabor analysis of $L^2(\mathbb{R},\mathfrak{Q})$. An orthonormal basis of $L^2(\mathbb{R},\mathfrak{Q})$ as the orbit of an operator, can be generated with the help of a given Gabor orthonormal basis $\{E_{\beta m}\mathscr{T}_{\alpha n}\mathfrak{g}\}_{m,n\in\mathbb{Z}}$ by reindexing it on $\mathbb{Z}$. In the following result, it is proved that for Riesz sequences the orbit representation can be provided with a bounded operator $\mathscr{T}$(does not hold for arbitrary sequences or frames).

Theorem 2.*For any right quaternionic Hilbert space $\mathfrak{H}$, the following statements hold:*

(a)For every Riesz sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$, there always exists a bounded bijective right-linear operator $\mathscr{T}$ on $\overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ such that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}=\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$.
(b)For a bounded operator $\mathscr{T}$ on $\mathfrak{H}$ such that $\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ forms a frame for $\mathfrak{H}$ for some $\mathfrak{h}_0\in\mathfrak{H}$, ran$(\mathscr{T})=\mathfrak{H}$.

Proof.(a) Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ be any Riesz sequence in $\mathfrak{H}$. Then, there exist some $A>0$ such that

$$A\sum_{n\in\mathbb{Z}}|\mathfrak{q}_n|^2\leq\Big\|\sum_{n\in\mathbb{Z}}\mathfrak{h}_n\mathfrak{q}_n\Big\|^2,$$

for all finite sequences $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{Z}}(\mathfrak{Q})$. Define $\mathscr{T}(\mathfrak{h}_n):=\mathfrak{h}_{n+1}$, $n\in\mathbb{Z}$. Now by Theorem 3.5 in [14], the operator $\mathscr{T}$ can extend right-linearly to a bounded operator $\mathscr{T}:\overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}\to\overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$. It is easy to observe that ran$(\mathscr{T})=\overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ and $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}=\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$.
(b) Let $\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$ for some $\mathfrak{h}_0\in\mathfrak{H}$. Then, the synthesis operator $\mathfrak{T}_{\mathfrak{h}}:\ell^2_{\mathbb{Z}}(\mathfrak{Q})\to\mathfrak{H}$ which is given by

$$\mathfrak{T}_{\mathfrak{h}}(\{\mathfrak{q}_n\}_{n\in\mathbb{Z}})=\sum_{n\in\mathbb{Z}}\mathscr{T}^n(\mathfrak{h}_0)\mathfrak{q}_n,\ \ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\in\ell^2_{\mathbb{Z}}(\mathfrak{Q})$$

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

is a surjective operator, indeed, $\operatorname{ran}(\mathfrak{T}_{\mathfrak{h}}) = \overline{\operatorname{rspan}}\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}} = \mathfrak{H}$. Also for $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$, we have

$$\sum_{n\in\mathbb{Z}} \mathscr{T}^n(\mathfrak{h}_0)\mathfrak{q}_n = \mathscr{T}\Big(\sum_{n\in\mathbb{Z}} \mathscr{T}^{n-1}(\mathfrak{h}_0)\mathfrak{q}_n\Big) \in \operatorname{ran}(\mathscr{T}).$$

This gives, $\operatorname{ran}(\mathscr{T}) = \mathfrak{H}$.

For a frame to be expressible as an orbit of some bounded operator is a more desired situation. So, in the following result a necessary condition for a frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ to have a representation of the form $\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ is obtained where, $\mathscr{T}$ is given to be a bounded operator.

Theorem 3._Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ be a frame for $\mathfrak{H}$, where $\mathscr{T} : rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is some bounded right-linear operator. Then, $\|\mathscr{T}\| \geq 1$. Moreover, if $\mathscr{T}^{-1}$ is also bounded, then $\|\mathscr{T}^{-1}\| \geq 1$._

*Proof.*Let r_1, $r_2 > 0$ be lower and upper frame bounds of the frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ respectively. From the frame inequality, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} r_1\|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle \mathscr{T}^n(\mathfrak{h}_0)|\mathfrak{h}\rangle|^2 &= \sum_{n\in\mathbb{Z}} |\langle \mathscr{T}^{n-m}(\mathfrak{h}_0)|(\mathscr{T}^m)^*(\mathfrak{h})\rangle|^2 \\ &= \sum_{n\in\mathbb{Z}} |\langle \mathscr{T}^{n}(\mathfrak{h}_0)|(\mathscr{T}^m)^*(\mathfrak{h})\rangle|^2 \\ &\leq r_2\|\mathscr{T}\|^{2m}\|\mathfrak{h}\|^2,\ \mathfrak{h} \in \mathfrak{H}. \end{aligned}$$

Thus $r_1 \leq r_2\|\mathscr{T}\|^{2m}$, $m \in \mathbb{N}$ and hence we get $\|\mathscr{T}\| \geq 1$. Moreover, if $\mathscr{T}^{-1}$ is also bounded then the result is obtained by replacing $\mathscr{T}$ with $\mathscr{T}^{-1}$.

*Remark.*The converse of the above theorem is not true, which can be visualized by taking the identity operator I on $\mathfrak{H}$. The system $\{I^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}} = \{\mathfrak{h}_0\}$ never forms a frame for $\mathfrak{H}$ with infinite-dimension.

In the next result, a characterization for the boundedness of the operator $\mathscr{T}$ is given, for the case when the frame $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ is expressible as $\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$. For this, we need to recall the definition of an invariant right-subspace. A right-subspace $\mathfrak{M} \subset \mathfrak{H}$ is said to be *invariant* under an operator $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ if $\mathscr{T}(\mathfrak{M}) \subseteq \mathfrak{M}$.

Theorem 4._Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ be a frame for $\mathfrak{H}$, for some right-linear operator $\mathscr{T} : rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ with lower and upper frame bounds r_1 and r_2 respectively and frame operator $\mathfrak{S}_{\mathfrak{h}}$. Then, the following statements are equivalent:_

(a)The operator $\mathscr{T}$ is bounded.
(b)$\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under the right-shift operator where $\mathfrak{T}_{\mathfrak{h}}$ is the synthesis operator for $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$.

*Proof.*Assume that for a bounded right-linear operator $\mathscr{T}$: $\operatorname{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to \operatorname{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$, the sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$. Then, $\mathscr{T}$ extends right-linearly to a bounded operator $\hat{\mathscr{T}} : \mathfrak{H} \to \mathfrak{H}$. For $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})$, we have

$$\begin{aligned} \mathfrak{T}_{\mathfrak{h}}(\mathscr{R}\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}) &= \sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_{n-1} = \sum_{n\in\mathbb{Z}} \mathfrak{h}_{n+1}\mathfrak{q}_n \\ &= \hat{\mathscr{T}}\Big(\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_n\Big) = 0. \end{aligned}$$

Thus, we conclude that $\mathscr{R}(\ker(\mathfrak{T}_{\mathfrak{h}})) \subseteq \ker(\mathfrak{T}_{\mathfrak{h}})$.

