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Applicability of Petryshyn's Fixed Point Theorem on the Existence of a Solution to Weakly Singular Integral Equations

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Abstract: Various approach have been introduced for the existence of solutions for integral equations but, for the most part, researchers have deal with the Darbo fixed point theorem as an extension of Schauder theorem. Under certain hypotheses, we establish the existence of solution for weakly singular integral equations by employing the generalization of Darbos fixed point theorem and measures of noncompactness in Banach space. Finally, some examples are given and with the help of MATLAB R2018a parameters is finding.

Keywords: Integral equation; Weakly singular integral equation; Petryshyn's fixed point theorem; Measure of noncompactness.

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1 Introduction

Mathematical modeling is the process of creating mathematical representation, or models of real-world phenomena. Many physical phenomena in plasma physics, physical chemistry, geophysics, fluid mechanics, nonlinear optics, electromagnetic theory are expressed by integral equations. Hence, the investigation of the existence of solutions for the both linear and nonlinear integral equations is of prime importance to the academic researches. For this reason, a lot of different techniques have been dealt with by them. One of the most commonly used method is the concept of noncompactness measure. The root of this concept go back to the famous work of Kuratowski [12]. This method plays a vital role in the publications of researches [2]. In 1955, an extension of this direction has been introduced by Italian mathematican Darbo [4] . He studied the existence of fixed point for condensing operators generalizing the Schauder fixed point theorem and Banach contraction princilpe. After this pioneering work, the number of researches dealing with Darbo fixed point theorem has increased considerably in the recent yaers([1,3,5,6,7,13,15,17,18,20]). In 2016, by the assitance of measure of noncompactness and Petryshyn's fixed point theorem Kazemi and Ezzati are established that the sublinear conditions in Darbo fixed point theorem is a additional condition [10]. We employ the idea of Kazemi and Ezzati to the existence of solutions for weakly singular integral equation as follows:

$$\pi(t) = \Lambda\left(t, f(t, \pi(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta} k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda\right), \tag{1}$$

2 A generalization of Darbos fixed point theorem

This section is devoted to collect some definitions and theorems which will be needed further on.
Assume that $(E, \|.\|)$ is a Banach space over R. Let $\overline{B}_\alpha$ denotes the closed ball, that is,

$$\overline{B}_\alpha = \{x \in E : \|x\| \leq \alpha\} \; and \; \partial\overline{B}_\alpha = \{x \in E : \|x\| = \alpha\}.$$

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In what follows we will work in the Banach space $C([0,a])$ consisting of all real functions defined and continuous on $[0,a]$. Now, we collect the construction of the measure of noncompactness which will be used in this paper.
Let us fix a nonempty and bounded subset T of $C([0,a])$. The Kuratowski measure of noncompactness[12] is

$$\phi(\mathrm{T}) = inf\left\{\rho > 0 : \mathrm{T} \subset \bigcup_{i=1}^{n} \mathrm{T}_i, Q_i \subset \mathbb{X}, diam(\mathrm{T}_i) < \rho, i = 1,\ldots,n\right\}. \tag{2}$$

Also, the ball measure of noncompactness[9] is

$$\mu(\mathrm{T}) = inf\left\{\rho > 0 : \mathrm{T} \subset \bigcup_{i=1}^{n} B_{\alpha_i}, \alpha_i < \rho\right\}. \tag{3}$$

This measures of noncompactness are mutually equvalent in the sense that $\mu(T) \leq \phi(T) \leq \mu(T)$.
For $\psi \in C([0,a])$ and $\rho \geq 0$ denoted by $\partial(\psi,\rho)$, the modulus of continuity of the function ψ, i.e.,

$$\partial(\psi,\rho) = sup\{|\psi(t) - \psi(\hat{t})| : |t - \hat{t}| \leq \rho\}.$$

The uniformly continuous ψ on $[0,a]$ implies that $\partial(\psi,\rho) \to 0$ as $\rho \to 0$.

