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Bijective Product and Product Square k-Cordial Labeling of Graphs

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Abstract: In this paper, we introduce two new concepts namely bijective product k-cordial labeling and bijective product square k-cordial labeling and show that some standard graphs admit bijective product k-cordial labeling, where $k = 2,3$. Also, we establish that the path, cycle, flower, helm and gear graphs are bijective product square 3-cordial graphs.

Keywords: Bijective product k-cordial labeling; Bijective product square k-cordial labeling; Bijective product 2-cordial graph; Bijective product square 3-cordial graph.

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1 Introduction and Terminology

Every graph considered here is simple, finite, connected, undirected and the order of the graph is greater than 1. We follow the basic notation and terminology of graph theory as in [3]. Graph labeling is an emerging area of research in graph theory. There are several labeling techniques introduced and studied by several authors over the last five decades. An interested reader can refer to [2]. One of the popular labelings called 'cordial labeling' was due to Cahit [1].

In this paper, we fixed the complete residue class of the ring of integers modulo k as $\mathbb{Z}_k = \{0,1,\dots,k-1\}$, $k \geq 2$.

Since the integer $|f(u)-f(v)|$ is the same as $f(u)+f(v) \pmod 2$, we give an equivalent definition for cordial labeling as follows: Let $f : V(G) \to \mathbb{Z}_2$. The induced labeling for each edge uv is defined by $f(u)+f(v) \pmod 2$. f is called a *cordial labeling* of G if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$.

Sundaram et al. [10] used this concept to product version and introduced product cordial labeling. A *product cordial labeling* f of a graph G is $f : V(G) \to \mathbb{Z}_2$ such that if each edge uv is assigned the label $f(u)f(v)$, $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$.

In 2012, Ponraj et al. [9] extended the concept of product cordial labeling and introduced k-product cordial labeling: Let $f : V(G) \to \mathbb{Z}_k$, where $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)$. f is called a *k-product cordial labeling* if $|v_f(i)-v_f(j)| \leq 1$, and $|e_f(i)-e_f(j)| \leq 1$, $i, j \in \mathbb{Z}_k$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with $x \in \mathbb{Z}_k$. Thus the concept of 2-product cordial is the same as that of product cordial.

Motivated by the results in [9, 10], we introduce the variations of 'cordial labeling', namely 'bijective product k-cordial labeling' and 'bijective product square k-cordial labeling'.

A *bijective product k-cordial labeling* f of a graph G with vertex set V and edge set E is a bijection from V to $\{1,2,\dots,|V|\}$ such that the induced edge labeling $f^{\times} : E(G) \to \mathbb{Z}_k = \{i \mid 0 \leq i \leq k-1\}$ defined as $f^{\times}(uv) \equiv f(u)f(v) \pmod k$ for every edge $uv \in E$ satisfies the condition $|e_f^{\times}(i) - e_f^{\times}(j)| \leq 1$, where $i, j \in \mathbb{Z}_k$ and $e_f^{\times}(i)$ is the number of edges labeled with i under $f^{\times}$. A graph which admits a bijective product k-cordial labeling is called *bijective product k-cordial graph*.

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A *bijective product square k-cordial labeling* f of a graph G with vertex set V and edge set E is a bijection from V to $\{1,2,\dots,|V|\}$ such that the induced edge labeling $f^*: E(G) \to S_k = \{i^2 \mod k \mid 0 \le i \le k-1\}$ defined as $f^*(uv) \equiv (f(u)f(v))^2 \pmod{k}$ for every edge $uv \in E$ satisfies the condition $|e^*_f(i) - e^*_f(j)| \le 1$, where $i,j \in S_k$ and $e^*_f(i)$ is the number of edges labeled with i under f^*. A graph which admits a bijective product square k-cordial labeling is called *bijective product square k-cordial graph*.

*Remark.*Since $a = a^2$ in $\mathbb{Z}_2$ for all $a \in \mathbb{Z}_2$, the concept of bijective product square 2-cordial is the same as bijective product 2-cordial. Thus, when $k = 2$, we consider bijective product 2-cordial instead of bijective product square 2-cordial.

*Remark.*Clearly, (i) x is even if and only if $x^2 \equiv 0 \pmod 2$ and $x^2 \equiv 0 \pmod 4$; (ii) x is odd if and only if $x^2 \equiv 1 \pmod 2$ and $x^2 \equiv 1 \pmod 4$. Note that $S_2 = \{0,1\}$ and $S_4 = \{0,1\}$. Thus, a labeling g is bijective product square 2-cordial if and only if g is bijective product square 4-cordial.

We use the following terminology and notations in the subsequent sequel.

Let $C_n = u_1u_2\cdots u_nu_1$ and $V(K_1) = \{w\}$. Let the wheel graph $W_n = C_n \vee K_1$. The *helm* graph H_n is obtained from W_n by attaching a pendent edge u_iv_i at u_i, where v_i are extra vertices, $1 \le i \le n$. For convenience, we let $u_{n+1} = u_1$.

The *flower graph* Fl_n, is obtained from the helm graph H_n by adding n edges wv_i, $1 \le i \le n$.

For $n \ge 2$, the *gear graph* G_n is obtained from a wheel W_n by subdividing each of its rim edge u_iu_{i+1}. Let the new inserted vertex at the edge u_iu_{i+1} be x_i, $1 \le i \le n$.