Conversely, let $\hat{\mathfrak{h}} \in \operatorname{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$. Then, $\hat{\mathfrak{h}} = \sum\limits_{n=M}^{N} \mathfrak{h}_n\hat{\mathfrak{q}_n}$ for some $M, N \in \mathbb{Z}$ and $\hat{\mathfrak{q}_n} \in \mathfrak{Q}$. We may consider the sequence $\{\hat{\mathfrak{q}_n}\}_{n=M}^{N}$ as an element $\{\hat{\mathfrak{q}_n}\}_{n\in\mathbb{Z}}$ of $\ell^2_{\mathbb{Z}}(\mathfrak{Q})$ by assigning zeros at $\hat{\mathfrak{q}_n}$ for $n > N$ and $n < M$. Also, we can split $\{\hat{\mathfrak{q}_n}\}_{n\in\mathbb{Z}}$ as $\{\hat{\mathfrak{q}_n}\}_{n\in\mathbb{Z}} = \{\mathfrak{p}_n\}_{n\in\mathbb{Z}} + \{\tilde{\mathfrak{p}_n}\}_{n\in\mathbb{Z}}$, for some $\{\mathfrak{p}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})$ and $\{\tilde{\mathfrak{p}_n}\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})^{\perp}$. Now using the fact that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$

form a frame for $\mathfrak{H}$ and $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under the right-shift operator, we have

$$\begin{aligned}
\|\mathscr{T}(\hat{\mathfrak{h}})\|^2 &= \Big\| \sum_{n=M}^{N} \mathfrak{h}_{n+1}\hat{\mathfrak{q}_n} \Big\|^2 \\
&= \Big\| \sum_{n\in\mathbb{Z}} \mathfrak{h}_{n+1}(\mathfrak{p}_n + \tilde{\mathfrak{p}_n}) \Big\|^2 \\
&= \Big\| \mathfrak{T}_{\mathfrak{h}}(\mathscr{R}\{\mathfrak{p}_n\}_{n\in\mathbb{Z}}) + \sum_{n\in\mathbb{Z}} \mathfrak{h}_{n+1}\tilde{\mathfrak{p}_n} \Big\|^2 \\
&= \Big\| \sum_{n\in\mathbb{Z}} \mathfrak{h}_{n+1}\tilde{\mathfrak{p}_n} \Big\|^2 \leq r_2 \sum_{n\in\mathbb{Z}} |\tilde{\mathfrak{p}_n}|^2. \qquad (1)
\end{aligned}$$

As the range of the synthesis operator $\operatorname{ran}(\mathfrak{T}_{\mathfrak{h}})$ is closed (being equal to $\mathfrak{H}$), Proposition 3.8 in [12] implies that the range of the analysis operator $\operatorname{ran}(\mathfrak{T}^*_{\mathfrak{h}})$ is closed as well. Therefore $\ker(\mathfrak{T}_{\mathfrak{h}})^{\perp} = \operatorname{ran}(\mathfrak{T}^*_{\mathfrak{h}})$. From the frame inequality, we have

$$\begin{aligned}
\Big(\sum_{n\in\mathbb{Z}} |\langle \mathfrak{h}_n|\mathfrak{h}\rangle|^2 \Big)^2 &= |\langle \mathfrak{S}_{\mathfrak{h}}(\mathfrak{h})|\mathfrak{h}\rangle|^2 \\
&\leq \|\mathfrak{S}_{\mathfrak{h}}(\mathfrak{h})\|^2 \|\mathfrak{h}\|^2 \\
&\leq \frac{1}{r_1}\|\mathfrak{S}_{\mathfrak{h}}(\mathfrak{h})\|^2 \sum_{n\in\mathbb{Z}} |\langle \mathfrak{h}_n|\mathfrak{h}\rangle|^2, \ \mathfrak{h}\in\mathfrak{H}.
\end{aligned}$$

This gives

$$r_1 \sum_{n\in\mathbb{Z}} |\langle \mathfrak{h}_n|\mathfrak{h}\rangle|^2 \leq \|\mathfrak{S}_{\mathfrak{h}}(\mathfrak{h})\|^2 = \|\mathfrak{T}_{\mathfrak{h}}\{\langle \mathfrak{h}_n|\mathfrak{h}\rangle\}_{n\in\mathbb{Z}}\|^2, \ \mathfrak{h}\in\mathfrak{H}.$$

As $\ker(\mathfrak{T}_{\mathfrak{h}})^{\perp} = \operatorname{ran}(\mathfrak{T}^*_{\mathfrak{h}})$, we get

$$r_1 \sum_{n\in\mathbb{Z}} |\mathfrak{q}_n|^2 \leq \|\mathfrak{T}_{\mathfrak{h}}\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}\|^2, \ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})^{\perp}. \qquad (2)$$

From (1) and (2), we obtain

$$\|\mathscr{T}(\hat{\mathfrak{h}})\|^2 \leq \frac{r_2}{r_1}\Big\| \sum_{n\in\mathbb{Z}} \mathfrak{h}_n\tilde{\mathfrak{p}_n} \Big\|^2 = \frac{r_2}{r_1}\Big\| \sum_{n\in\mathbb{Z}} \mathfrak{h}_n\hat{\mathfrak{q}_n} \Big\|^2 = \frac{r_2}{r_1}\|\hat{\mathfrak{h}}\|^2.$$

Thus, $\mathscr{T}$ is a bounded operator with $\|\mathscr{T}\| \leq \sqrt{\frac{r_2}{r_1}}$.

Corollary 1. *Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ be a frame for $\mathfrak{H}$, where $\mathscr{T}: rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is a right-linear operator. Then, the operator $\mathscr{T}^{-1}$ is bounded if and only if $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under the left-shift operator where $\mathfrak{T}_{\mathfrak{h}}$ is the synthesis operator for $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$.*

Proof. We may write $\{\mathfrak{h}_{-n}\}_{n\in\mathbb{Z}} = \{(\mathscr{T}^{-1})^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$. Let $\mathscr{U}$ be the synthesis operator for $\{\mathfrak{h}_{-n}\}_{n\in\mathbb{Z}}$. It is easy to observe that $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})$ if and only if $\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_n = 0$. Also we may write $\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_n = \sum_{n\in\mathbb{Z}} \mathfrak{h}_{-n}\mathfrak{q}_{-n} = \mathscr{U}(\{\mathfrak{q}_{-n}\}_{n\in\mathbb{Z}})$. Therefore, $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})$ if and only if $\{\mathfrak{q}_{-n}\}_{n\in\mathbb{Z}} \in \ker(\mathscr{U})$. This implies that $\ker(\mathscr{U})$ is right-shift invariant if and only if $\ker(\mathfrak{T}_{\mathfrak{h}})$ is left-shift invariant. Hence, from Theorem 4, $\mathscr{T}^{-1}$ is bounded if and only if $\ker(\mathfrak{T}_{\mathfrak{h}})$ is left-shift invariant.

