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On a class of Caputo modified fractional differential equations with advanced arguments

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Abstract: In this paper by using Banach's contraction principle, we prove the existence and uniqueness of solutions for a class of fractional differential equations with advanced arguments has a unique solution. Moreover, we give several examples illustrating the application of our results.

Keywords: Liouville modified fractional Integral; Caputo modified fractional derivative; advanced arguments; contraction principle; successive iteration.

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1 Introduction

The focus of this paper is to study the existence of solutions to the problem

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = f\left(x, y(x), y(\theta(x))\right), x \in [1, +\infty), \\ \lim\limits_{x \to +\infty} y(x) = L_{\infty}, \end{cases} \tag{1}$$

where ${}^{C}D_{-}^{\alpha}$ is the Caputo modified fractional derivative of order α with $0 < \alpha < 1$, $f : [1, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$, $\theta : [1, +\infty) \to [1, +\infty)$ continuous with $\theta(x) \geq x$, for all $x \geq 1$ and $L_{\infty} \in \mathbb{R}$.

Differential equations with advanced arguments arises in the light absorption in an interstellar medium, biology and theoretical physics (see [1], [2], [6], [18]).

On the other hand fractional differential equations occur in many areas like viscoelasticity, anelasticity, electroanalytical chemistry, electromagnetic theory, medicine and those kinds of things (see [3], [4], [9], [12], [13], [15] and the references cited in [20]).

Fractional differential equations with Liouville modified fractional derivative on infinite intervals have been studied only by [9] and [5] using fixed point theorems.

The purpose of this work is to show the existence and uniqueness of solutions for the problem (1) and therefore our result improve and generalize certain results obtained in [5] and [9]. We also note that, as far as we know, this is the first paper which study the existence and uniqueness of solutions for problems of type (1).

This paper is organized as follows: Section 2 provides some definitions and preliminary results that will be used throughout the rest of this manuscript. In Section 3, we present and prove the main result of this work. In Section 4, we give several examples and lastly in Section 5 we give a conclusion.

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2 Preliminary results

Definition 1. *The equation in (1) is called a fractional differential equation with advanced argument because $\theta(x) \geq x$, for all $x \geq 1$.*

Definition 2(See [16, Chapter 1 page 21]). *For $\beta > 0$, we define the Mittag-Leffler function E_β by*

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}, \tag{2}$$

where $z \in \mathbb{R}$ and Γ is the Euler gamma function.

Definition 3(See [16, Chapter 1 page 21]). *For $\beta_1 > 0$ and $\beta_2 > 0$, we define the Mittag-Leffler function E_{β_1,β_2} by*

$$E_{\beta_1,\beta_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta_2+\beta_1 n)}, \tag{3}$$

where $z \in \mathbb{R}$.

Definition 4. *([17]) The deformed Mittag-Leffler function F_β is defined by*

$$F_\beta(z_1,z_2) = \sum_{k=0}^{+\infty} \frac{z_1^k}{\Gamma(1+\beta k)} z_2^{\frac{k(k-1)}{2}}, \tag{4}$$

where $\beta > 0$, $z_1 \in \mathbb{R}$ and $z_2 \in [-1,1]$.

Definition 5. *For $\beta_1 > 0$ and $\beta_2 > 0$, we define the deformed Mittag-Leffler function F_{β_1,β_2} by*

$$F_{\beta_1,\beta_2}(z_1,z_2) = \sum_{k=0}^{+\infty} \frac{z_1^k}{\Gamma(\beta_2+\beta_1 k)} z_2^{\frac{k(k+1)}{2}\alpha-k}, \tag{5}$$

where $z_1 \in \mathbb{R}$ and $z_2 \in [-1,1]$.

Remark. To the best of our knowledge this is the first work which done the definition of deformed Mittag-Leffler function F_{β_1,β_2}.

Definition 6(See [8] and [10, Chapter 1 page 38]). *The function $\lambda_{\gamma,\sigma}^{(\varepsilon)}$ is defined by*

$$\lambda_{\gamma,\sigma}^{(\varepsilon)}(z) = \frac{\varepsilon}{\Gamma\left(1+\gamma-\frac{1}{\varepsilon}\right)} \int_1^{+\infty} (t^\varepsilon - 1)^{\gamma-\frac{1}{\varepsilon}} t^\sigma e^{-zt} dt, \tag{6}$$

where $\sigma \in \mathbb{R}$, $\gamma > \frac{1}{\varepsilon} - 1$ with $\varepsilon > 0$ and $z > 0$.

Definition 7(See [9]). *Let $g : [1,+\infty) \to \mathbb{R}$ be a function and let $\alpha > 0$. We define the Liouville modified fractional Integral of order α of g by*

$$(J_-^\alpha g)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{g(\tau)}{\tau^2} d\tau. \tag{7}$$

Examples For all $\alpha > 0$, $\beta < 1$, $\mu > 0$ and $x \geq 1$, one has

$$(J_-^\alpha t^\beta)(x) = \frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)} x^{\beta-\alpha}. \tag{8}$$

$$(J_-^\alpha e^{-\mu t})(x) = x^{-\alpha} \lambda_{\alpha,-\alpha-1}^{(1)}(\mu x). \tag{9}$$

$$(J_{-}^{\alpha}E_{\alpha}\left(\mu t^{-\alpha}\right))(x)=\frac{1}{\mu}\left(E_{\alpha}\left(\mu x^{-\alpha}\right)-1\right). \tag{10}$$

Notation ([9, Page 71])**.** We note $\pounds\left(1;+\infty\right):=L_{1}\left(\left[1,+\infty\right);x^{-2}\right)$ that is

$$\pounds\left(1;+\infty\right)=\left\{h:\ \left\|h\right\|_{\pounds\left(1;+\infty\right)}=\int\limits_{1}^{+\infty}\frac{\left|h\left(x\right)\right|}{x^{2}}dx<\infty\right\}. \tag{11}$$

Lemma 1._([9, Lemma 2 page 70]). Modified fractional integration has the semi-group property_

$$J_{-}^{\alpha}J_{-}^{\beta}g=J_{-}^{\alpha+\beta}g, \tag{12}$$

where $\alpha>0$, $\beta>0$ _and_ $g\in\pounds\left(1;+\infty\right)$.