We present this paper in three sections. In the first section, we provide a brief introduction about the development of the concepts, the required definitions and the terminology used for our study. In the second section, we deal with bijective product k-cordial labeling of graphs and in the third section we study the bijective product square 3-cordial labeling.

2 Bijective Product k-Cordial Labeling of Graphs

Here, we give some necessary and sufficient conditions for bijective product k-cordial labeling and also show that some standard graphs are bijective product k-cordial.

Theorem 1.*If a graph G is bijective product k-cordial for $k \ge 2$, then G is k-product cordial.*

*Proof.*Let f be a bijective product k-cordial labeling of G. Define $\bar{f}: V(G) \to \mathbb{Z}_k$ by $\bar{f}(u) \equiv f(u) \pmod{k}$. Since f is bijective then $|v_{\bar{f}}(i) - v_{\bar{f}}(j)| \le 1$ for all $i,j \in \mathbb{Z}_k$. Moreover, $\bar{f}(u)\bar{f}(v) = f^{\times}(uv)$.

Theorem 2.*A graph G is bijective product 2-cordial if and only if G admits a 2-product cordial labeling g with $v_g(1) \ge v_g(0)$.*

*Proof.*By Theorem 1 we need to show the sufficiency only. Let g be a 2-product cordial labeling of G. Let L_1 and L_0 be the set of vertices in G whose label is 1 and 0, respectively. Note that, an edge uv with $g(u)g(v) = 1$ if and only if $u,v \in L_1$.

We shall define a bijective product 2-cordial labeling f for G.

(a) Suppose the order of G is even, say $2n$ then $|L_1| = |L_0| = n$. Label all the vertices in L_1 by odd integers in $\{1,2,\dots,2n\}$ injectively, and all the vertices in L_0 by even integers in $\{1,2,\dots,2n\}$ injectively. Denote this labeling by f. Now, for an edge uv, $g(u)g(v) = 1$ if and only if $f(u)$ and $f(v)$ are odd if and only if $f^{\times}(uv) = 1$. Hence $|e^{\times}_f(0) - e^{\times}_f(1)| = |e_g(0) - e_g(1)|$. Since g is a 2-product cordial labeling then $|e^{\times}_f(0) - e^{\times}_f(1)| \le 1$. Hence f is a bijective product 2-cordial labeling.

(b) Suppose the order of G is odd, say $2n+1$ then by the hypothesis, $|L_1| = n+1$ and $|L_0| = n$. Label all the vertices in L_1 by odd integers in $\{1,2,\dots,2n+1\}$ injectively, and all the vertices in L_0 by even integers in $\{1,2,\dots,2n+1\}$ injectively. Denote this labeling by f. Now, for an edge uv, $g(u)g(v) = 1$ if and only if $f(u)$ and $f(v)$ are odd if and only if $f^{\times}(uv) = 1$. Hence $|e^{\times}_f(0) - e^{\times}_f(1)| = |e_g(0) - e_g(1)|$. Since g is a 2-product cordial labeling then $|e^{\times}_f(0) - e^{\times}_f(1)| \le 1$. Hence, we get a bijective product 2-cordial labeling f.

Sundaram et al. [10] showed that cycles with odd order, trees, unicycle graphs of odd order, triangle T_n, tadpole graphs, the one point union of t cycles of length n, $C_n(t)$ (if t is even or both t and n are odd), $K_{1,m} \vee K_1$ (the join graph of $K_{1,m}$ and K_1) with odd m, $K_2 \vee mK_1$ and helm graphs are 2-product cordial. Moreover, all these graphs satisfy $v_g(1) \ge v_g(0)$. Therefore, we say that these graphs are bijective product 2-cordial.

Next, we consider the bijective product k-cordiality for a unicycle graph G, where $k \ge 2$. Note that, there are $\left\lfloor \frac{n}{k} \right\rfloor$ multiples of k in $\{1,2,\dots,n\}$.

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Suppose G is a unicycle of order n with girth greater than $n - \left\lfloor \frac{n}{k} \right\rfloor$ and f is a bijective product k-cordial labeling of G. Let V' be the set of vertices with labels which are not multiple of k. Let $H = G[V']$, the subgraph of G induced by V'. Note that $|V(H)| = n - \left\lfloor \frac{n}{k} \right\rfloor$. Since the girth of G is greater than $|V(H)|$, $H = \sum_{i=1}^{\omega} T_{n_i}$ for some $\omega \geq 1$, where T_{n_i} is a tree of order n_i. So $n - \left\lfloor \frac{n}{k} \right\rfloor = \sum_{i=1}^{\omega} n_i$.