We know from Theorem 1 that every right-linearly independent sequence $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ is expressible as $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$. Also, if $\mathscr{T}$ is a bounded operator, it extends uniquely to a bounded operator $\hat{\mathscr{T}}: \overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to \overline{\text{rspan}}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ such that

$$\hat{\mathscr{T}}\Big(\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_n \Big) = \sum_{n\in\mathbb{Z}} \mathfrak{h}_{n+1}\mathfrak{q}_n.$$

Now, given that $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$, for some operator $\mathscr{T}$ on $\text{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$, the following result gives a necessary and sufficient condition for the operator $\mathscr{T}$ to have a bounded bijective extension on $\mathfrak{H}$.

Lemma 1. *Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ be a frame for $\mathfrak{H}$, for some right-linear operator $\mathscr{T} : rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$. Then, $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under both the right and left-shift operators if and only if the operator $\mathscr{T}$ extends to a bounded right-linear bijective operator $\hat{\mathscr{T}} : \mathfrak{H} \to \mathfrak{H}$ where $\mathfrak{T}_{\mathfrak{h}}$ is the synthesis operator for $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$.*

Proof. Let us assume that $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under both the right and left-shift operators. Then from Theorem 4, the operators $\mathscr{T}$ and $\mathscr{T}^{-1}$ are bounded and $\mathscr{T}\mathscr{T}^{-1} = \mathscr{T}^{-1}\mathscr{T} = I$ on $\text{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$. Hence, $\mathscr{T}$ and $\mathscr{T}^{-1}$ can extend to bounded right-linear operators $\hat{\mathscr{T}}$ and $\hat{\mathscr{T}}^{-1}$ respectively on $\mathfrak{H}$ such that $\hat{\mathscr{T}}\hat{\mathscr{T}}^{-1} = \hat{\mathscr{T}}^{-1}\hat{\mathscr{T}} = I$ on $\mathfrak{H}$.

Conversely, let $\mathscr{T}$ extends to a bounded right-linear bijective operator $\hat{\mathscr{T}} : \mathfrak{H} \to \mathfrak{H}$. As $\mathscr{T}$ is invertible, $\mathscr{T}$ and $\mathscr{T}^{-1}$ as a restriction of $\hat{\mathscr{T}}$ and $\hat{\mathscr{T}}^{-1}$ are again bounded which further implies that $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under both the right and left-shift operators using Theorem 4 and Corollary 1.

We now obtain an important consequence of Theorem 4 which shows that if a redundant frame with finite excess is expressible as an orbit of some operator $\mathscr{T}$, then $\mathscr{T}$ must be unbounded.

Lemma 2. *Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ be a redundant frame for $\mathfrak{H}$, where $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ is a bounded right-linear operator. Then, $\ker(\mathfrak{T}_{\mathfrak{h}})$ is infinite-dimensional where $\mathfrak{T}_{\mathfrak{h}}$ is the synthesis operator for $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$.*

Proof. Let $\mathfrak{p} = \{\mathfrak{p}_n\}_{n\in\mathbb{Z}}$ be a non-zero element of $\ker(\mathfrak{T}_{\mathfrak{h}})$. As $\mathscr{T}$ is given to be a bounded operator, Theorem 4 implies that $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under the right-shift operator and hence $\mathscr{R}^n(\mathfrak{p}) \in \ker(\mathfrak{T}_{\mathfrak{h}})$, $n \in \mathbb{N}$. Now, we claim that the sequence $\{\mathscr{R}^n(\mathfrak{p})\}_{n\in\mathbb{N}}$ is a right-linearly independent sequence in $\ker(\mathfrak{T}_{\mathfrak{h}})$. Define an operator $\Phi : \ell^2_{\mathbb{Z}}(\mathfrak{Q}) \to L^2([0,1],\mathfrak{Q})$ such that

$$\Phi(\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} E_n\mathfrak{q}_n,\ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$$

where $E_n : [0,1] \to \mathfrak{Q}$ be such that $E_n(x) = e^{-2\pi jnx}, n \in \mathbb{Z}$. One may observe that Φ is a well-defined operator. Indeed, for $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$

$$\begin{aligned}\int_0^1 \Big|\sum_{n\in\mathbb{Z}} E_n\mathfrak{q}_n(x)\Big|^2 dx &= \int_0^1 \Big|\sum_{n\in\mathbb{Z}} e^{-2\pi jnx}\mathfrak{q}_n\Big|^2 dx \\ &\leq \int_0^1 \sum_{n\in\mathbb{Z}} |e^{-2\pi jnx}\mathfrak{q}_n|^2 dx \\ &= \int_0^1 \sum_{n\in\mathbb{Z}} |\mathfrak{q}_n|^2 dx < \infty.\end{aligned}$$

Also for $\mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$,

$$\begin{aligned}\Phi(\mathscr{R}(\mathfrak{q})) &= \sum_{n\in\mathbb{Z}} E_n\mathfrak{q}_{n-1} = \sum_{n\in\mathbb{Z}} E_{n+1}\mathfrak{q}_n = \sum_{n\in\mathbb{Z}} (E_1.E_n)\mathfrak{q}_n \\ &= E_1.\Big(\sum_{n\in\mathbb{Z}} E_n\mathfrak{q}_n\Big) = E_1.\Phi(\mathfrak{q}).\end{aligned}$$

Thus, $\Phi(\mathscr{R}(\mathfrak{q})) = E_1.\Phi(\mathfrak{q}),\ \mathfrak{q} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$. Let us assume that there exists some $M \in \mathbb{N}$ and $\{\mathfrak{r}_n\}_{n=1}^M \subset \mathfrak{Q}$ such that $\sum\limits_{n=1}^M \mathscr{R}^n(\mathfrak{p})\mathfrak{r}_n = 0$. Then

$$0 = \Phi\Big(\sum_{n=1}^M \mathscr{R}^n(\mathfrak{p})\mathfrak{r}_n\Big) = \sum_{n=1}^M \Big(\Phi(\mathscr{R}^n(\mathfrak{p}))\Big)\mathfrak{r}_n = \sum_{n=1}^M \Big(E_n.\Phi(\mathfrak{p})\Big)\mathfrak{r}_n.$$

This gives $\sum\limits_{n=1}^M \Big((E_n.\Phi(\mathfrak{p}))(x)\Big)\mathfrak{r}_n = 0$, for almost all $x \in [0,1]$ by the definition of $L^2([0,1],\mathfrak{Q})$ (as the elements of $L^2([0,1],\mathfrak{Q})$ are equivalence classes). Since $\mathfrak{p} \neq 0$, we have $\Phi(\mathfrak{p}) \neq 0$ and hence $\Phi(\mathfrak{p})$ has support of positive measure. Thus, we conclude that $\sum\limits_{n=1}^M E_n(x)\mathfrak{r}_n = 0$, for almost all $x \in [0,1] \cap$ supp $\Phi(\mathfrak{p})$, which further gives that $\mathfrak{r}_n = 0$ for $1 \leq n \leq M$. Therefore, $\{\mathscr{R}^n(\mathfrak{p})\}_{n\in\mathbb{N}}$ is a right-linearly independent sequence in $\ker(\mathfrak{T}_{\mathfrak{h}})$ and hence $\ker(\mathfrak{T}_{\mathfrak{h}})$ is infinite-dimensional.