Notation We note $\mathfrak{C}\left(1;+\infty\right)$ and $\mathfrak{C}^{1}\left(1;+\infty\right)$ the following spaces

$$\mathfrak{C}\left(1;+\infty\right)=\left\{h:\left[1,+\infty\right)\rightarrow\mathbb{R}:\ h\text{ continuous and }\lim_{x\rightarrow+\infty}h\left(x\right)\text{ exists and finite}\right\}, \tag{13}$$

and

$$\mathfrak{C}^{1}\left(1;+\infty\right)=\left\{h\in\mathfrak{C}\left(1;+\infty\right),\ h\text{ derivable}\right\}. \tag{14}$$

Note that $\left(\mathfrak{C}\left(1;+\infty\right),\left\|.\right\|_{0}\right)$ and $\left(\mathfrak{C}^{1}\left(1;+\infty\right),\left\|.\right\|_{1}\right)$ are Banach spaces with

$$\left\|h\right\|_{0}=\sup_{x\in\left[1,+\infty\right)}\left|h\left(x\right)\right|, \tag{15}$$

and

$$\left\|h\right\|_{1}=\sup_{x\in\left[1,+\infty\right)}\left|h\left(x\right)\right|+\sup_{x\in\left[1,+\infty\right)}\left|h'\left(x\right)\right|. \tag{16}$$

Lemma 2.. _For all_ $\alpha>0$ _and_ $g\in\mathfrak{C}\left(1;+\infty\right)$, _we have_

$$\left\|J_{-}^{\alpha}g\right\|_{0}\leq\frac{\left\|g\right\|_{0}}{\Gamma\left(1+\alpha\right)}. \tag{17}$$

_Proof._We have

$$\begin{aligned}\left|\left(J_{-}^{\alpha}g\right)\left(x\right)\right|&=\frac{1}{\Gamma\left(\alpha\right)}\left|\int\limits_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-1+\alpha}\frac{g\left(\tau\right)}{\tau^{2}}d\tau\right|\\&\leq\frac{\left\|g\right\|_{0}}{\Gamma\left(\alpha\right)}\int\limits_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-1+\alpha}\frac{d\tau}{\tau^{2}}\\&=\frac{\left\|g\right\|_{0}}{x^{\alpha}\Gamma\left(\alpha+1\right)}\\&\leq\frac{\left\|g\right\|_{0}}{\Gamma\left(\alpha+1\right)};\end{aligned}$$

and consequently, we obtain

$$\left\|J_{-}^{\alpha}g\right\|_{0}\leq\frac{\left\|g\right\|_{0}}{\Gamma\left(1+\alpha\right)}.$$

Definition 8._([9]). Let_ $0<\alpha<1$ _and_ $g:\left[1,+\infty\right)\rightarrow\mathbb{R}$ _be a function. We define Liouville's modified fractional derivative of the_ α _order of_ g _by_

$$\left(D_{-}^{\alpha}g\right)\left(x\right)=-x^{2}\frac{d}{dx}\left(J_{-}^{1-\alpha}g\right)\left(x\right). \tag{18}$$

*Example 1.*For $b \in \mathbb{R}$, we have

$$\begin{aligned} \left(D_{-}^{\alpha} b\right)(x) &= -x^{2} \frac{d}{dx}\left(J_{-}^{1-\alpha} b\right)(x) \\ &= -b x^{2} \frac{d}{dx} \frac{x^{\alpha-1}}{\Gamma(2-\alpha)} \\ &= \frac{b}{\Gamma(1-\alpha)} x^{\alpha}. \end{aligned}$$

Example 2.([9, Page 71]). For $\beta < 1$ with $\alpha + \beta < 1$, one has

$$\left(D_{-}^{\alpha} t^{\beta}\right)(x) = \frac{\Gamma(-\beta+1)}{\Gamma(-\beta-\alpha+1)} x^{\beta+\alpha}. \tag{19}$$

Notation In [9, Page 71], the authors define the space of functions $\mathscr{AC}[1;+\infty)$ as follows

$$\mathscr{AC}[1;+\infty) = \left\{ h : [1;+\infty) \to \mathbb{R} : \ h(x) = c + \int_{x}^{+\infty} \frac{\varphi(t)}{t^{2}} dt \right\}, \tag{20}$$

where c is an arbitrary constant and $\varphi \in \pounds(1;+\infty)$.

Definition 9.*Let $0 < \alpha < 1$ and $g \in \mathscr{AC}[1;+\infty) : [1,+\infty) \to \mathbb{R}$ with $\lim_{x \to +\infty} g(x) = g_{\infty}$. We define Caputo's modified fractional derivative of order α of g by*

$$\begin{aligned} \left({}^{C}D_{-}^{\alpha} g\right)(x) &= \left(D_{-}^{\alpha}\left(g - g_{\infty}\right)\right)(x) \\ &= -\frac{x^{2}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-\alpha} \frac{\left(g(\tau) - g_{\infty}\right)}{\tau^{2}} d\tau. \end{aligned} \tag{21}$$

Lemma 3.*([9, Lemma 5]) Let $0 < \alpha < 1$. The function $g \in \mathscr{AC}[1;+\infty)$ if and only if Qg is absolutely continuous on $[0;1]$, where*

$$(Qg)(x) = g\left(\frac{1}{x}\right), \text{for all } x \geq 1. \tag{22}$$

Definition 10.*([16, Chapter 1 page 33]) Let $\alpha > 0$ and $h \in L_{1}(0;1]$. The Riemann-Liouville integral of order α of h is defined by*

$$\left(I_{0^{+}}^{\alpha} h\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\tau)^{\alpha-1} h(\tau) d\tau, \text{ for a.e. } x \in (0,1]. \tag{23}$$

Definition 11.*([9]) Let $0 < \alpha < 1$ and $h : [0,1] \to \mathbb{R}$ absolutely continuous. The Riemann-Liouville fractional derivative of order α of h is defined by*

$$\begin{aligned} \left({}^{RL}\mathfrak{D}_{0^{+}}^{\alpha} h\right)(x) &= \frac{d}{dx}\left(I_{0^{+}}^{1-\alpha} h\right)(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{0}^{x} (x-\tau)^{-\alpha} h(\tau) d\tau. \end{aligned} \tag{24}$$

Definition 12.*([9]) Let $0 < \alpha < 1$ and $h : [0,1] \to \mathbb{R}$ absolutely continuous. The Caputo fractional derivative of order α of h is defined by*

$$\left({}^{C}\mathfrak{D}_{0^{+}}^{\alpha} h\right)(x) = \left({}^{RL}\mathfrak{D}_{0^{+}}^{\alpha}\left(h - h(0)\right)\right)(x), \text{ for a.e. } x \in [0,1]. \tag{25}$$

*Remark.*In [9, page 70], the authors gave the following properties of modified fractional integrals and derivatives

$$J_{-}^{\alpha} g = Q I_{0^{+}}^{\alpha} Q g, \text{ for all } g \in \pounds(1;+\infty), \tag{26}$$

$$I_{0^+}^{\alpha}g = QJ_{-}^{\alpha}Qg,\ \text{for all } g \in L_1(0;1], \tag{27}$$

$$D_{-}^{\alpha}h = Q^{RL}\mathfrak{D}_{0^+}^{\alpha}Qh,\ \text{for all } h \in \mathscr{A}\mathscr{C}[1;+\infty), \tag{28}$$

$$^{Rl}\mathfrak{D}_{0^+}^{\alpha}h = QD_{-}^{\alpha}Qh,\ \text{for all } h:[0,1]\to\mathbb{R} \text{ absolutely continuous.} \tag{29}$$