Proposition 1.[16] *Let $T,\hat{T} \subset E$ then*

1.$\mu(T\bigcup\hat{T}) = max\left\{\mu(T),\mu(\hat{T})\right\}$;
2.$\mu(T+\hat{T}) \leq \mu(T) + \pi(\hat{T})$;
3.$\mu(\lambda T) = |\lambda|\mu(T)$, where $\lambda T = \left\{\lambda m : m \in T\right\}$;
4.$\mu(T) \leq \mu(\hat{T})$, for $T \subset \hat{T}$;
5.$\mu(\overline{co}T) = \mu(T)$;

Proposition 2.[11] *For all bounded subset $T \subset [0,a]$ the measures of noncompactness (2) are equivalent to $\mu(T) = limsup_{\rho\to 0}\partial(\psi,\rho), \quad \psi \in T$.*

Let $\Xi \in C(E)$. Ξ is called a κ-set contraction if for all bounded subset $K \subset E$, Ξ is bounded and $\phi(\Xi K) \leq \kappa\phi(K)$ for all $0 < \kappa < 1$. If $\phi(\Xi K) \leq \phi(K)$ for all $\phi(K) > 0$, then Ξ is called condensing map. In fact, a condensing mapping is a mapping for which the image of any set in a certain sense more compact than the set itself, the degree of noncompactness of a set is measured by means of functions called measure of noncompactness [8].
Let us recall the following important result, which is called Petryshyn's fixed point theorem:

Proposition 3.[16] *Let $\Xi : B_\alpha \to E$ to be a condensing mapping which satisfying the boundary condition, if $\Xi(x) = \kappa x$, for some $x \in \partial B_\alpha$, then $\kappa \leq 1$, then $F(\Xi) \neq \emptyset$.*

3 An existence theorem

In this section we denote some notations as follows:

–$M = sup\left\{\left|k(t,\lambda,\pi\right|) : t,\lambda \in [0,a], \pi \in [-\alpha,\alpha]\right\}$,
–$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is called beta function, one can easily to show that
$= \int_0^t \frac{s^\gamma}{(t^\sigma - s^\sigma)^\alpha} ds = \frac{1}{\sigma}B(1-\alpha, \frac{1+\gamma}{\sigma})t^{1+\gamma-\sigma\alpha}$,
–$\partial(\Lambda,\rho) = sup\left\{\left|\Lambda(t,\lambda,\pi) - \Lambda(\hat{t},\lambda,\pi)\right| : |\hat{t} - t| \leq \rho, \lambda \in [-\alpha,\alpha], |\pi| \leq Mt^{1+\beta-\zeta}B(1-\zeta,1+\beta)\right\}$,
–$\partial(k,\rho) = sup\left\{\left|k(t,\lambda,\pi) - k(t,\lambda,\hat{\pi})\right| : t,\lambda \in [0,a], \pi,\hat{\pi} \in [-\alpha,\alpha], |\pi - \hat{\pi}| \leq \rho\right\}$,
–$\hat{\partial}(k,\rho) = sup\left\{\left|k(t,\lambda,\pi) - k(\hat{t},\lambda,\pi)\right| : t,\hat{t},\lambda \in [0,a], \pi \in [-\alpha,\alpha], |t - \hat{t}| \leq \rho\right\}$,
–$\partial(f,\rho) = sup\left\{\left|f(t,\pi) - f(\hat{t},\pi)\right| : |\hat{t} - t| \leq \rho, t,\hat{t} \in [0,a], \pi \in [-\alpha,\alpha]\right\}$.

To set our main results we introduce the following assumptions:

$(\mathbf{A1}): \Lambda \in C([0,a]\times R\times R,R), f\in C([0,a]\times R,R), k\in C([0,a]\times[0,a]\times R,R) and\ \ \theta,\hat{\theta}:[0,a]\to[0,a]$ $is\ \ continous,$

$(\mathbf{A2}): \left|\Lambda(t,\lambda,\pi)-\Lambda(t,\hat{\lambda},\hat{\pi})\right| \le m_1\left|\lambda-\hat{\lambda}\right| + m_2\left|\pi-\hat{\pi}\right|, \quad \left|f(t,x)-f(t,\hat{x})\right| \le \hat{m}_1\left|x-\hat{x}\right|,$

$m_1,m_2,\hat{m}_1\ \ nonnegative\ \ constants\ \ and\ \ m_1\hat{m}_1<1,$

$(\mathbf{A3}):\ sup\left\{\left|\Lambda(t,\lambda,\pi)\right| : t\in[0,a], \lambda\in[-\alpha,\alpha], |\pi|\le Ma^{1+\beta-\zeta}B(1-\zeta,1+\beta)\right\}\le\alpha.$

Proposition 4.*Under control conditions* $(\mathbf{A1})-(\mathbf{A3})$*, the weak singular fractional integral equation (1) has at least one solution in* $C([0,a])$.