We may observe that the excess of a frame in equal to the dimension of the kernel of its synthesis operator. Now Lemma 2 directs to a significant result that a redundant frame having finite excess is expressible as the orbit of some unbounded operator only (if the operator representation is possible).
Note that, the converse of Lemma 2 is not true in general; i.e., even if $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ forms a redundant frame for $\mathfrak{H}$ and $\ker(\mathfrak{T}_{\mathfrak{h}})$ is infinite-dimensional, the operator $\mathscr{T}$ may not be bounded. The following example demonstrates this statement.

*Example 2.*Let $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ denote an orthonormal basis for $\mathfrak{H}$. We now execute a repeated procedure to construct another sequence $\{\mathfrak{u}_\mathfrak{n}\}_{n\in\mathbb{N}}$ in $\mathfrak{H}$ as follows:
We divide $\mathbb{N}$ into two disjoint and infinite subsets A_1 and B_1 such that $\mathbb{N} = A_1 \cup B_1$. Let $\mathfrak{u}_1 := \sum_{k\in A_1} \frac{1}{\sqrt{2}^k}\mathfrak{e}_k$. Clearly, $\{\mathfrak{e}_n\}_{n\in A_1} \cup \{\mathfrak{u}_1\}$ is a right-linearly independent redundant frame sequence with excess 1. Now, starting with the orthonormal sequence $\{\mathfrak{e}_n\}_{n\in B_1}$, we continue this process. In the m^{th}- step, we define $B_{m-1} = A_m \cup B_m$ and define $\mathfrak{u}_m := \frac{1}{m}\sum_{k\in A_m} \frac{1}{\sqrt{2}^k}\mathfrak{e}_k$. Again, the construction of $\mathfrak{u}_m$ gives that $\{\mathfrak{e}_n\}_{n\in A_m} \cup \{\mathfrak{u}_m\}$ is a right-linearly independent redundant frame with excess 1. As $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ is a right-linearly independent sequence (being the orthonormal basis), the construction of the sequence $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ implies that $\{\mathfrak{h}_n\}_{n\in\mathbb{N}} := \{\mathfrak{e}_n\}_{n\in\mathbb{N}} \cup \{\mathfrak{u}_n\}_{n\in\mathbb{N}} = \cup_{n=1}^{\infty}(\{\mathfrak{e}_k\}_{k\in A_n} \cup \{\mathfrak{u}_n\})$ form a right-linearly independent frame for $\mathfrak{H}$ with infinite excess.
Reordering the frame $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$, we now define

$$\mathscr{T}(\mathfrak{h}_n) = \mathfrak{h}_{n+1},\ n \in \mathbb{N}$$

and extends right-linearly to $\mathrm{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$. For any $n > 1$, there exists some $k \in \mathbb{N}$, such that $\|\mathfrak{h}_k\| \leq \frac{1}{n}$ and $\|\mathfrak{h}_{k+1}\| = 1$. Therefore,

$$\left\|\mathscr{T}\left(\frac{\mathfrak{h}_k}{\|\mathfrak{h}_k\|}\right)\right\| = \frac{\|\mathfrak{h}_{k+1}\|}{\|\mathfrak{h}_k\|} \geq n.$$

Thus, we conclude that $\mathscr{T}$ is an unbounded operator.

3 Stability of Frames as Operator Orbits

For practical purposes, we need to check for the stability of the frames obtained as the orbits of operators, under some perturbation conditions. Here we give a perturbation condition that preserves or retains the existence of the expression of frames as orbit of some operator which is a generalization of a result by Christensen [5]. Firstly, a lemma for the invertibility of an operator $\mathscr{T}$ on $\mathfrak{H}$ is given, which will be used in the main result.

Lemma 3.*Let $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ be a right-linear operator and there exist some $\alpha, \beta \in [0,1)$ satisfying*

$$\|\mathscr{T}(\mathfrak{h}) - \mathfrak{h}\| \leq \alpha\|\mathfrak{h}\| + \beta\|\mathscr{T}(\mathfrak{h})\|,\ \mathfrak{h} \in \mathfrak{H}.$$

Then, $\mathscr{T}$ is a bounded invertible operator such that

$$\frac{1-\alpha}{1+\beta}\|\mathfrak{h}\| \leq \|\mathscr{T}(\mathfrak{h})\| \leq \frac{1+\alpha}{1-\beta}\|\mathfrak{h}\|, \quad \frac{1-\beta}{1+\alpha}\|\mathfrak{h}\| \leq \|\mathscr{T}^{-1}(\mathfrak{h})\| \leq \frac{1+\beta}{1-\alpha}\|\mathfrak{h}\|,\ \mathfrak{h} \in \mathfrak{H}.$$

*Proof.*Follows directly on the same lines as in complex Hilbert spaces [4].

Theorem 5.*Let $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ be a frame for $\mathfrak{H}$, for some right-linear operator $\mathscr{T} : rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to rspan\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ with lower and upper frame bounds r_1 and r_2 respectively. Now let $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ be any sequence in $\mathfrak{H}$ and suppose there exist two constants $\alpha, \beta \in [0,1)$ satisfying*

$$\left\|\sum_{n\in\mathbb{Z}}(\mathfrak{h}_n - \mathfrak{g}_n)\mathfrak{q}_n\right\| \leq \alpha\left\|\sum_{n\in\mathbb{Z}}\mathfrak{h}_n\mathfrak{q}_n\right\| + \beta\left\|\sum_{n\in\mathbb{Z}}\mathfrak{g}_n\mathfrak{q}_n\right\| \tag{3}$$

for all the finite sequences $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$. Then, $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$ with lower and upper frame bounds $r_1\left(\frac{1-\alpha}{1+\beta}\right)^2$ and $r_2\left(\frac{1+\alpha}{1-\beta}\right)^2$ respectively. Further, $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ can be expressed as the orbit of a right-linear operator $\mathfrak{P}$: $rspan\{\mathfrak{g}_n\}_{n\in\mathbb{Z}} \to rspan\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$, i.e., $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}} = \{\mathfrak{P}^n(\mathfrak{g}_0)\}_{n\in\mathbb{Z}}$. Moreover, if $\mathscr{T}$ is bounded then so is $\mathfrak{P}$.