Then from the previous equalities, we have

$$^{C}D_{-}^{\alpha}h = Q^{C}\mathfrak{D}_{0^+}^{\alpha}Qh,\ \text{for all } h \in \mathscr{A}\mathscr{C}[1;+\infty), \tag{30}$$

$$^{C}\mathfrak{D}_{0^+}^{\alpha}h = Q^{C}D_{-}^{\alpha}Qh,\ \text{for all } h:[0,1]\to\mathbb{R} \text{ absolutely continuous.} \tag{31}$$

Proposition 1._If_ $g \in \mathfrak{C}^1(1;+\infty)$ _such that_ $t^2g' \in L_1(0;+\infty)$ _and_ $\lim\limits_{x\to+\infty} g(x) = g_\infty$ _and_ $0<\alpha<1$, _then we have_

$$\begin{aligned}\left(^{C}D_{-}^{\alpha}g\right)(x) &= -(J_{-}^{1-\alpha}(t^2g'(t)))(x)\\ &= -\frac{1}{\Gamma(-\alpha+1)}\int_x^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha}g'(\tau)\,d\tau.\end{aligned} \tag{32}$$

*Proof.*We have

$$(^{C}D_{-}^{\alpha}g)(x) = -\frac{x^2}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^{+\infty}(\frac{1}{x}-\frac{1}{\tau})^{-\alpha}\frac{(g(\tau)-g_\infty)}{\tau^2}d\tau.$$

Integrating by parts, we get

$$(^{C}D_{-}^{\alpha}g)(x) = -\frac{x^2}{\Gamma(-\alpha+1)}\frac{d}{dx}\left(-\frac{1}{-\alpha+1}\int_x^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha+1}g'(\tau)\,d\tau\right).$$

Since it is not difficult to prove that

$$\frac{d}{dx}\left(-\frac{1}{-\alpha+1}\int_x^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha+1}g'(\tau)\,d\tau\right) = \frac{1}{x^2}\int_x^{+\infty}\left(\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha}g'(\tau)\right)d\tau,$$

we obtain

$$(^{C}D_{-}^{\alpha}g)(x) = -\frac{1}{\Gamma(-\alpha+1)}\int_x^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha}g'(\tau)\,d\tau.$$

Lemma 4._If_ $g \in \mathfrak{C}^1(1;+\infty)$ _such that_ $t^2g' \in L_1(0;+\infty)$ _and_ $\lim\limits_{x\to+\infty} g(x) = g_\infty$ _and_ $0<\alpha<1$, _then_ $^{C}D_{-}^{\alpha}g \in \mathfrak{C}(1;+\infty)$ _and we have_

$$\left\|^{C}D_{-}^{\alpha}g\right\|_0 \le \frac{\left\|t^2g'\right\|_0}{\Gamma(2-\alpha)}. \tag{33}$$

*Proof.*The proof follows from the preceding Proposition and Lemma 2.

Lemma 5._If_ $g \in \mathfrak{C}^1(1;+\infty)$ _such that_ $t^2g' \in L_1(0;+\infty)$ _and_ $\lim\limits_{x\to+\infty} g(x) = g_\infty$ _and_ $0<\alpha<1$, _then_

$$\lim_{x\to+\infty}\left(^{C}D_{-}^{\alpha}g\right)(x) = 0. \tag{34}$$

*Proof.*We have

$$\left|\left({}^{C}D_{-}^{\alpha}g\right)(x)\right| = \frac{1}{\Gamma(-\alpha+1)}\left|\int_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha}g'(\tau)\,d\tau\right|$$
$$\leq \frac{\left\|t^{2}g'\right\|_{0}}{\Gamma(-\alpha+1)}\int_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-\alpha}\frac{d\tau}{\tau^{2}}$$
$$= \frac{\left\|t^{2}g'\right\|_{0}}{x^{1-\alpha}\Gamma(-\alpha+1)}\int_{1}^{+\infty}\left(1-\frac{1}{v}\right)^{-\alpha}v^{-2}dv$$
$$\leq \frac{\left\|t^{2}g'\right\|_{0}}{\Gamma(2-\alpha)}x^{\alpha-1}.$$

Since

$$\lim_{x\to+\infty}\frac{\left\|t^{2}g'\right\|_{0}}{\Gamma(2-\alpha)}x^{\alpha-1}=0;$$

we obtain

$$\lim_{x\to+\infty}\left({}^{C}D_{-}^{\alpha}g\right)(x)=0.$$

Examples For all $0<\alpha<1$, $\beta<0$, $b\in\mathbb{R}$ and $x\geq 1$, one has

$$\left({}^{C}D_{-}^{\alpha}b\right)(x)=0. \tag{35}$$

$$\left({}^{C}D_{-}^{\alpha}t^{\beta}\right)(x)=\frac{\Gamma(-\beta+1)}{\Gamma(-\beta-\alpha+1)}x^{\beta+\alpha}. \tag{36}$$

Lemma 6.*([9, Lemma 3]) Let $g\in\pounds(1;+\infty)$ and let $0<\alpha<1$, then we have*

$$(D_{-}^{\alpha}\circ J_{-}^{\alpha})g=g. \tag{37}$$

*Remark.*In the remainder of this paper, we use the definition of the modified Caputo fractional derivative given in Definition 9.

*Remark.*If $g\in\mathfrak{C}(1;+\infty)$, then we have $\lim_{x\to+\infty}(J_{-}^{\alpha}g)(x)=0$, for all $0<\alpha<1$.

We have these results.

Lemma 7.*If $g\in\mathfrak{C}(1;+\infty)$ and $0<\alpha<1$, then we have*

$$\left({}^{C}D_{-}^{\alpha}\circ J_{-}^{\alpha}\right)g=g. \tag{38}$$

*Proof.*Let $g\in\mathfrak{C}(1;+\infty)$, then we have $\lim_{x\to+\infty}(J_{-}^{\alpha}g)(x)=0$, for all $0<\alpha<1$.

Which implies that

$$\left({}^{C}D_{-}^{\alpha}\circ J_{-}^{\alpha}\right)g=(D_{-}^{\alpha}\circ J_{-}^{\alpha})g$$
$$=g.$$

Theorem 1.*([9, Theorem 1])Suppose that $h\in\pounds(1;+\infty)$ and $J_{-}^{1-\alpha}h\in\mathscr{AC}[1;+\infty)$ and let $0<\alpha<1$, then one has*

$$\left(J_{-}^{\alpha}\circ D_{-}^{\alpha}\right)h(x)=h(x)-\frac{x^{1-\alpha}}{\Gamma(\alpha)}\lim_{x\to+\infty}(J_{-}^{1-\alpha}h)(x). \tag{39}$$

Proposition 2.*Suppose that $h\in\mathfrak{C}(1;+\infty)$ with $\lim_{x\to+\infty}h(x)=h_{\infty}$ and $J_{-}^{1-\alpha}h\in\mathscr{AC}[1;+\infty)$ and let $0<\alpha<1$, then one has*

$$\left(J_{-}^{\alpha}\circ{}^{C}D_{-}^{\alpha}\right)h(x)=h(x)-h_{\infty}. \tag{40}$$