*Proof.*Let

$$\begin{cases}\Sigma : E_\alpha \to C([0,a])\\[2ex] (\Sigma\pi)(t) = \Lambda\left(t, f(t,\pi(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda\right),\end{cases}$$

where $E_\alpha = \left\{\pi\in C([0,a]), \|\pi\|\le\alpha\right\}$. We divided the proof into three steps:

Step1. We show that Σ is continuous on E_α. Assume that $\pi,\hat{\pi}\in E_\alpha$ and $\rho>0$ such that $|\pi-\hat{\pi}|<\rho$, we have

$$\left|(\Sigma\pi)(t)-(\Sigma\hat{\pi})(t)\right| =$$

$$\Lambda\left(t, f(t,\pi(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda\right) - \Lambda\left(t, f(t,\hat{\pi}(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\hat{\pi}(\hat{\theta}(\lambda)))d\lambda\right)\Bigg|$$

$$\le \left|\Lambda\left(t, f(t,\pi(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda\right) - \Lambda\left(t, f(t,\hat{\pi}(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda\right)\right|$$

$$+\left|\Lambda\left(t, f(t,\hat{\pi}(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda\right) - \Lambda\left(t, f(t,\hat{\pi}(\theta(t))), \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\hat{\pi}(\hat{\theta}(\lambda)))d\lambda\right)\right|$$

$$\le m_1\left|f(t,\pi(\theta(t)))-f(t,\hat{\pi}(\theta(t)))\right| + m_2\int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}\left|k(t,\lambda,\pi(\hat{\theta}(\lambda)))-k(t,\lambda,\hat{\pi}(\hat{\theta}(\lambda)))\right|d\lambda$$

$$\le m_1\hat{m}_1\left|\pi(\theta(t))-\hat{\pi}(\theta(t))\right| + m_2\partial(k,\rho)\int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}d\lambda$$

$$\le m_1\hat{m}_1\left\|\pi-\hat{\pi}\right\| + m_2\partial(k,\rho)B(1-\zeta,1+\beta)t^{1+\beta-\zeta}$$

$$\le m_1\hat{m}_1\left\|\pi-\hat{\pi}\right\| + m_2\partial(k,\rho)B(1-\zeta,1+\beta)a^{1+\beta-\zeta},$$

since $k(t,\lambda,\pi)$ is uniform continuity on the subset $[0,a]\times[0,a]\times[-\alpha,\alpha]$ then $lim_{\rho\to 0}\partial(k,\rho)=0$, which implies that the map of Σ is continuous on E_α.

Step 2. We will show that Σ is a condensing map. Let $\Gamma\subset E$ to be a bounded set. For $\rho>0$ and $\pi\in\Gamma$ and $t_1,t_2\in[0,a]$ with $t_2-t_1\le\rho$. We get:

$$\Big|(\Sigma\pi)(t_2)-(\Sigma\pi)(t_1)\Big|=$$

$$\Lambda\Bigg(t,f(t_2,\pi(\theta(t_2))),\int_0^{t_2}\tfrac{\lambda^\beta}{(t_2-\lambda)^\zeta}k(t_2,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)-\Lambda\Bigg(t_1,f(t_1,\pi(\theta(t_1))),\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)\Bigg|$$

$$\leq\Bigg|\Lambda\Bigg(t,f(t_2,\pi(\theta(t_2))),\int_0^{t_2}\tfrac{\lambda^\beta}{(t_2-\lambda)^\zeta}k(t_2,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)-\Lambda\Bigg(t,f(t_2,\pi(\theta(t_2))),\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)\Bigg|$$

$$+\Bigg|\Lambda\Bigg(t,f(t_2,\pi(\theta(t_2))),\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)-\Lambda\Bigg(t,f(t_2,\pi(\theta(t_1))),\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)\Bigg|$$

$$+\Bigg|\Lambda\Bigg(t,f(t_2,\pi(\theta(t_1))),\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)-\Lambda\Bigg(t,f(t_1,\pi(\theta(t_1))),\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)\Bigg|$$

$$\leq m_2\Bigg|\int_0^{t_2}\tfrac{\lambda^\beta}{(t_2-\lambda)^\zeta}k(t_2,\lambda,\pi(\hat\theta(\lambda)))d\lambda-\int_0^{t_1}\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg|$$

$$+m_1\Bigg|f(t_2,\pi(\theta(t_2)))-f(t_1,\pi(\theta(t_1)))\Bigg|+\partial(f,\rho)$$

$$\leq m_2\Bigg\{\int_0^{t_1}\tfrac{\lambda^\beta}{(t_2-\lambda)^\zeta}\Bigg|k(t_2,\lambda,\pi(\hat\theta(\lambda)))-k(t_1,\lambda,\pi(\hat\theta(\lambda)))\Bigg|d\lambda$$