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

*Proof.*Let $\mathscr{I} \subset \mathbb{Z}$ be a finite set. Then, by (3), we have

$$\begin{aligned}\Big\|\sum_{n\in\mathscr{I}} \mathfrak{g}_n\mathfrak{q}_n\Big\| &\le \Big\|\sum_{n\in\mathscr{I}} (\mathfrak{h}_n-\mathfrak{g}_n)\mathfrak{q}_n\Big\| + \Big\|\sum_{n\in\mathscr{I}} \mathfrak{h}_n\mathfrak{q}_n\Big\| \\ &\le (1+\alpha)\Big\|\sum_{n\in\mathscr{I}} \mathfrak{h}_n\mathfrak{q}_n\Big\| + \beta\Big\|\sum_{n\in\mathscr{I}} \mathfrak{g}_n\mathfrak{q}_n\Big\|.\end{aligned}$$

This implies

$$\Big\|\sum_{n\in\mathscr{I}} \mathfrak{g}_n\mathfrak{q}_n\Big\| \le \frac{(1+\alpha)}{(1-\beta)}\Big\|\sum_{n\in\mathscr{I}} \mathfrak{h}_n\mathfrak{q}_n\Big\|. \tag{4}$$

Also,

$$\begin{aligned}\Big\|\sum_{n\in\mathscr{I}} \mathfrak{h}_n\mathfrak{q}_n\Big\| &= \sup_{\|\mathfrak{f}\|=1}\Big|\Big\langle \sum_{n\in\mathscr{I}} \mathfrak{h}_n\mathfrak{q}_n\Big|\mathfrak{f}\Big\rangle\Big| \\ &\le \Big(\sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2\Big)^{\frac{1}{2}} \sup_{\|\mathfrak{f}\|=1}\Big(\sum_{n\in\mathscr{I}} |\langle \mathfrak{h}_n|\mathfrak{f}\rangle|^2\Big)^{\frac{1}{2}} \\ &\le \sqrt{r_2}\Big(\sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2\Big)^{\frac{1}{2}}.\end{aligned} \tag{5}$$

From (4) and (5), we get

$$\Big\|\sum_{n\in\mathscr{I}} \mathfrak{g}_n\mathfrak{q}_n\Big\| \le \frac{(1+\alpha)\sqrt{r_2}}{(1-\beta)}\Big(\sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2\Big)^{\frac{1}{2}}.$$

Define $\mathfrak{U} : \ell^2_{\mathbb{Z}}(\mathfrak{Q}) \to \mathfrak{H}$ by

$$\mathfrak{U}(\{\mathfrak{q}_n\}_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \mathfrak{g}_n\mathfrak{q}_n,\ \{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q}).$$

Then, $\|\mathfrak{U}\| \le \frac{(1+\alpha)\sqrt{r_2}}{(1-\beta)}$ and hence $\mathfrak{U}$ is a bounded operator. Thus by Theorem 3.4 [13], $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ form a Bessel sequence for $\mathfrak{H}$ with Bessel bound $r_2\Big(\frac{1+\alpha}{1-\beta}\Big)^2$.
Now, let $\mathfrak{T}_\mathfrak{h}$ be the synthesis operator and $\mathfrak{S}_\mathfrak{h} = \mathfrak{T}_\mathfrak{h}\mathfrak{T}^*_\mathfrak{h}$ be the frame operator for $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$. Define an operator $\mathfrak{V} : \mathfrak{H} \to \ell^2_{\mathbb{Z}}(\mathfrak{Q})$ by

$$\mathfrak{V}(\mathfrak{h}) = (\mathfrak{T}^*_\mathfrak{h}(\mathfrak{T}_\mathfrak{h}\mathfrak{T}^*_\mathfrak{h})^{-1})(\mathfrak{h}),\ \mathfrak{h} \in \mathfrak{H}.$$

As $\{(\mathfrak{T}_\mathfrak{h}\mathfrak{T}^*_\mathfrak{h})^{-1}(\mathfrak{h}_n)\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$ with lower and upper frame bounds $\frac{1}{r_2}$ and $\frac{1}{r_1}$ respectively, we have

$$\|\mathfrak{V}(\mathfrak{h})\|^2 = \sum_{n\in\mathbb{Z}} |\langle \mathfrak{S}^{-1}_\mathfrak{h}(\mathfrak{h}_n)|\mathfrak{h}\rangle|^2 \le \frac{1}{r_1}\|\mathfrak{h}\|^2,\ \mathfrak{h} \in \mathfrak{H}.$$

Now for $\mathfrak{h} \in \mathfrak{H}$, let $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} = \mathfrak{V}(\mathfrak{h})$. Then from (3),

$$\begin{aligned}\|\mathfrak{h} - \mathfrak{U}\mathfrak{V}(\mathfrak{h})\| &\le \Big\|\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\langle \mathfrak{S}^{-1}_\mathfrak{h}(\mathfrak{h}_n)|\mathfrak{h}\rangle - \sum_{n\in\mathbb{Z}} \mathfrak{g}_n\langle \mathfrak{S}^{-1}_\mathfrak{h}(\mathfrak{h}_n)|\mathfrak{h}\rangle\Big\| \\ &= \Big\|\sum_{n\in\mathbb{Z}} (\mathfrak{h}_n-\mathfrak{g}_n)\langle \mathfrak{S}^{-1}_\mathfrak{h}(\mathfrak{h}_n)|\mathfrak{h}\rangle\Big\| \\ &\le \alpha\|\mathfrak{h}\| + \beta\|\mathfrak{U}\mathfrak{V}(\mathfrak{h})\|,\ \mathfrak{h} \in \mathfrak{H}.\end{aligned}$$

From Lemma 3, $\mathfrak{U}\mathfrak{V}$ is invertible and $\|(\mathfrak{U}\mathfrak{V})^{-1}\| \le \frac{1+\beta}{1-\alpha}$. Now for each $\mathfrak{h} \in \mathfrak{H}$, we have

$$\begin{aligned}\mathfrak{h} &= (\mathfrak{U}\mathfrak{V})(\mathfrak{U}\mathfrak{V})^{-1}(\mathfrak{h}) \\ &= \sum_{n\in\mathbb{Z}} \mathfrak{g}_n\langle \mathfrak{S}^{-1}_\mathfrak{h}(\mathfrak{h}_n)|(\mathfrak{U}\mathfrak{V})^{-1}\mathfrak{h}\rangle.\end{aligned}$$

Therefore for each $\mathfrak{h} \in \mathfrak{H}$, we have

$$\begin{aligned}\|\mathfrak{h}\|^4 &= \Big|\Big\langle \sum_{n\in\mathbb{Z}} \mathfrak{g}_n \langle \mathfrak{S}_{\mathfrak{h}}^{-1}(\mathfrak{h}_n)|(\mathfrak{U}\mathfrak{V})^{-1}\mathfrak{h}\rangle \Big| \mathfrak{h} \Big\rangle\Big|^2 \\ &\le \sum_{n\in\mathbb{Z}} |\langle \mathfrak{S}_{\mathfrak{h}}^{-1}(\mathfrak{h}_n)|(\mathfrak{U}\mathfrak{V})^{-1}\mathfrak{h}\rangle|^2 \Big(\sum_{n\in\mathbb{Z}} |\langle \mathfrak{g}_n|\mathfrak{h}\rangle|^2\Big) \\ &\le \frac{1}{r_1}\Big(\frac{1+\beta}{1-\alpha}\Big)^2 \|\mathfrak{h}\|^2 \Big(\sum_{n\in\mathbb{Z}} |\langle \mathfrak{g}_n|\mathfrak{h}\rangle|^2\Big).\end{aligned}$$