*Proof.*From Theorem 1, we have

$$\left(J_{-}^{\alpha}\circ{}^{C}D_{-}^{\alpha}\right)h(x)=h(x)-h_{\infty}-\frac{x^{1-\alpha}}{\Gamma(\alpha)}\lim_{x\to+\infty}\left(J_{-}^{1-\alpha}(h-h_{\infty})\right)(x)$$
$$=h(x)-h_{\infty}-\frac{x^{1-\alpha}}{\Gamma(\alpha)}\lim_{x\to+\infty}\left(\left(J_{-}^{1-\alpha}h\right)(x)-\frac{h_{\infty}}{\Gamma(\alpha+1)}x^{-\alpha}\right).$$

Since $\lim_{x\to+\infty}x^{-\alpha}=0$ and using Remark 2, we obtain

$$\left(J_{-}^{\alpha}\circ{}^{C}D_{-}^{\alpha}\right)h(x)=h(x)-h_{\infty}.$$

Now for $0<\alpha<1$, we consider the problem

$$\begin{cases}\left({}^{C}D_{-}^{\alpha}y\right)(x)=F(x,y(x),y(\theta(x))),\ x\in[1,+\infty),\\ \lim_{x\to+\infty}y(x)=l,\end{cases}\tag{41}$$

$F:[1,+\infty)\times O\to\mathbb{R}$ is a function such that $F(x,z)\in\mathfrak{C}(1;+\infty)$ for any $z\in O$ with O is an open set in $\mathbb{R}^2$ and l is a real number.

Theorem 2.*Let $y\in\mathfrak{C}(1;+\infty)$, then y is a solution for the problem (41) if, and only if, y is a solution of the integral equation*

$$y(x)=l+\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-1+\alpha}\frac{F(\tau,y(\tau),y(\theta(\tau)))}{\tau^{2}}d\tau.\tag{42}$$

*Proof.*First suppose that $y\in\mathfrak{C}(1;+\infty)$ is a solution for the problem (41). Since $F(x,z)\in\mathfrak{C}(1;+\infty)$ for all $z\in O$, we have ${}^{C}D_{-}^{\alpha}y\in\mathfrak{C}(1;+\infty)$ and since

$$\left({}^{C}D_{-}^{\alpha}y\right)(x)=-\frac{x^{2}}{\Gamma(1-\alpha)}\frac{d}{dx}\left(J_{-}^{1-\alpha}(y-l)\right)(x),$$

we obtain

$$J_{-}^{1-\alpha}(y-l)\in\mathfrak{C}^{1}(1;+\infty).$$

Which implies that

$$J_{-}^{1-\alpha}y\in\mathfrak{C}^{1}(1;+\infty),$$

and then from Proposition 2, we get

$$\left(J_{-}^{\alpha}\circ{}^{C}D_{-}^{\alpha}y\right)(x)=y(x)-l.\tag{43}$$

Now applying the operator J_{-}^{α} to both sides of first equation in (41) and using the equality (43), we obtain

$$y(x)=l+\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-1+\alpha}\frac{F(\tau,y(\tau),y(\theta(\tau)))}{\tau^{2}}d\tau.\tag{44}$$

Conversely applying the operator ${}^{C}D_{-}^{\alpha}$ to the integral equation (42) and from Lemma 7, we get

$$\left({}^{C}D_{-}^{\alpha}y\right)(x)=F(x,y(x),y(\theta(x))).\tag{45}$$

Also, one has

$$\lim_{x\to+\infty}y(x)=\lim_{x\to+\infty}\left(l+\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-1+\alpha}\frac{F(\tau,y(\tau),y(\theta(\tau)))}{\tau^{2}}d\tau\right)$$
$$=l+\frac{1}{\Gamma(\alpha)}\lim_{x\to+\infty}\int_{x}^{+\infty}\left(\frac{1}{x}-\frac{1}{\tau}\right)^{-1+\alpha}\frac{F(\tau,y(\tau),y(\theta(\tau)))}{\tau^{2}}d\tau,$$

and then using Remark 2, we obtain

$$\lim_{x\to+\infty}y(x)=l.\tag{46}$$

*Remark.*The idea of proving Theorem (2) is analogous to that of Theorem 3.24 in [10].

3 Main result

In this section, the main result of this work is stated and proven.

Consider problem (1) and suppose that f fulfills the assumption

There exist $K_1 > 0$ and $K_2 > 0$ such that for all $x \in [1,+\infty)$ and y_j, $z_j \in \mathbb{R}$ for $j = 1,2$, we have

$$|f(x,y_1,z_1) - f(x,y_2,z_2)| \leq K_1 |y_1 - y_2| + K_2 |z_1 - z_2|. \tag{47}$$

Theorem 3. *Suppose that the assumption (47) is satisfied, then the problem (1) admits a solution $y \in \mathfrak{C}(1;+\infty)$ such that ${}^{C}D_{-}^{\alpha}y \in \mathfrak{C}(1;+\infty)$ and this solution is unique.*

Proof. Consider the following operator

$$\begin{aligned} T :& \mathfrak{C}(1;+\infty) \to \mathfrak{C}(1;+\infty) \\ y \mapsto& (Ty)(x) = L_\infty + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{f(\tau, y(\tau), y(\theta(\tau)))}{\tau^2} d\tau, \end{aligned}$$

and we consider the following norm which is used in [11].

$$\|z\|_* = \sup_{x\in[1,+\infty)} \frac{|z(x)|}{E_\alpha(\mu x^{-\alpha})}, \tag{48}$$

with $\mu > 0$ and $z \in \mathfrak{C}(1;+\infty)$.

Now let's prove operator T is a contraction on the Banach space $(\mathfrak{C}(1;+\infty), \|.\|_*)$.

For all y_1, $y_2 \in \mathfrak{C}(1;+\infty)$ and all $x \in [1,+\infty)$, one has

$$\begin{aligned} &\frac{|((Ty_1)(x) - (Ty_2)(x))|}{E_\alpha(\mu x^{-\alpha})} \\ &= \frac{\left| \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{(f(\tau, y_1(\tau), y_1(\theta(\tau))) - f(\tau, y_2(\tau), y_2(\theta(\tau))))}{\tau^2} d\tau \right|}{\Gamma(\alpha) E_\alpha(\mu x^{-\alpha})} \\ &\leq \frac{K_1}{\Gamma(\alpha) E_\alpha(\mu x^{-\alpha})} \left| \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{|y_1(\tau) - y_2(\tau)|}{\tau^2} d\tau \right| \\ &\quad + \frac{K_2}{\Gamma(\alpha) E_\alpha(\mu x^{-\alpha})} \left| \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{|y_1(\theta(\tau)) - y_2(\theta(\tau))|}{\tau^2} d\tau \right| \\ &\leq \frac{(K_1 + K_2)\|y_1 - y_2\|_*}{\Gamma(\alpha) E_\alpha(\mu x^{-\alpha})} \int_x^{+\infty} E_\alpha(\mu \tau^{-\alpha}) \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{dt}{\tau^2} \\ &= \frac{(K_1 + K_2)\|y_1 - y_2\|_*}{\mu E_\alpha(\mu x^{-\alpha})} \left(E_\alpha(\mu x^{-\alpha}) - 1\right) \\ &= \frac{(K_1 + K_2)\|y_1 - y_2\|_*}{\mu} \left(1 - \frac{1}{E_\alpha(\mu x^{-\alpha})}\right) \\ &< \frac{(K_1 + K_2)\|y_1 - y_2\|_*}{\mu}. \end{aligned}$$