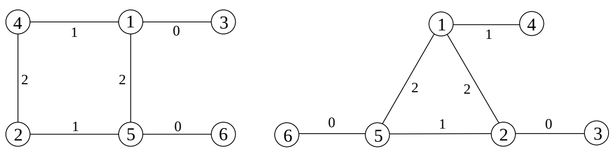
Now $|E(H)| = \sum_{i=1}^{\omega} |E(T_{n_i})| = \sum_{i=1}^{\omega} (n_i - 1) = \left(\sum_{i=1}^{\omega} n_i \right) - \omega = n - \left\lfloor \frac{n}{k} \right\rfloor - \omega$. On the other hand, since H may contains some edges which induced label is zero, $|E(H)| \geq n - e_f^{\times}(0)$. Thus $e_f^{\times}(0) \geq \left\lfloor \frac{n}{k} \right\rfloor + \omega \geq \left\lfloor \frac{n}{k} \right\rfloor + 1$. Since $e_f^{\times}(0) \leq \left\lceil \frac{n}{k} \right\rceil$, $\left\lceil \frac{n}{k} \right\rceil \geq \left\lfloor \frac{n}{k} \right\rfloor + 1$. Hence n is not a multiple of k. So we have

Theorem 3. *Let G be a unicycle of order n and $k \geq 2$. If G is bijective product k-cordial and the girth of G is greater than $n - \left\lfloor \frac{n}{k} \right\rfloor$, then $n \not\equiv 0 \pmod{k}$.*

Corollary 1. *For $k \geq 2$, if $n \equiv 0 \pmod{k}$, then C_n is not bijective product k-cordial.*

Remark. From the above discussion, if f is a bijective product k-cordial labeling for a unicycle graph G of order n with the girth of G greater than $n - \left\lfloor \frac{n}{k} \right\rfloor$, then H must be a tree.

Remark. Theorem 3 may not be true when the girth of G is less than or equals to $n - \left\lfloor \frac{n}{k} \right\rfloor$. Following are the examples for $k = 3$ and $n = 6$.



Theorem 4([10, Theorem 3.1]). *For $n \geq 3$, C_n is product cordial if and only if n is odd.*

Thus we have

Theorem 5. *For $n \geq 3$, C_n is bijective product 2-cordial if and only if n is odd.*

Conjecture 1. For any 2-product cordial graph, there is a 2-product cordial labeling g satisfies $v_g(1) \geq v_g(0)$.

Theorem 6. *A graph G is bijective product 3-cordial if and only if G admits a 3-product cordial labeling g with $v_g(1) \geq v_g(0)$ and $v_g(2) \geq v_g(0)$.*

Proof. By Theorem 1, we only need to show the sufficiency. Let g be a 3-product cordial labeling of G. Let L_i be the set of vertices in G whose label is $i \in \mathbb{Z}_3$. Note that, an edge uv with $g(u)g(v) = 1$ if and only if $u, v \in L_1$ or $u, v \in L_2$. An edge uv with $g(u)g(v) = 2$ if and only if $u \in L_1$ and $v \in L_2$ or $u \in L_2$ and $v \in L_1$.

We define a bijective product 3-cordial labeling f for G.

a) Suppose $|V(G)| = 3n$ then $|L_1| = |L_2| = |L_0| = n$. For each $i \in \mathbb{Z}_3$, label all the vertices in L_i by k, where $k \equiv i \pmod{3}$ in $\{1, 2, \ldots, 3n\}$ injectively.

b) Suppose $|V(G)| = 3n + 2$ then by the hypothesis $|L_1| = |L_2| = n + 1$ and $|L_0| = n$. By the similar argument in part (a), we get a bijective product 3-cordial labeling of G.

c) Suppose $|V(G)| = 3n + 1$ then by the hypothesis $|L_0| = |L_2| = n$ and $|L_1| = n + 1$ or $|L_0| = |L_1| = n$ and $|L_2| = n + 1$.
  - c-1) If $|L_0| = |L_2| = n$ and $|L_1| = n + 1$, then by the similar argument in part (a), we get a bijective product 3-cordial labeling of G.
  - c-2) If $|L_0| = |L_1| = n$ and $|L_2| = n + 1$, then label all the vertices in L_0, L_1 and L_2 by $i \equiv 0 \pmod{3}$, $i \equiv 2 \pmod{3}$ and $i \equiv 1 \pmod{3}$ in $\{1, 2, \ldots, 3n + 1\}$ injectively, respectively.

From the above cases, for an edge uv, $g(u)g(v) = 1$ if and only if $f(u) \equiv 1 \pmod{3}$ and $f(v) \equiv 1 \pmod{3}$ or $f(u) \equiv 2 \pmod{3}$ and $f(v) \equiv 2 \pmod{3}$ if and only if $f^{\times}(uv) = 1$. For an edge uv, $g(u)g(v) = 2$ if and only if $f(u) \equiv 1 \pmod{3}$ and $f(v) \equiv 2 \pmod{3}$ or $f(u) \equiv 2 \pmod{3}$ and $f(v) \equiv 1 \pmod{3}$ if and only if $f^{\times}(uv) = 2$. Hence $|e_f^{\times}(i) - e_f^{\times}(j)| = |e_g(i) - e_g(j)|$ where $i, j \in \mathbb{Z}_3$. Since g is a 3-product cordial labeling then $|e_f^{\times}(i) - e_f^{\times}(j)| \leq 1$. Hence f is a bijective product 3-cordial labeling.

Jeyanthi et al. [4,5,6,7,8] showed that some families of graphs are 3-product cordial. Since these graphs satisfy $v_g(1) \geq v_g(0)$ and $v_g(2) \geq v_g(0)$, we say that they are bijective product 3-cordial. Furthermore, combining with Corollary 1, we have

Theorem 7. *P_n is bijective product 3-cordial for $n \geq 2$; and C_m is bijective product 3-cordial if and only if $m \not\equiv 0 \pmod{3}$.*

3 Bijective Product Square 3-Cordial Graphs

Here, we show that the path, cycle, flower, helm and gear graphs are bijective product square 3-cordial. We denote $e^*_f(x)$ by e_x when f is fixed.