This gives

$$r_1\Big(\frac{1-\alpha}{1+\beta}\Big)^2 \|\mathfrak{h}\|^2 \le \sum_{n\in\mathbb{Z}} |\langle \mathfrak{g}_n|\mathfrak{h}\rangle|^2,\ \mathfrak{h}\in\mathfrak{H}.$$

Therefore, $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$ with lower and upper frame bounds $r_1\left(\frac{1-\alpha}{1+\beta}\right)^2$ and $r_2\left(\frac{1+\alpha}{1-\beta}\right)^2$ respectively.
Now, from (3) it follows that for all finite sequences $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$,

$$\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_n = 0 \iff \sum_{n\in\mathbb{Z}} \mathfrak{g}_n\mathfrak{q}_n = 0,$$

and since $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ is right-linearly independent, therefore $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ is also right-linearly independent. Thus by Theorem 1, there exists a bijective right-linear operator $\mathfrak{P} : \text{rspan}\{\mathfrak{g}_n\}_{n\in\mathbb{Z}} \to \text{rspan}\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ such that $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}} = \{\mathfrak{P}^n(\mathfrak{g}_0)\}_{n\in\mathbb{Z}}$. Now let us assume that $\mathscr{T}$ is bounded and $\mathfrak{T}_{\mathfrak{h}}$, $\mathfrak{T}_{\mathfrak{g}} : \ell^2_{\mathbb{Z}}(\mathfrak{Q}) \to \mathfrak{H}$ be the synthesis operators for $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$ and $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ respectively. Let $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{g}})$ be arbitrary i.e., $\sum_{n\in\mathbb{Z}} \mathfrak{g}_n\mathfrak{q}_n = 0$ and hence $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ker(\mathfrak{T}_{\mathfrak{h}})$. Since $\mathscr{T}$ is a bounded operator, Theorem 4 gives that $\ker(\mathfrak{T}_{\mathfrak{h}})$ is invariant under the right-shift operator i.e., $\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_{n-1} = 0$ which further implies that $\sum_{n\in\mathbb{Z}} \mathfrak{g}_n\mathfrak{q}_{n-1} = 0$. This implies that $\ker(\mathfrak{T}_{\mathfrak{g}})$ is invariant under the right-shift operator. Again from Theorem 4, we conclude that $\mathfrak{P}$ is a bounded operator.

One may observe that (3) is a particular case of the condition

$$\Big\|\sum_{n\in\mathbb{Z}} (\mathfrak{h}_n - \mathfrak{g}_n)\mathfrak{q}_n\Big\| \le \alpha\Big\|\sum_{n\in\mathbb{Z}} \mathfrak{h}_n\mathfrak{q}_n\Big\| + \beta\Big\|\sum_{n\in\mathbb{Z}} \mathfrak{g}_n\mathfrak{q}_n\Big\| + \gamma\Big(\sum_{n\in\mathbb{Z}} |\mathfrak{q}_n|^2\Big)^{\frac{1}{2}}, \tag{6}$$

with $\gamma = 0$. On the similar lines as in Theorem 5, it can be easily proved that if for some right-linear operator $\mathscr{T} : \text{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} \to \text{rspan}\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$, $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}} = \{\mathscr{T}^n(\mathfrak{h}_0)\}_{n\in\mathbb{Z}}$ form a frame for $\mathfrak{H}$, with lower and upper frame bounds r_1 and r_2 respectively and $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}$ satisfy (6) for all the finite sequences $\{\mathfrak{q}_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathfrak{Q})$ with $\alpha,\beta,\gamma \ge 0$ and $\max\left(\alpha + \frac{\gamma}{\sqrt{r_1}}, \beta\right) < 1$, then $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ also form a frame for $\mathfrak{H}$ with lower and upper frame bounds $r_1\left(1 - \frac{\alpha+\beta+\frac{\gamma}{\sqrt{r_1}}}{1+\beta}\right)^2$ and $r_2\left(1 + \frac{\alpha+\beta+\frac{\gamma}{\sqrt{r_2}}}{1-\beta}\right)^2$ respectively. But the problem with the perturbation condition (6) is that it doesn't guarantee the preservation of the existence of the representation as an orbit of some operator.

*Example 3.*Let $\{\mathfrak{e}_n\}_{n\in\mathbb{Z}}$ denote an orthonormal basis for $\mathfrak{H}$. For some index set J, consider the right-linearly independent frame sequence $\{\mathfrak{h}_n\}_{n\in J} := \{\mathfrak{e}_n\}_{n\in\mathbb{Z}} \cup \{\frac{1}{3}\sum_{k\in\mathbb{N}} \frac{1}{3^k}\mathfrak{e}_k\}$. One can easily verify that in view of the inequality (6), $\{\mathfrak{g}_n\}_{n\in J} := \{\mathfrak{e}_n\}_{n\in\mathbb{Z}} \cup \{0\}$ is a perturbation of $\{\mathfrak{h}_n\}_{n\in J}$ with $\alpha = \beta = 0$ and $\gamma = \frac{1}{6}$. Clearly by Theorem 1, $\mathfrak{g}$ cannot be expressed as orbit of any operator being right-linearly dependent.

4 Approximating Frames via sub-orbits of Operators

Since now, we have observed that only a special category of frames can be expressed as operator orbits. In this section, we approximate an arbitrary frame of $\mathfrak{H}$ with a frame as a sub-orbit of some hypercyclic operator. Now similar to the case of complex Hilbert spaces, we call a bounded operator $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ as a *hypercyclic operator* if there exists an element $\mathfrak{h}_0 \in \mathfrak{H}$ such that $\overline{\{\mathscr{T}^n(\mathfrak{h}_0)\}_{n=0}^{\infty}} = \mathfrak{H}$. Firstly we define, what we mean by an approximation of a frame in $\mathfrak{H}$.

Definition 4. *Let $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ be a frame for $\mathfrak{H}$ and $\gamma > 0$. A sequence $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ in $\mathfrak{H}$ is said to be a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ if for all finite sequences $\{\mathfrak{q}_n\}_{n\in\mathscr{I}} \subseteq \mathfrak{Q}$*

$$\Big\| \sum_{n\in\mathscr{I}} (\mathfrak{h}_n - \mathfrak{g}_n)\mathfrak{q}_n \Big\|^2 \leq \gamma \sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2. \tag{7}$$

Example 4. As in Example 3, $\{\mathfrak{g}_n\}_{n\in\mathbb{Z}}$ forms a $\frac{1}{6}$-approximation of $\{\mathfrak{h}_n\}_{n\in\mathbb{Z}}$.