Which implies that

$$\|Ty_1 - Ty_2\|_* \leq \frac{(L_1 + L_2)}{\mu} \|y_1 - y_2\|_*. \tag{49}$$

Now if we choose $\mu > L_1 + L_2$, we obtain

$$\|Ty_1 - Ty_2\|_* < \|y_1 - y_2\|_*. \tag{50}$$

Then according to Banach's fixed point theorem (see Theorem 1.9 in [10]), it follows the existence of a unique fixed point for T and consequently from Theorem 2, we conclude that the problem (1) admits a solution and this solution is unique.

*Remark.*The assumption (47) for Theorem 3 is a sufficient condition as shown in the following example

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = \Gamma(\alpha+1)x^{\alpha}y(x), x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} y(x) = 0, \end{cases} \tag{51}$$

where $0 < \alpha < 1$.

The problem (51) admits two solutions $y \equiv 0$ and $y(x) = \dfrac{x^{-\alpha}}{\Gamma(\alpha+1)}$, for all $x \geq 1$.

*Remark.*The assumption (47) for Theorem 3 is not a necessary condition as shown in the following example

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = y^{2}(x), x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} y(x) = 0, \end{cases} \tag{52}$$

where $0 < \alpha < 1$.

The problem (52) admits the unique solution $y \equiv 0$.

*Remark.*Theorem 3 can be generalized to the following problems

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = f\left(x, y(x), y(\theta_1(x)), \ldots, y(\theta_n(x))\right), x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} y(x) = L_{\infty}, \end{cases} \tag{53}$$

$$\begin{cases} \left(D_{-}^{\alpha}y\right)(x) = f\left(x, y(x), y(\theta_1(x)), \ldots, y(\theta_n(x))\right), x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} x^{\alpha-1}y(x) = L_{\infty}, \end{cases} \tag{54}$$

where $0 < \alpha < 1$, L_∞ a real number, $\theta_i : [1,+\infty) \to [1,+\infty)$ are continuous and $\theta_i(x) \geq x$ for all $i = 1,\ldots,n$ and $f : [1,+\infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous and satisfies the following condition.

There exist $K_i > 0$ for $i = 1,\ldots,n$ such that

$$\left|f(x,u_1,\ldots,u_n) - f(x,v_1,\ldots,v_n)\right| \leq \sum_{i=1}^{n} K_i \left|u_i - v_i\right|, \tag{55}$$

for all $x \in [1,+\infty)$ and u_i, $v_i \in \mathbb{R}$ for $i = 1,\ldots,n$.

4 Examples

*Example 3.*Consider the problem

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = \lambda y(x),\ x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} y(x) = L_{\infty}, \end{cases} \tag{56}$$

where $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and Ł$_\infty \in \mathbb{R}$.

First, we note that the function $(x,y,z) \mapsto \lambda.y$ satisfy the assumption (47) and then from Theorem 3 the problem (56) admits a unique solution.

Now using the technique of successive approximations, we have

$$\begin{aligned} y_0(x) &= L_\infty, \\ y_1(x) &= L_\infty + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_0(\tau)}{\tau^2} d\tau \\ &= L_\infty + \frac{\lambda L_\infty}{\Gamma(\alpha+1)} x^{-\alpha}, \end{aligned}$$

$$y_2(x) = L_\infty + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{\lambda y_1(\tau)}{\tau^2} d\tau$$
$$= L_\infty + \frac{\lambda L_\infty}{\Gamma(\alpha+1)} x^{-\alpha} + \frac{\lambda^2 L_\infty}{\Gamma(2\alpha+1)} x^{-2\alpha}.$$

By recurrence, we obtain

$$y_n(x) = L_\infty \sum_{k=0}^{n} \frac{\lambda^k x^{-k\alpha}}{\Gamma(1+k\alpha)},$$

and the unique solution for problem (56) is given by

$$y(x) = L_\infty E_\alpha\left(\lambda x^{-\alpha}\right). \tag{57}$$

*Example 4.*Consider the problem

$$\begin{cases} \left({}^C D_-^\alpha y\right)(x) = \lambda y(x) + \widetilde{\mathfrak{F}}(x),\ x \in [1,+\infty), \\ \lim\limits_{x \to +\infty} y(x) = L_\infty, \end{cases} \tag{58}$$

where $0 < \alpha < 1$, λ and L_∞ are real numbers and $\widetilde{\mathfrak{F}} \in \mathfrak{C}(1;+\infty)$.

First, we note that the function $(x,y,z) \mapsto \lambda.y + \widetilde{\mathfrak{F}}(x)$ satisfy the assumption (47) and then from Theorem 3 the problem (58) admits a unique solution.

Now using the technique of successive approximations, we show that the problem (58) admits a unique solution given by

$$y(x) = L_\infty E_\alpha\left(\lambda x^{-\alpha}\right) + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{\tau - x}{x\tau}\right)^{-1+\alpha} E_{\alpha,\alpha}\left(\lambda\left(\frac{\tau - x}{x\tau}\right)^\alpha\right) \frac{\widetilde{\mathfrak{F}}(\tau)}{\tau^2} d\tau. \tag{59}$$

*Example 5.*We consider the following problem

$$\begin{cases} \left({}^C D_-^\alpha y\right)(x) = \lambda y(qx),\ x \in [1,+\infty), \\ \lim\limits_{x \to +\infty} y(x) = L_\infty, \end{cases} \tag{60}$$

where $0 < \alpha < 1$, $q > 1$, $\lambda \in \mathbb{R}$ and $L_\infty \in \mathbb{R}$.