First, we prove that the path P_n is a bijective product square 3-cordial graph for all $n \geq 2$. Note that $S_3 = \{0,1\}$.

Theorem 8. *The path P_n is a bijective product square 3-cordial graph for all $n \geq 2$.*

Proof. Let $P_n = v_1v_2\cdots v_n$. We define $f : V(P_n) \to \{1,2,\ldots,n\}$ as follows:

a) Suppose $n = 6k+5$, $k \geq 0$. Define

$$\begin{aligned} f(v_{2i-1}) &= i, & 1 \leq i \leq 3k+1, \\ f(v_{2i}) &= 3k+1+i, & 1 \leq i \leq 3k+1, \\ f(v_{6k+3+i}) &= 6k+3+i, & 0 \leq i \leq 2 \text{ if } k \geq 1. \end{aligned}$$

That is, the vertices v_1, v_2, $\ldots$, v_{6k+5} are labeled by 1, $3k+2$, 2, $3k+3$, $\ldots$, $6k+1$, $3k+1$, $6k+2$, $6k+3$, $6k+4$, $6k+5$. Clearly f is a bijection. The induced labels for edges v_1v_2, v_2v_3, v_3v_4, $\ldots$, $v_{6k+2}v_{6k+3}$, $v_{6k+3}v_{6k+4}$, $v_{6k+4}v_{6k+5}$ are $1,1;0,0,0,1,1,1,\ldots,0,0,0;1,1,0,0,1$, i.e., $f^*(v_jv_{j+1}) = 1$ if $j \equiv 0,1,2 \pmod 6$ and $f^*(v_jv_{j+1}) = 0$ if $j \equiv 3,4,5 \pmod 6$, where $1 \leq j \leq 6k+1$; and $f^*(v_{6k+2}v_{6k+3}) = f^*(v_{6k+3}v_{6k+4}) = 0$, $f^*(v_{6k+4}v_{6k+5}) = 1$. Hence $e_1 = 3k+2$ and $e_0 = 3k+2$.

b) Suppose $n = 6k+4$, $k \geq 0$. Use the labeling defined in Case a) and delete the vertex v_{6k+5} then we get $e_1 = 3k+1$ and $e_0 = 3k+2$.

c) Suppose $n = 6k+3$, $k \geq 0$. Use the labeling defined in Case a) and delete the vertices v_{6k+5} and v_{6k+4} then we get $e_1 = 3k+1$ and $e_0 = 3k+1$.

d) Suppose $n = 6k+2$, $k \geq 0$. When $k = 0$. It is clear that P_2 is bijective product square 3-cordial graph by labeling v_1 and v_2 by 1 and 2, respectively.
When $k \geq 1$. Define

$$\begin{aligned} f(v_{2i-1}) &= i, & \text{if } 1 \leq i \leq 3k, \\ f(v_{2i}) &= 3k+i, & \text{if } 1 \leq i \leq 3k, \\ f(v_{6k+i}) &= 6k+i, & \text{if } 1 \leq i \leq 2. \end{aligned}$$

That is, the vertices v_1, v_2, $\ldots$, v_{6k+2} are labeled by 1, $3k+1$, 2, $3k+2$, $\ldots$, $6k-1$, $3k$, $6k$, $6k+1$, $6k+2$. Clearly f is a bijection.
One can see that the induced labels for edges v_1v_2, v_2v_3, v_3v_4, $\ldots$, $v_{6k-1}v_{6k}$, $v_{6k}v_{6k+1}$, $v_{6k+1}v_{6k+2}$ are $1,1,1,0,0,0,1,1,1,0,0,0,\ldots,0,0,0,1$, i.e., $f^*(v_jv_{j+1}) = 1$ if $j \equiv 1,2,3 \pmod 6$ and $f^*(v_jv_{j+1}) = 0$ if $j \equiv 4,5,0 \pmod 6$, where $1 \leq j \leq 6k+1$. Hence $e_1 = 3k+1$ and $e_0 = 3k$.

e) Suppose $n = 6k+1$, $k \geq 1$. Use the labeling defined in Case d) and delete the vertex v_{6k+2} then we get $e_1 = 3k$ and $e_0 = 3k$.

f) Suppose $n = 6k$, $k \geq 1$. Use the labeling defined in Case d) and delete the vertices v_{6k+2} and v_{6k+1} then we get $e_1 = 3k$ and $e_0 = 3k-1$.

All the above cases show that the path P_n admits a bijective product square 3-cordial labeling. Therefore, the path P_n is a bijective product square 3-cordial graph for all $n \geq 2$.

Example 1. Consider P_{17}. By Theorem 8 the sequence of the labels of vertices of P_{17} is 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 15, 16, 17, and then the sequence of induced labels for edges is 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, which is in the natural order.

Thus the sequences of the labels of vertices of P_{16} and P_{15} are 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 15, 16 and 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 15, respectively.

Consider P_{14}. By Theorem 8 the sequence of the labels of vertices of P_{14} is 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12, 13, 14, and then the sequence of induced labels for edges is 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, which is in the natural order.