Now in the next result, it is shown that for good enough small values of γ, a γ-approximation $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ of $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$, also form a frame for $\mathfrak{H}$ having the same excess. Moreover, its synthesis and frame operators can also be approximated using the synthesis and frame operators of $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ respectively.

Theorem 6. *Let $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ be a frame for $\mathfrak{H}$ with lower and upper frame bounds r_1 and r_2 respectively and $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ be a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ for some $\gamma \in (0, r_1)$. Then, the following statements hold:*

(a) $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ form a frame for $\mathfrak{H}$ with lower and upper frame bounds $r_1\left(1-\sqrt{\frac{\gamma}{r_1}}\right)^2$ and $r_2\left(1+\sqrt{\frac{\gamma}{r_2}}\right)^2$ respectively.

(b) Let $\mathfrak{T}_{\mathfrak{h}}$, $\mathfrak{T}_{\mathfrak{g}}$ be the synthesis operators and $\mathfrak{S}_{\mathfrak{h}}$, $\mathfrak{S}_{\mathfrak{g}}$ be the frame operators for $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ and $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ respectively. Then

$$\|\mathfrak{T}_{\mathfrak{h}} - \mathfrak{T}_{\mathfrak{g}}\| \leq \sqrt{\gamma}, \quad \|\mathfrak{S}_{\mathfrak{h}} - \mathfrak{S}_{\mathfrak{g}}\| \leq \sqrt{\gamma r_2}\left(2+\sqrt{\frac{\gamma}{r_2}}\right),$$

and

$$\|\mathfrak{S}_{\mathfrak{h}}^{-1} - \mathfrak{S}_{\mathfrak{g}}^{-1}\| \leq \frac{\sqrt{\gamma r_2}\left(2+\sqrt{\frac{\gamma}{r_2}}\right)}{r_1^2\left(1-\sqrt{\frac{\gamma}{r_1}}\right)^2}.$$

Proof. (a) One may easily observe that inequality (7) is a particular case of the condition (6) where $\alpha = \beta = 0$ and hence, $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ form a frame for $\mathfrak{H}$ on the similar lines as in Theorem 5.
(b) Let $\{\mathfrak{q}_n\}_{n\in\mathscr{I}} \in \ell^2_{\mathscr{I}}(\mathfrak{Q})$. Then, we have

$$\|\mathfrak{T}_{\mathfrak{h}}(\{\mathfrak{q}_n\}_{n\in\mathscr{I}}) - \mathfrak{T}_{\mathfrak{g}}(\{\mathfrak{q}_n\}_{n\in\mathscr{I}})\|^2 = \Big\| \sum_{n\in\mathscr{I}} (\mathfrak{h}_n - \mathfrak{g}_n)\mathfrak{q}_n \Big\|^2 \leq \gamma \sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2.$$

This gives $\|\mathfrak{T}_{\mathfrak{h}} - \mathfrak{T}_{\mathfrak{g}}\| \leq \sqrt{\gamma}$. Now for the frame operators, we have

$$\mathfrak{S}_{\mathfrak{h}} - \mathfrak{S}_{\mathfrak{g}} = \mathfrak{T}_{\mathfrak{h}}\mathfrak{T}_{\mathfrak{h}}^* - \mathfrak{T}_{\mathfrak{g}}\mathfrak{T}_{\mathfrak{g}}^* = (\mathfrak{T}_{\mathfrak{h}} - \mathfrak{T}_{\mathfrak{g}})\mathfrak{T}_{\mathfrak{h}}^* + \mathfrak{T}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{h}}^* - \mathfrak{T}_{\mathfrak{g}}^*). \tag{8}$$

Since $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ and $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ are frames for $\mathfrak{H}$ with upper bounds r_2 and $r_2\left(1+\sqrt{\frac{\gamma}{r_2}}\right)^2$ respectively, we have

$$\|\mathfrak{T}_{\mathfrak{h}}\| \leq \sqrt{r_2} \quad \text{and} \quad \|\mathfrak{T}_{\mathfrak{g}}\| \leq \sqrt{r_2}\left(1+\sqrt{\frac{\gamma}{r_2}}\right). \tag{9}$$

From (8) and (9),

$$\begin{aligned}\|\mathfrak{S}_{\mathfrak{h}} - \mathfrak{S}_{\mathfrak{g}}\| &\leq \|(\mathfrak{T}_{\mathfrak{h}} - \mathfrak{T}_{\mathfrak{g}})\|\|\mathfrak{T}_{\mathfrak{h}}^*\| + \|\mathfrak{T}_{\mathfrak{g}}\|\|(\mathfrak{T}_{\mathfrak{h}}^* - \mathfrak{T}_{\mathfrak{g}}^*)\| \\ &\leq (\sqrt{\gamma})(\sqrt{r_2}) + (\sqrt{\gamma})(\sqrt{r_2})\left(1+\sqrt{\frac{\gamma}{r_2}}\right) \\ &= \sqrt{\gamma r_2}\left(2+\sqrt{\frac{\gamma}{r_2}}\right).\end{aligned}$$

Since $\|\mathfrak{S}_{\mathfrak{h}}^{-1}\| \leq \frac{1}{r_1}$, we obtain

$$\|\mathfrak{S}_{\mathfrak{h}}^{-1} - \mathfrak{S}_{\mathfrak{g}}^{-1}\| = \|\mathfrak{S}_{\mathfrak{h}}^{-1}(\mathfrak{S}_{\mathfrak{h}} - \mathfrak{S}_{\mathfrak{g}})\mathfrak{S}_{\mathfrak{g}}^{-1}\| \leq \frac{1}{r_1}\sqrt{\gamma r_2}\left(2+\sqrt{\frac{\gamma}{r_2}}\right)\frac{1}{r_1\left(1-\sqrt{\frac{\gamma}{r_1}}\right)^2}$$

as desired.

Next, we give an equivalent condition of (7), for being a γ-approximation of a frame of $\mathfrak{H}$.

Lemma 4. *Let $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$ be a frame for $\mathfrak{H}$ with lower bound r_1 and $\gamma \in [0, r_1)$ be given. Then a sequence $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ in $\mathfrak{H}$ form a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$, if*

$$\sum_{n\in\mathscr{I}} \|\mathfrak{h}_n - \mathfrak{g}_n\|^2 \leq \gamma.$$

Proof. For any finite sequence $\{\mathfrak{q}_n\}_{n\in\mathscr{I}} \subseteq \mathfrak{Q}$, we have

$$\begin{aligned} \Big\| \sum_{n\in\mathscr{I}} (\mathfrak{h}_n - \mathfrak{g}_n)\mathfrak{q}_n \Big\|^2 &\leq \Big(\sum_{n\in\mathscr{I}} \|\mathfrak{h}_n - \mathfrak{g}_n\| |\mathfrak{q}_n| \Big)^2 \\ &\leq \Big(\sum_{n\in\mathscr{I}} \|\mathfrak{h}_n - \mathfrak{g}_n\|^2 \Big) \Big(\sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2 \Big) \\ &\leq \gamma \sum_{n\in\mathscr{I}} |\mathfrak{q}_n|^2. \end{aligned}$$

Thus, $\{\mathfrak{g}_n\}_{n\in\mathscr{I}}$ is a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathscr{I}}$.