First, we note that the function $(x,y,z) \mapsto \lambda.z$ satisfy the assumption (47) and then from Theorem 3 the problem (60) admits a unique solution and using the technique of successive approximations, we have

$$y_0(x) = L_\infty,$$
$$y_1(x) = L_\infty + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_0(q\tau)}{\tau^2} d\tau$$
$$= L_\infty + \frac{\lambda L_\infty}{\Gamma(1+\alpha)} x^{-\alpha},$$
$$y_2(x) = L_\infty + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_1(q\tau)}{\tau^2} d\tau$$
$$= L_\infty + \frac{\lambda L_\infty}{\Gamma(1+\alpha)} x^{-\alpha} + \frac{\lambda^2 L_\infty q^{-\alpha}}{\Gamma(1+2\alpha)} x^{-2\alpha},$$
$$y_3(x) = L_\infty + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_2(q\tau)}{\tau^2} d\tau$$
$$= L_\infty + \frac{\lambda L_\infty}{\Gamma(\alpha+1)} x^{-\alpha} + \frac{\lambda^2 L_\infty q^{-\alpha}}{\Gamma(2\alpha+1)} x^{-2\alpha}$$
$$+ \frac{\lambda^3 \mathfrak{L}_\infty q^{-3\alpha}}{\Gamma(3\alpha+1)} x^{-3\alpha}.$$

By recurrence, we get

$$y_k(x) = L_\infty \sum_{j=0}^{k} \frac{q^{\frac{-j(j-1)\alpha}{2}} \left(\lambda x^{-\alpha}\right)^j}{\Gamma(j\alpha+1)},$$

and consequently the unique solution for the problem (60) is given by

$$y(x) = L_\infty F_\alpha\left(\lambda x^{-\alpha}, q^{-\alpha}\right). \tag{61}$$

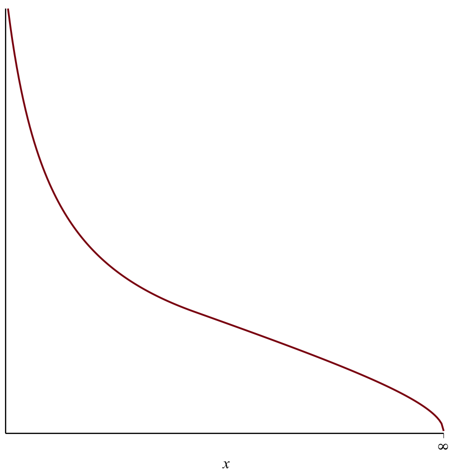

Fig. 1: The graph of $F_{\frac{1}{2}}(x^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$

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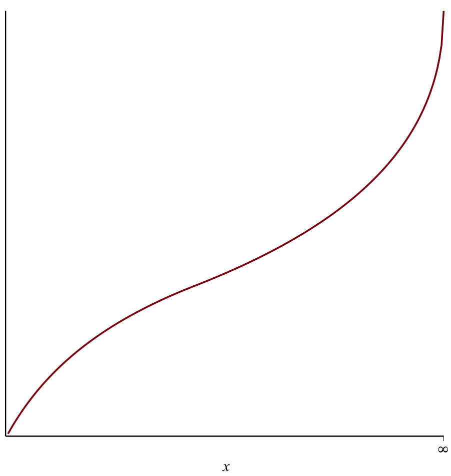

Fig. 2: The graph of $F_{\frac{1}{2}}(-x^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$

*Example 6.*Consider the problem

$$\begin{cases} \left({}^C D_-^\alpha y\right)(x) = \lambda y(qx) + \widetilde{\mathfrak{F}}(x), \; x \in [1, +\infty), \\ \lim\limits_{x \to +\infty} y(x) = L_\infty, \end{cases} \tag{62}$$

where $0<\alpha<1<q$, λ and L_∞ are real numbers and $\widetilde{\mathfrak{F}}\in\mathfrak{C}(1;+\infty)$.

First, we note that the function $(x,y,z)\mapsto\lambda.z+\widetilde{\mathfrak{F}}(x)$ satisfy the assumption (47) and then from Theorem 3 the problem (62) admits a unique solution and using the technique of successive approximations, we show that the unique solution of (62) is

$$y(x)=L_\infty F_\alpha(\lambda x^{-\alpha},q^{-\alpha}) + \sum_{k=0}^{+\infty}\int_x^{+\infty}\frac{1}{\Gamma(k\alpha+\alpha)}\left(\frac{q^k x\tau}{\tau-q^k x}\right)^{1-k\alpha}\chi_{]q^k x;+\infty)}(\tau)\frac{\widetilde{\mathfrak{F}}(\tau)}{\tau^2}d\tau, \tag{63}$$

where $\chi_{]q^k x;+\infty)}$ is the indicator function of the set $]q^k x;+\infty)$.

*Remark.*If we consider the following problem

$$\begin{cases}\left({}^C D_-^{\frac{1}{2}}y\right)(x)=\Gamma\left(\frac{3}{2}\right)y(2x)-\frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{2x}},\ x\in[1,+\infty),\\ \lim\limits_{x\to+\infty}y(x)=1.\end{cases} \tag{64}$$

The problem (64) admits a unique solution given by $y(x)=\frac{1}{\sqrt{x}}+1$, for all $x\geq 1$.

*Remark.*Consider the problem

$$\left({}^C D_-^{\alpha}y\right)(x)=y(qx)+E_\alpha\left(\mu x^{-\alpha}\right),\ x\in[1,+\infty), \tag{65}$$

where $0<\alpha<1\leq q$, $\lambda\in\mathbb{R}$ and μ is a strictly negative real number.

We can obtain a particular solution for the problem (65) by using the following iteration used by Georges Valiron in [21]

$$\begin{cases}y_0(x)=-E_\alpha\left(\mu x^{-\alpha}\right),\\ y_{n+1}(qx)=\left({}^C D_-^{\alpha}y_n\right)(x),\ n\geq 1.\end{cases}$$

Then the particular solution is defined by the following Dirichlet series

$$y(x)=\sum_{n=0}^{+\infty}y_n(x)=\begin{cases}-\sum\limits_{n=0}^{+\infty}\mu^n q^{\frac{n(n-1)}{2}\alpha}E_\alpha\left(\mu q^{n\alpha}x^{-\alpha}\right)\ \text{if } q>1,\\ -\dfrac{E_\alpha\left(\mu x^{-\alpha}\right)}{1-\mu}\ \text{if } q=1 \text{ and } \mu\in[-1,0[.\end{cases} \tag{66}$$

Remark.

*Example 7.*Consider the problem

$$\begin{cases}\left({}^C D_-^{\alpha}y\right)(x)=\mu y(x)+\sum\limits_{j=1}^{m}\lambda_j y(q_j x),\ x\in[1,+\infty),\\ \lim\limits_{x\to+\infty}y(x)=L_\infty,\end{cases} \tag{67}$$

with $0<\alpha<1$, $\mu\in\mathbb{R}$, λ_j and $q_j>1$ for all $j=1,\ldots,m$ and $L_\infty\in\mathbb{R}$.

First, we note that the function $(x,y,z_1,\ldots,z_m)\mapsto\lambda.y+\sum\limits_{j=1}^{m}\lambda_j z_j$ satisfy the assumption (55) and then from Remark 3 the problem (67) admits a unique solution and using the technique of successive approximations, we show that the unique solution of (67) is given by

$$y(x)=L_\infty+L_\infty\sum_{k=1}^{+\infty}\frac{x^{-k\alpha}}{\Gamma(1+k\alpha)}\prod_{j=0}^{k-1}\left(\mu+\sum_{i=2}^{m}\lambda_i q_i^{-j\alpha}\right). \tag{68}$$

*Example 8.*Consider the problem

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = \sum_{k=2}^{n} \left(y(kx+1) - y(kx-1)\right), \; x \in [1,+\infty), \\ \lim_{x\to+\infty} y(x) = L_{\infty}, \end{cases} \tag{69}$$

where $0 < \alpha < 1$ and $L_{\infty} \in \mathbb{R}$.