Hence the sequences of the labels of vertices of P_{13} and P_{12} are 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12, 13 and 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12, respectively.

Next, we prove that the cycle C_n is a bijective product square 3-cordial graph for all $n \geq 3$.

Theorem 9. *The cycle C_n is a bijective product square 3-cordial graph for all $n \geq 3$.*

*Proof.*Let f be the bijective product square 3-cordial labeling of P_n defined in Theorem 8. We use the same labeling f for the vertices of $C_n = v_1v_2\cdots v_nv_1$. Then we have

a) Suppose $n = 6k+5$, $k \geq 0$. $e_1 = 3k+3$ and $e_0 = 3k+2$.
b) Suppose $n = 6k+4$, $k \geq 0$. $e_1 = 3k+2$ and $e_0 = 3k+2$.
c) Suppose $n = 6k+3$, $k \geq 0$. $e_1 = 3k+1$ and $e_0 = 3k+2$.
d) Suppose $n = 6k+2$, $k \geq 0$.
We redefine $f(v_{6k+2}) = 6k$ and $f(v_{6k}) = 6k+2$. Then we have $e_1 = 3k+1$ and $e_0 = 3k+1$.
e) Suppose $n = 6k+1$, $k \geq 1$. $e_1 = 3k+1$ and $e_0 = 3k$.
f) Suppose $n = 6k$, $k \geq 1$. $e_1 = 3k$ and $e_0 = 3k$.

This completes the proof.

*Example 2.*By Theorem 9,

- ❖ the sequence of the labels of vertices of C_{17} is 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 15, 16, 17, and then the sequence of induced labels for edges is 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, which is in the natural order;
- ❖ the sequences of the labels of vertices of C_{16} and C_{15} are 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 15, 16 and 1, 8, 2, 9, 3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 15, respectively;
- ❖ the sequence of the labels of vertices of C_{14} is 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 14, 13, 12, and then the sequence of induced labels for edges is 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, which is in the natural order;
- ❖ the sequences of the labels of vertices of C_{13} and C_{12} are 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12, 13 and 1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12, respectively.

In the following proofs, $a \equiv b$ means $a \equiv b \pmod 3$ and $a \not\equiv b$ means $a \not\equiv b \pmod 3$. Also note that $i^2 \equiv 1$ if and only if $i \not\equiv 0$ or equivalently $i^3 \equiv i$ for all $i \in \mathbb{Z}$. In Theorem 10, we show that the helm graph H_n is a bijective product square 3-cordial graph for all $n \geq 3$.

Theorem 10.*The helm graph H_n is a bijective product square 3-cordial graph for all $n \geq 3$.*

*Proof.*We define a bijective product square 3-cordial labeling f for H_n.
a) $n = 6k+3$, $k \geq 0$. Define $f(w) = 12k+7$; $f(u_i) = i$ for $1 \leq i \leq 6k+3$; and

$$f(v_i) = \begin{cases} 9k+3+i, & \text{if } 1 \leq i \leq 3k+3, \\ 12k+7-i, & \text{if } 3k+4 \leq i \leq 6k+3. \end{cases}$$

Clearly f is bijective. Now, for $1 \leq i \leq 6k+3$,

$$f^*(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 1, \\ 0, & \text{if } i \not\equiv 1, \end{cases} \text{(including } i = 6k+3).$$

$$f^*(wu_i) = \begin{cases} 1, & \text{if } i \not\equiv 0, \\ 0, & \text{if } i \equiv 0. \end{cases}$$

$$f^*(u_iv_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 0, & \text{if } i \equiv 1 \text{ and } 3k+4 \leq i \leq 6k+3, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $e_1 = 9k+5$ and $e_0 = 9k+4$. So f is a bijective product square 3-cordial labeling.
b) $n = 6k$, $k \geq 1$. Define $f(w) = 12k+1$; $f(u_i) = i$ for $1 \leq i \leq 6k$; and

$$f(v_i) = \begin{cases} 9k+i, & \text{if } 1 \leq i \leq 3k, \\ 12k+1-i, & \text{if } 3k+1 \leq i \leq 6k. \end{cases}$$

Clearly f is bijective. Now $f^*(u_iu_{i+1})$ and $f^*(wu_i)$ have the same values in case a).

$$f^*(u_iv_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 0, & \text{if } i \equiv 1 \text{ and } 3k+1 \leq i \leq 6k, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $e_1 = e_0 = 9k$. So f is a bijective product square 3-cordial labeling.

c) $n = 6k+1$, $k \geq 1$. Define $f(w) = 12k+2$; $f(u_i) = i$ for $1 \leq i \leq 6k+1$; and

$$f(v_i) = \begin{cases} 6k+1+i, & \text{if } 1 \leq i \leq 3k+1, \\ 15k+3-i, & \text{if } 3k+2 \leq i \leq 6k, \\ 12k+3, & \text{if } i = 6k+1. \end{cases}$$

Clearly f is bijective. Hence, we have $e_1 = 9k+2$ and $e_0 = 9k+1$. So f is a bijective product square 3-cordial labeling.