In the following result, we approximate a frame with sub-orbit of a given operator $\mathscr{T}$.

Corollary 2. *Let $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ be a frame for $\mathfrak{H}$ with lower bound r_1 and $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ be a given bounded operator. Suppose $\mathfrak{h}_0 \in \mathfrak{H}$ and for some $\gamma \in (0, r_1)$ and each $n \in \mathbb{N}$, there exists a non-negative integer ϕ_n satisfying*

$$\|\mathfrak{h}_n - \mathscr{T}^{\phi_n}(\mathfrak{h}_0)\|^2 \leq \frac{\gamma}{2^n},$$

then $\{\mathscr{T}^{\phi_n}(\mathfrak{h}_0)\}_{n\in\mathbb{N}}$ is a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ and hence form a frame for $\mathfrak{H}$.

Proof. Consider

$$\sum_{n\in\mathbb{N}} \|\mathfrak{h}_n - \mathscr{T}^{\phi_n}(\mathfrak{h}_0)\|^2 \leq \sum_{n\in\mathbb{N}} \frac{\gamma}{2^n} = \gamma.$$

Thus, Lemma 4 and Theorem 6 implies that $\{\mathscr{T}^{\phi_n}(\mathfrak{h}_0)\}_{n\in\mathbb{N}}$ is a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ and hence form a frame for $\mathfrak{H}$.

Remark. Let $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ be a frame for $\mathfrak{H}$ and $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ be a hypercyclic operator on $\mathfrak{H}$. Then for every frame element $\mathfrak{h}_n$ and sufficiently small $\gamma > 0$, there exists a non-negative integer ϕ_n such that

$$\|\mathfrak{h}_n - \mathscr{T}^{\phi_n}(\mathfrak{h}_0)\|^2 \leq \frac{\gamma}{2^n}.$$

Therefore, by Corollary 2, $\{\mathscr{T}^{\phi_n}(\mathfrak{h}_0)\}_{n\in\mathbb{N}}$ is a γ-approximation of $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ and hence form a frame for $\mathfrak{H}$.

5 Conclusion

Frames expressed as operator orbits contributes a lot in the field of dynamical sampling which has immense applications in the field of data center temperature sensing, neuron-imaging and satellite remote sensing but is specially known for its applications in Wireless Sensor Networks (WSN) to gather information about some physical quantity such as pressure, temperature etc., where measurements devices are distributed at some locations that exploits the evolutionary structure and the positions of sensors to recover an unknown function.

In this paper, we discussed frames which can be expressed as operator orbits under a single generator in quaternionic Hilbert spaces. A necessary and sufficient condition is discussed for the representation of the system with a bounded operator. Testing of the stability of a system under various perturbations is an important aspect. Keeping this in mind, we obtain conditions for the stability of frames as operator orbits. Approximation of an arbitrary frame with a frame as a sub-orbit of an operator is also taken into consideration. These results are motivated by there applications in the area of dynamical sampling. The idea of dynamical sampling is to ensure the recovery of an unknown function $\mathfrak{h}$ which is evolving in time through some operator $\mathscr{T}$ from the roughly sampled states $\mathscr{T}^n(\mathfrak{h})$ at each time n.

Declarations

Competing interests: The authors declare that they have no conflicts of interest that are related to the contents of this paper.
Authors' contributions: All authors contributed equally to the research paper.
Funding: The research of the first author is sponsored by the Council of Scientific and Industrial Research (File No: 09/045(1679)/2019-EMR-I).
Availability of data and materials: No datasets were generated or analyzed during the current study and hence data sharing is not applicable to this paper.
Acknowledgments: The authors express their gratitude to the reviewers for their valuable comments, which enhanced the quality of the paper.

References

[1] Aldroubi, Akram, et al. Dynamical sampling. *Applied and Computational Harmonic Analysis* 42.3 (2017): 378-401.
[2] Aldroubi, Akram, et al. Iterative actions of normal operators. *Journal of Functional Analysis* 272.3 (2017): 1121-1146.
[3] Aldroubi, Akram, and Armenak Petrosyan. Dynamical sampling and systems from iterative actions of operators. *Frames and Other Bases in Abstract and Function Spaces.* Birkhuser, Cham, 2017. 15-26.
[4] Cazassa, Peter G., and Ole Christensen. Perturbation of operators and applications to frame theory. *Journal of Fourier Analysis and Applications* 3.5 (1997): 543-557.
[5] Christensen, Ole, and Marzieh Hasannasab. Operator representations of frames: boundedness, duality, and stability. *Integral Equations and Operator Theory* 88.4 (2017): 483-499.
[6] Christensen, Ole, Marzieh Hasannasab, and Ehsan Rashidi. Dynamical sampling and frame representations with bounded operators. *Journal of Mathematical Analysis and Applications* 463.2 (2018): 634-644.
[7] Christensen, Ole, and Marzieh Hasannasab. Approximate frame representations via iterated operator systems. *arXiv preprint arXiv*:1910.03535 (2019).
[8] Christensen, Ole, and Marzieh Hasannasab. Frame properties of systems arising via iterated actions of operators. *Applied and Computational Harmonic Analysis* 46.3 (2019): 664-673.
[9] Christensen, Ole, Marzieh Hasannasab, and Friedrich Philipp. Frame properties of operator orbits. *Mathematische Nachrichten* 293.1 (2020): 52-66.
[10] Ghiloni, Riccardo, Valter Moretti, and Alessandro Perotti. Continuous slice functional calculus in quaternionic Hilbert spaces. *Reviews in Mathematical Physics* 25.04 (2013): 1350006.
[11] Khokulan, Mohananathan, Kengatharam Thirulogasanthar, and Sivakolundu Srisatkunarajah. Discrete frames on finite dimensional left quaternion Hilbert spaces. *Axioms* 6.1 (2017): 3.
[12] Muraleetharan, B., and K. Thirulogasanthar. Fredholm operators and essential S-spectrum in the quaternionic setting. *Journal of Mathematical Physics* 59.10 (2018).
[13] Sharma, S.K., Goel, S., Frames in quaternionic Hilbert spaces. *J. Math. Phy. Anal. Geo.*, **15(3)** 2019, 395-411.
[14] Sharma, S. K., and S. K. Kaushik. Riesz bases in quaternionic Hilbert spaces. *arXiv preprint arXiv*:1909.06364 (2019).