$$+\int_0^{t_1}\Bigg|\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))-\tfrac{\lambda^\beta}{(t_1-\lambda)^\zeta}k(t_1,\lambda,\pi(\hat\theta(\lambda)))\Bigg|d\lambda+\int_{t_1}^{t_2}\Bigg|\tfrac{\lambda^\beta}{(t_2-\lambda)^\zeta}k(t_2,\lambda,\pi(\hat\theta(\lambda)))\Bigg|d\lambda\Bigg\}$$

$$+m_1\Bigg|f(t_2,\pi(\theta(t_2)))-f(t_1,\pi(\theta(t_1)))\Bigg|+\partial(f,\rho)$$

$$\leq m_2\hat\partial(k,\rho)B(1-\zeta,1+\beta)t_1^{1+\beta-\zeta}+m_2\,M\,B(1-\zeta,1+\beta)\Bigg\{t_2^{1+\beta-\zeta}-t_1^{1+\beta-\zeta}\Bigg\}$$

$$+m_1\Bigg|f(t_2,\pi(\theta(t_2)))-f(t_2,\pi(\theta(t_1)))\Bigg|+m_1\Bigg|f(t_2,\pi(\theta(t_1)))-f(t_1,\pi(\theta(t_1)))\Bigg|+\partial(f,\rho)$$

$$\leq m_2\hat\partial(k,\rho)B(1-\zeta,1+\beta)a^{1+\beta-\zeta}+2\,m_2\,M\,B(1-\zeta,1+\beta)\rho^{1+\beta-\zeta}+m_1\hat m_1\Bigg|\pi(\theta(t))-\pi(\theta(t_1))\Bigg|$$

$$+m_1\hat\partial(f,\rho)+\partial(f,\rho)$$

$$\leq m_2\hat\partial(k,\rho)B(1-\zeta,1+\beta)a^{1+\beta-\zeta}+2\,m_2\,M\,B(1-\zeta,1+\beta)\rho^{1+\beta-\zeta}$$

$$+m_1\hat m_1\partial\Bigg(u,\partial(\beta,\rho)\Bigg)+m_1\hat\partial(f,\rho)+\partial(f,\rho).$$

Thus $\partial\big(\Sigma\pi,\rho\big)\leq m_1\hat m_1\partial\big(\pi,\rho\big)$ as $\rho\to 0$. This implies that $\mu\big(\Sigma\Gamma\big)\leq m_1\hat m_1\mu\big(\hat\theta\big)$. Thus, Σ is a condensing map.

Step 3. Finally, we investigation of Petryshyn conditions. Let $\pi\in\partial E_\alpha$, if $\Sigma\pi=\hat k\pi$ then $\|\Sigma\pi\|=\hat k\ \|\pi\|=\hat k\alpha$. The condition(A1) implies that

$$\Bigg|\Lambda\pi(t)\Bigg|=\Bigg|\Lambda\Bigg(t,f(t,\pi(\theta(t))),\int_0^t\frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat\theta(\lambda)))d\lambda\Bigg)\Bigg|\leq\alpha,$$

for any $t\in[0,a]$. Thus, $\|\Sigma\pi\|\leq\alpha$, so this show $\hat k\leq 1$. This completes the proof.

Proposition 5. *Let m_1, m_2 to be nonnegativeconstant and $k \in C([0,a]\times[0,a]\times R, R), f \in C([0,a]\times R, R)$. If*

$(C1): |f(t,\pi)-f(t,\hat{\pi})| \leq m_1\Big|\pi-\hat{\pi}|, |g(t,\pi)-g(t,\hat{\pi})| \leq m_2|\pi-\hat{\pi}|,$

$\Big|f(t,0)\Big| \leq b_1, |g(t,0)\Big| \leq b_2,$

$(C2):$ *There exist nonnegative constants* c_1, c_2 *such that* $\Big|k(t,\lambda,\pi)| \leq c_1 + c_2|\pi|$

for $t,\lambda \in [0,a], \pi \in R,$

$C3): \; m_1 + b_1 + (m_2+b_2)(c_1+c_2)a^{1+\beta-\zeta}B(1-\zeta, 1+\beta) < 1.$

Then

$$\pi(t) = f(t,\pi(\theta(t))) + g(t,\pi(\beta(t)))\int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda, \tag{4}$$

has at least one solution in $C([0,a])$.