d) $n = 6k+4$. For $k = 0$, define $f(u_i) = i$ for $1 \leq i \leq 4$, $f(w) = 8$ and $f(v_j) = 4+j$ for $1 \leq j \leq 3$ and $f(v_4) = 9$. Hence $e_0 = e_1 = 6$.
For $k \geq 1$. Define $f(w) = 12k+8$; $f(u_i) = i$ for $1 \leq i \leq 6k+4$; and

$$f(v_i) = \begin{cases} 6k+4+i, & \text{if } 1 \leq i \leq 3k+4, \\ 15k+12-i, & \text{if } 3k+5 \leq i \leq 6k+3, \\ 12k+9, & \text{if } i = 6k+4. \end{cases}$$

Clearly f is bijective. Hence, we have $e_1 = e_0 = 9k+6$. So f is a bijective product square 3-cordial labeling.

e) $n = 6k+5$, $k \geq 0$.
Suppose $k = 0$. Figure 1 shows a bijective labeling for H_5. Clearly, $e_1 = 7$ and $e_0 = 8$. Hence it is bijective product square 3-cordial labeling for H_5.

Suppose $k \geq 1$. Define $f(w) = 12k+11$; $f(u_i) = i$ for $1 \leq i \leq 6k+4$; $f(u_{6k+5}) = 6k+6$ and

$$f(v_i) = \begin{cases} 6k+6+i, & \text{if } 1 \leq i \leq 3k+8, \\ 15k+19-i, & \text{if } 3k+9 \leq i \leq 6k+4, \\ 6k+5, & \text{if } i = 6k+5. \end{cases}$$

Clearly f is bijective. Hence, we have $e_1 = 9k+8$ and $e_0 = 9k+7$. So f is a bijective product square 3-cordial labeling.

f) $n = 6k+2$, $k \geq 1$.
Suppose $k = 1$. Figure 1 shows a bijective labeling for H_8.

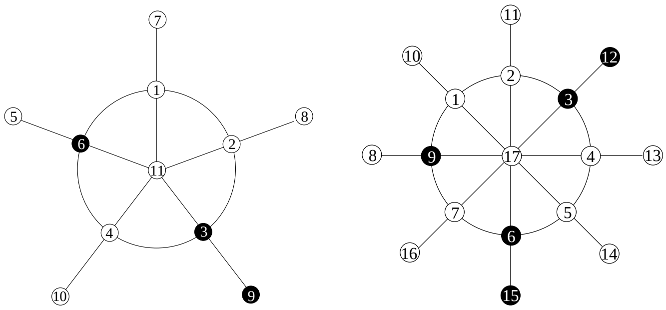


Fig. 1: Bijective product square 3-cordial labelings for H_5 and H_8.

Clearly, $e_1 = 12$ and $e_0 = 12$. Hence it is bijective product square 3-cordial labeling for H_8.

Suppose $k \geq 2$. Define $f(w) = 12k+5$; $f(u_i) = i$ for $1 \leq i \leq 6k+1$; $f(u_{6k+2}) = 6k+3$ and

$$f(v_i) = \begin{cases} 6k+3+i, & \text{if } 1 \leq i \leq 3k+5, \\ 15k+10-i, & \text{if } 3k+6 \leq i \leq 6k+1, \\ 6k+2, & \text{if } i = 6k+2. \end{cases}$$

Clearly f is bijective. Hence, we have $e_1 = e_0 = 9k+3$. So f is a bijective product square 3-cordial labeling.

Our next result shows that the flower graph Fl_n is a bijective product square 3-cordial graph for all $n \geq 3$.

Theorem 11. *The flower graph Fl_n is a bijective product square 3-cordial graph for all $n \geq 3$.*

Proof. We define a bijective product square 3-cordial labeling f for Fl_n.

a) $n = 3k$, $k \geq 1$. Define $f(w) = 6k+1$; $f(u_i) = i$ for $1 \leq i \leq 3k$; and $f(v_i) = 6k+1-i$. Clearly f is bijective. Now, for $1 \leq i \leq 3k$,

$$f^*(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 1, \\ 0, & \text{if } i \not\equiv 1; \end{cases} \text{ (including } i = 3k).$$

$$f^*(wu_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 1, & \text{if } i \not\equiv 0. \end{cases}$$

$$f^*(wv_i) = \begin{cases} 0, & \text{if } i \equiv 1, \\ 1, & \text{if } i \not\equiv 1. \end{cases}$$

$$f^*(u_iv_i) = \begin{cases} 1, & \text{if } i \equiv 2, \\ 0, & \text{if } i \not\equiv 2. \end{cases}$$

Thus, $e_1 = e_0 = 6k$. Thus f is a bijective product square 3-cordial labeling.

b) $n = 3k+1$, $k \geq 1$. Define $f(w) = 6k+2$; $f(u_i) = i$ for $1 \leq i \leq 3k+1$; $f(v_j) = 3k+1+j$ for $1 \leq j \leq 3k$ and $f(v_{3k+1}) = 6k+3$. Clearly f is bijective. Now, for $1 \leq i \leq 3k$,

$$f^*(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 1, \\ 0, & \text{if } i \not\equiv 1. \end{cases}$$

$$f^*(wu_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 1, & \text{if } i \not\equiv 0. \end{cases}$$

$$f^*(wv_i) = \begin{cases} 0, & \text{if } i \equiv 2, \\ 1, & \text{if } i \not\equiv 2. \end{cases}$$

$$f^*(u_iv_i) = \begin{cases} 1, & \text{if } i \equiv 1, \\ 0, & \text{if } i \not\equiv 1. \end{cases}$$