From Theorem 3 it follows that the problem (69) admits only the constant solution $y = L_{\infty}$.

*Example 9.*Consider the terminal problem

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = \lambda y\left(x^2\right), \; x \in [1,+\infty), \\ \lim_{x\to+\infty} y(x) = L_{\infty}, \end{cases} \tag{70}$$

where $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and $L_{\infty} \in \mathbb{R}$.

First, we note that the function $(x,y,z) \mapsto \lambda.z$ satisfy the assumption (47) and then from Theorem 3 the problem (62) admits a unique solution and using the technique of successive approximations, we show that the unique solution of (70) is

$$y(x) = L_{\infty} + L_{\infty} \sum_{k=1}^{+\infty} \lambda^k \left[\prod_{j=1}^{k} \frac{\Gamma\left(1+\left(2^j-2\right)\alpha\right)}{\Gamma\left(1+\left(2^j-1\right)\alpha\right)}\right] x^{\left(1-2^k\right)\alpha}. \tag{71}$$

*Example 10.*We consider the following problem

$$\begin{cases} \left({}^{C}D_{-}^{\alpha}y\right)(x) = \lambda x^{-\beta} y(qx), \; x \in [1,+\infty), \\ \lim_{x\to+\infty} y(x) = L_{\infty}, \end{cases} \tag{72}$$

where $0 < \alpha < 1$, $\beta > 0$, $q \geq 1$ and λ and $L_{\infty} \in \mathbb{R}$ are real numbers.

First, we note that the function $(x,y,z) \mapsto \lambda.x^{-\beta}.y$ satisfy the assumption (47) and then from Theorem 3 the problem (72) admits a unique solution.

Now using the method of successive approximations, one has

$$\begin{aligned} y_0(x) &= L_{\infty}, \\ y_1(x) &= L_{\infty} + \frac{\lambda}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{\tau^{-\beta} y_0(q\tau)}{\tau^2} d\tau \\ &= L_{\infty} + \lambda L_{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha)} x^{-\beta-\alpha}, \end{aligned}$$

$$\begin{aligned} y_2(x) &= L_{\infty} + \frac{\lambda}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{\tau^{-\beta} y_1(q\tau)}{\tau^2} d\tau \\ &= L_{\infty} + \lambda L_{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha)} x^{-\beta-\alpha} \\ &\quad + \lambda^2 L_{\infty} \frac{\Gamma(1+\beta)\Gamma(1+2\beta+\alpha)}{\Gamma(1+\beta+\alpha)\Gamma(1+2\beta+2\alpha)} q^{-(\beta+\alpha)} x^{-2(\beta+\alpha)}, \\ y_3(x) &= L_{\infty} + \frac{\lambda}{\Gamma(\alpha)} \int_{x}^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{\alpha-1} \frac{\tau^{-\beta} y_2(q\tau)}{\tau^2} d\tau \\ &= L_{\infty} + \lambda L_{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha)} x^{-\beta-\alpha} \\ &\quad + \lambda^2 L_{\infty} \frac{\Gamma(1+\beta)\Gamma(1+2\beta+\alpha)}{\Gamma(1+\beta+\alpha)\Gamma(1+2\beta+2\alpha)} q^{-(\beta+\alpha)} x^{-2(\beta+\alpha)} \\ &\quad + \lambda^3 L_{\infty} \frac{\Gamma(1+\beta)\Gamma(1+2\beta+\alpha)\Gamma(1+3\beta+2\alpha)}{\Gamma(1+\beta+\alpha)\Gamma(1+2\beta+2\alpha)\Gamma(1+3\beta+3\alpha)} q^{-2(\beta+\alpha)} x^{-3(\beta+\alpha)} \end{aligned}$$

Then by recurrence, for all $n \geq 1$, one has

$$y_n(x) = L_\infty + L_\infty \sum_{k=1}^{n} \left[\prod_{j=1}^{k} \frac{\Gamma(1 + j(\alpha+\beta) - \alpha)}{\Gamma(1 + j(\alpha+\beta))} \right] \lambda^k q^{-(k-1)(\beta+\alpha)} x^{-k(\beta+\alpha)},$$

and consequently the unique solution to problem (72) is

$$y(x) = L_\infty \left(1 + \sum_{k=1}^{+\infty} \left[\prod_{j=1}^{k} \frac{\Gamma(1 + j(\alpha+\beta) - \alpha)}{\Gamma(1 + j(\alpha+\beta))} \right] \lambda^k q^{-(k-1)(\beta+\alpha)} x^{-k(\beta+\alpha)} \right). \tag{73}$$

*Example 11.*Consider the problem

$$\begin{cases} (D_-^\alpha y)(x) = \lambda y(qx), \ x \in [1, +\infty), \\ \lim\limits_{x \to +\infty} x^{\alpha-1} y(x) = \dfrac{L_\infty}{\Gamma(\alpha)}, \end{cases} \tag{74}$$

where $0 < \alpha < 1 \leq q$, λ and L_∞ are real numbers.

By using the method of successive approximations, we have

$$\begin{aligned} y_0(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)}, \\ y_1(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau} \right)^{\alpha-1} \frac{\lambda y_0(q\tau)}{\tau^2} d\tau \\ &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{\lambda L_\infty q^{1-\alpha}}{\Gamma(2\alpha)} x^{1-2\alpha}, \end{aligned}$$

$$\begin{aligned} y_2(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau} \right)^{-1+\alpha} \frac{\lambda y_1(q\tau)}{\tau^2} d\tau \\ &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{\lambda L_\infty q^{1-\alpha}}{\Gamma(2\alpha)} x^{1-2\alpha} + \frac{\lambda^2 L_\infty q^{2-3\alpha}}{\Gamma(3\alpha)} x^{1-3\alpha}, \\ y_3(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau} \right)^{-1+\alpha} \frac{\lambda y_2(q\tau)}{\tau^2} d\tau \\ &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{\lambda L_\infty q^{1-\alpha}}{\Gamma(2\alpha)} x^{1-2\alpha} + \frac{\lambda^2 L_\infty q^{2-3\alpha}}{\Gamma(3\alpha)} x^{1-3\alpha} \\ &\quad + \frac{\lambda^3 L_\infty q^{3-6\alpha}}{\Gamma(4\alpha)} x^{1-4\alpha}. \end{aligned}$$