Proof. Let $\Lambda(t,s,v,w) = f(t,s)+vw$, where $v = g(t,\pi(\beta(t)))$ and $w = \int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda$. For $\|\pi\| \leq \alpha$ we have

$$\begin{aligned}|\pi(t)| &= |f(t,\pi(\theta(t))) + g(t,\pi(\beta(t)))\textstyle\int_0^t \frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))d\lambda| \\ &\leq \Big|f(t,\pi(\theta(t)))-f(t,0)\Big| + |f(t,0)| + \Big(|g(t,\pi(\beta(t)))-g(t,0)| + |g(t,0)|\Big)\textstyle\int_0^t \Big|\frac{\lambda^\beta}{(t-\lambda)^\zeta}k(t,\lambda,\pi(\hat{\theta}(\lambda)))\Big|d\lambda \\ &\leq m_1\|\pi\| + b_1 + (m_2\|\pi\|+b_2)(c_1+c_2\|\pi\|)a^{1+\beta-\zeta}B(1-\zeta,1+\beta),\end{aligned}$$

This implies that

$$m_1\alpha + b_1 + (m_2\alpha+b_2)(c_1+c_2\alpha)a^{1+\beta-\zeta}B(1-\zeta,1+\beta) \leq \alpha. \tag{5}$$

Using mean value theorem, one can easily prove that the inequality (5) has a solution in $(0,1)$. Therefore the condition (A2) is obtain. On the other hand, the controls $(C1),(C2)$ implies that $(A3)$ is holds.

4 Examples

In this section, some examples have been presented using proposition (4). With the help of MATLAB R2018a, we obtain the parameter α.

Example 1. Consider the following weakly singular integral equation

$$\pi(t) = \frac{t^4\pi(\sqrt{t})}{6(1+t^4)} + \frac{sint}{8(e^{t^2}+4cos\sqrt{t})}\int_0^t \frac{\lambda^{\frac{1}{4}}}{(t-\lambda)^{\frac{3}{4}}} \times \frac{1+cos\sqrt{\lambda}+|\pi(\sqrt{\lambda})|}{1+\lambda t^2+lnt}d\lambda, \quad t\in[0,1] \tag{6}$$

the assumptions (A1) and (A2) of proposition (4) are satisfied. Now, we check that (A3) also holds. Suppose that $\|\pi(t)\| \leq r$, then

$$|\pi(t)| = \left|\frac{t^4\pi(\sqrt{t})}{6(1+t^4)} + \frac{sint}{8(e^{t^2}+4cos\sqrt{t}}\int_0^t \frac{\lambda^{\frac{1}{4}}}{(t-\lambda)^{\frac{3}{4}}} \times \frac{1+cos\sqrt{\lambda}+|\pi(\sqrt{\lambda})|}{1+\lambda t^2+lnt}\right|d\lambda \leq r.$$

The (A3) holds if,

$$\frac{1}{6}r + \frac{1}{8}B(\frac{1}{4},\frac{5}{4})(2+r) \leq r,$$

By choosing $r \geq 1.3483$ the condition (A3) holds. This implies that the equation (**??**) has at least one solution in $C[0,1]$.

Example 2. In 1971, Miller [14] introduced a nonlinear integral equation as follow:

$$\pi(t) = 1 - \frac{\sqrt{3}}{\pi}\int_0^t \frac{\lambda^{\frac{1}{3}}\pi(t)^4}{(t-\lambda)^{\frac{2}{3}}}d\lambda. \tag{7}$$

Now, by using theorem 4 we will prove that the equation of 7 has a one solution. Suppose that $\|\pi(t)\| \leq \alpha$, then

$$|\pi(t)| = \left|1 - \frac{\sqrt{3}}{\pi} \int_0^t \frac{\lambda^{\frac{1}{3}} \pi(t)^4}{(t-\lambda)^{\frac{2}{3}}}\right| d\lambda \leq \alpha,$$

hence,

$$1 + \frac{\sqrt{3}}{\pi} \alpha B(\frac{1}{3}, \frac{4}{3}) \leq \alpha.$$

By choosing $\alpha \geq 2.169098$ the condition (A3) holds. This implies that 7 has at least one solution in $C[0,1]$.

5 Conclusion

The basic tools used in this paper is the techniques of measure of non compactness and Petryshyn's fixed point theorem which is a generalization of Darbo's Fixed Point theorem. In this work, we studied the existence of solutions for weakly singular integral equations by Petryshyns fixed point theorem. By using Petryshyns fixed point theorem, it is not necessary to verify that the involved operator maps a closed convex subset onto itself. Finally, some examples are given

Declarations

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