For $i = 3k+1$, $f^*(u_{3k+1}u_1) \equiv 1$; $f^*(wu_{3k+1}) = [(6k+2)(3k+1)]^2 \equiv 1$; $f^*(wv_{3k+1}) \equiv 0$ and $f^*(u_{3k+1}v_{3k+1}) \equiv 0$. Hence, we have $e_1 = e_0 = 6k+2$. Thus f is a bijective product square 3-cordial labeling.

c) $n = 3k+2$, $k \geq 1$. Define $f(w) = 6k+5$; $f(u_i) = i$ for $1 \leq i \leq 3k+1$; $f(u_{3k+2}) = 3k+3$; $f(v_j) = 3k+2+j$ for $2 \leq j \leq 3k+1$; $f(v_1) = 6k+4$ and $f(v_{3k+2}) = 3k+2$. Clearly f is bijective. Now, for $2 \leq i \leq 3k$,

$$f^*(u_iu_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 1, \\ 0, & \text{if } i \not\equiv 1. \end{cases}$$

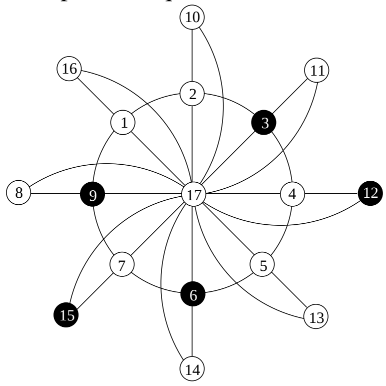
$$f^*(wu_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 1, & \text{if } i \not\equiv 0. \end{cases}$$

$$f^*(wv_i) = \begin{cases} 0, & \text{if } i \equiv 1, \\ 1, & \text{if } i \not\equiv 1. \end{cases}$$

$$f^*(u_iv_i) = \begin{cases} 1, & \text{if } i \equiv 2, \\ 0, & \text{if } i \not\equiv 2. \end{cases}$$

For $i = 1, 3k+1, 3k+2$,
$f^*(u_1u_2) \equiv 1$; $f^*(wu_1) \equiv 1$; $f^*(wv_1) \equiv 1$; $f^*(u_1v_1) \equiv 1$;
$f^*(u_{3k+1}u_{3k+2}) \equiv 0$; $f^*(wu_{3k+1}) \equiv 1$; $f^*(wv_{3k+1}) \equiv 0$; $f^*(u_{3k+1}v_{3k+1}) \equiv 0$;
$f^*(u_{3k+2}u_1) \equiv 0$; $f^*(wu_{3k+2}) \equiv 0$; $f^*(wv_{3k+2}) \equiv 1$; $f^*(u_{3k+2}v_{3k+2}) \equiv 0$.
Hence, we have $e_1 = e_0 = 6k+4$. Hence f is a bijective product square 3-cordial labeling.

Example 3. By Theorem 11, we have a bijective product square 3-cordial labeling for Fl_8 as follows:



In the subsequent theorem we prove that the gear graph G_n is a bijective product square 3-cordial graph for all $n \geq 3$.

Theorem 12. *The gear graph G_n is a bijective product square 3-cordial graph for all $n \geq 3$.*

Proof. We define a bijective product square 3-cordial labeling f for G_n.

a) $n = 3$. Define $f(w) = 7$; $f(u_i) = i$ and $f(x_i) = 3+i$ for $1 \leq i \leq 3$. It is easy to check that f is a bijective product square 3-cordial labeling for G_3.

b) $n = 6k+3$, $k \geq 1$. Define $f(w) = 12k+7$; $f(u_i) = i$ for $1 \leq i \leq 6k+3$;

$$f(x_i) = \begin{cases} 6k+3+i, & \text{if } 1 \leq i \leq 3k+3, \\ 15k+10-i, & \text{if } 3k+4 \leq i \leq 6k+3. \end{cases}$$

Clearly f is bijective. Now, for $1 \leq i \leq 6k+3$,

$$f^*(wu_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 1, & \text{if } i \not\equiv 0. \end{cases}$$

$$f^*(u_ix_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 0, & \text{if } i \equiv 1 \text{ and } 3k+4 \leq i \leq 6k+3, \\ 1, & \text{otherwise.} \end{cases}$$

$$f^*(x_iu_{i+1}) = \begin{cases} 0, & \text{if } i \equiv 0 \text{ and } 1 \leq i \leq 3k+3, \\ 0, & \text{if } i \equiv 1 \text{ and } 3k+4 \leq i \leq 6k+2, \\ 0, & \text{if } i \equiv 2, \\ 1, & \text{otherwise.} \end{cases}$$

We have $e_0 = 9k+4$ and $e_1 = 9k+5$. Thus f is a bijective product square 3-cordial labeling.

c) $n = 6k$, $k \geq 1$. Define $f(w) = 12k+1$; $f(u_i) = i$ for $1 \leq i \leq 6k$;

$$f(x_i) = \begin{cases} 6k+i, & \text{if } 1 \leq i \leq 3k, \\ 15k+1-i, & \text{if } 3k+1 \leq i \leq 6k. \end{cases}$$

Clearly f is bijective. Hence, we have $e_0 = e_1 = 9k$. Thus f is a bijective product square 3-cordial labeling.

d) $n = 6k+1$, $k \geq 1$. Define $f(w) = 12k+2$; $f(u_i) = i$ for $1 \leq i \leq 6k+1$;

$$f(x_i) = \begin{cases} 6k+1+i, & \text{if } 1 \leq i \leq 3k-1, \\ 15k+1-i, & \text{if } 3k \leq i \leq 6k, \\ 12k+3, & \text{if } i = 6k+1. \end{cases}$$

Clearly f is bijective. Hence, we have $e_0 = 9k+2$ and $e_1 = 9k+1$. Thus f is a bijective product square 3-cordial labeling.

e) $n = 6k+4$, $k \geq 0$. Define $f(w) = 12k+8$; $f(u_i) = i$ for $1 \leq i \leq 6k+4$;

$$f(x_i) = \begin{cases} 6k+4+i, & \text{if } 1 \leq i \leq 3k+2, \\ 15k+10-i, & \text{if } 3k+3 \leq i \leq 6k+3, \\ 12k+9, & \text{if } i = 6k+4. \end{cases}$$

Clearly f is bijective. Hence, we have $e_0 = e_1 = 9k+6$. Thus f is a bijective product square 3-cordial labeling.
f) $n = 6k+5$, $k \geq 0$. Define $f(w) = 12k+11$;

$$f(u_i) = \begin{cases} i, & \text{if } 2 \leq i \leq 6k+4, \\ 6k+6, & \text{if } i = 1, \\ 1, & \text{if } i = 6k+5. \end{cases}$$

$$f(x_i) = \begin{cases} 6k+6+i, & \text{if } 1 \leq i \leq 3k+2, \\ 15k+13-i, & \text{if } 3k+3 \leq i \leq 6k+4, \\ 6k+5, & \text{if } i = 6k+5. \end{cases}$$

Clearly f is bijective. Now, $f^*(wu_1) \equiv 0$; $f^*(u_1x_1) \equiv 0$; $f^*(x_{6k+5}u_1) \equiv 0$;
$f^*(wu_{6k+5}) \equiv 1$; $f^*(u_{6k+5}x_{6k+5}) \equiv 1$; $f^*(x_{6k+4}u_{6k+5}) \equiv 0$.
For $2 \leq i \leq 6k+4$,

$$f^*(wu_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 1, & \text{otherwise.} \end{cases}$$

$$f^*(u_ix_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 0, & \text{if } i \equiv 1 \text{ and } 3k+3 \leq i \leq 6k+4, \\ 1, & \text{otherwise.} \end{cases}$$

$$f^*(x_{i-1}u_i) = \begin{cases} 0, & \text{if } i \equiv 0, \\ 0, & \text{if } i \equiv 1 \text{ and } 2 \leq i \leq 3k+3, \\ 0, & \text{if } i \equiv 2 \text{ and } 3k+4 \leq i \leq 6k+4, \\ 1, & \text{otherwise.} \end{cases}$$

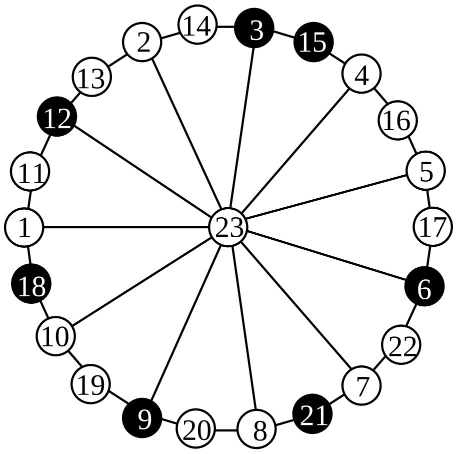
We have $e_0 = 9k+8$ and $e_1 = 9k+7$. Thus f is a bijective product square 3-cordial labeling.
g) $n = 6k+2$, $k \geq 1$. Define $f(w) = 12k+5$;

$$f(u_i) = \begin{cases} i, & \text{if } 2 \leq i \leq 6k+1, \\ 6k+3, & \text{if } i = 1, \\ 1, & \text{if } i = 6k+2. \end{cases}$$

$$f(x_i) = \begin{cases} 6k+3+i, & \text{if } 1 \leq i \leq 3k+2, \\ 15k+7-i, & \text{if } 3k+3 \leq i \leq 6k+1, \\ 6k+2, & \text{if } i = 6k+2. \end{cases}$$

Clearly f is bijective. Hence, we have $e_0 = e_1 = 9k+3$. Thus f is a bijective product square 3-cordial labeling.

*Example 4.*By Theorem 12, we have a bijective product square 3-cordial labeling for G_{11}.



Here $e_0 = 17$ and $e_1 = 16$.

Declarations

Competing interests: The authors declare that they have no conflict of interest.
Authors' contributions: The basic concept was initiated by R. Santrin Sabibha and redrafted by W.C. Shiu. The first draft of the manuscript is prepared by R. Santrin Sabibha and P. Jeyanthi. All the authors have read, checked the analysis part and commented on all the previous versions of the manuscript. Their suggestions for improvement are incorporated in the manuscript and they approved the final manuscript.
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