By recurrence, we obtain

$$y_n(x) = L_\infty x^{1-\alpha} \sum_{k=0}^{n} \frac{q^{k - \frac{k(k+1)}{2}\alpha} \lambda^k x^{-\alpha k}}{\Gamma(\alpha + k\alpha)},$$

and then problem (74) admits a unique solution that is given by

$$\begin{aligned} y(x) &= L_\infty x^{1-\alpha} \sum_{k=0}^{+\infty} \frac{q^{k - \frac{k(k+1)}{2}\alpha} \lambda^k x^{-k\alpha}}{\Gamma(\alpha + k\alpha)} \\ &= \begin{cases} L_\infty x^{1-\alpha} F_{\alpha,\alpha}\left(\lambda x^{-\alpha}, q^{-1} \right) \text{ if } q > 1, \\ L_\infty x^{1-\alpha} E_{\alpha,\alpha}\left(\lambda x^{-\alpha} \right) \text{ if } q = 1. \end{cases} \end{aligned}$$

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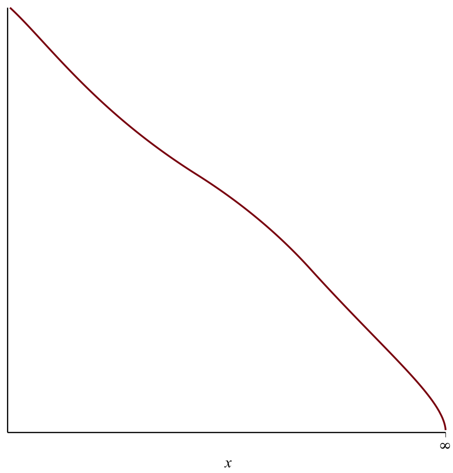

Fig. 3: The graph of $\sqrt{x}F_{\frac{1}{2},\frac{1}{2}}\left(x^{-\frac{1}{2}},2^{-1}\right)$

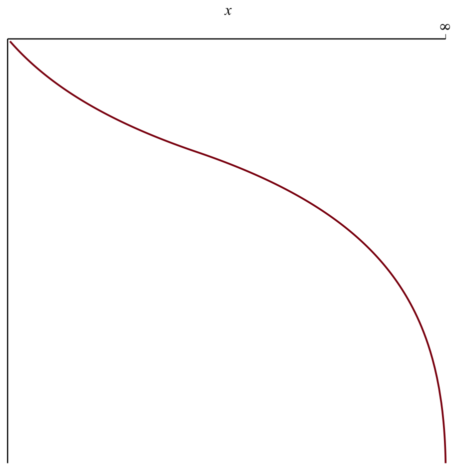

Fig. 4: The graph of $\sqrt{x}F_{\frac{1}{2},\frac{1}{2}}\left(-x^{-\frac{1}{2}},2^{-1}\right)$

*Example 12.*We consider the following problem

$$\begin{cases} \left(D_{-}^{\alpha}y\right)(x) = \lambda y(qx) + \widetilde{F}(x),\ x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} x^{\alpha-1}y(x) = \dfrac{L_\infty}{\Gamma(\alpha)}, \end{cases} \tag{75}$$

with $\widetilde{F} \in \mathfrak{C}(1;+\infty)$, $0 < \alpha < 1 < q$, λ and L_∞ are real numbers.

Using the technique of successive approximations, we show that the unique solution of (75) is given by

$$\begin{aligned} y(x) &= L_\infty x^{1-\alpha}F_{\alpha,\alpha}\left(\lambda x^{-\alpha},q^{-1}\right) \\ &\quad + \sum_{n=0}^{+\infty} \frac{1}{\Gamma(\alpha+n\alpha)} \int_x^{+\infty} \left(\frac{q^n x\tau}{\tau - q^n x}\right)^{1-n\alpha} \chi_{]q^n x;+\infty)}(t)\, \frac{\widetilde{F}(\tau)}{\tau^2} d\tau. \end{aligned} \tag{76}$$

*Example 13.*Consider the problem

$$\begin{cases} \left(D_{-}^{\alpha}y\right)(x) = \mu y(x) + \sum_{j=1}^{m} \lambda_j y(q_j x),\ x \in [1,+\infty), \\ \lim_{x\to+\infty} x^{\alpha-1} y(x) = \frac{L_\infty}{\Gamma(\alpha)}, \end{cases} \tag{77}$$

where $0 < \alpha < 1$, $\mu \in \mathbb{R}$, λ_j and $q_j > 1$ for all $j = 1,\dots,m$ and $L_\infty \in \mathbb{R}$.

First, we note that the function $(x,y,z_1,\dots,z_m) \mapsto \mu.y + \sum_{j=1}^{m} \lambda_j z_j$ satisfy the assumption (55) and then from Remark 3 the problem (77) admits a unique solution and using the technique of successive approximations, we show that the unique solution of (77) is

$$y(x) = \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + L_\infty x^{1-\alpha} \sum_{k=1}^{+\infty} \frac{x^{-k\alpha}}{\Gamma(\alpha+k\alpha)} \prod_{j=0}^{k-1} \left(\mu + \sum_{i=2}^{m} \lambda_i q_i^{1-(j+1)\alpha}\right). \tag{78}$$

*Example 14.*Consider the problem

$$\begin{cases} \left(D_{-}^{\alpha}y\right)(x) = \lambda y\left(x^2\right),\ x \in [1,+\infty), \\ \lim_{x\to+\infty} x^{\alpha-1} y(x) = \frac{L_\infty}{\Gamma(\alpha)}, \end{cases} \tag{79}$$

where $0 < \alpha < 1$ with $\alpha \neq \frac{1}{2}$, λ and L_∞ are real numbers.

First, we note that the function $(x,y) \mapsto \lambda.z$ satisfy the assumption (55) and then from Remark 3 the problem (79) admits a unique solution.

Now using the method of successive approximations, we have

$$\begin{aligned} y_0(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)}, \\ y_1(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_0\left(\tau^2\right)}{\tau^2} d\tau \\ &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{\lambda L_\infty \Gamma(-1+2\alpha)}{\Gamma(-1+3\alpha)} x^{2-3\alpha}, \\ y_2(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_1\left(\tau^2\right)}{\tau^2} d\tau \\ &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{\lambda L_\infty \Gamma(-1+2\alpha)}{\Gamma(-1+3\alpha)} x^{2-3\alpha} \\ &\quad + \frac{\lambda^2 L_\infty \Gamma(-1+2\alpha)\Gamma(-3+6\alpha)}{\Gamma(-1+3\alpha)\Gamma(-3+7\alpha)} x^{4-7\alpha}, \end{aligned}$$

$$\begin{aligned} y_3(x) &= \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} \left(\frac{1}{x} - \frac{1}{\tau}\right)^{-1+\alpha} \frac{\lambda y_2\left(\tau^2\right)}{\tau^2} dt \\ &= L_\infty x^{1-\alpha} \sum_{k=0}^{3} \lambda^k \left[\prod_{j=0}^{k} \frac{\Gamma\left(\left(2^j-1\right)(2\alpha-1)\right)}{\Gamma\left(\left(2^j-1\right)(2\alpha-1)+\alpha\right)}\right] x^{\left(-1+2^k\right)(-2\alpha+1)} \end{aligned}$$

By recurrence, we obtain

$$y_n(x) = \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + L_\infty x^{1-\alpha} \sum_{k=1}^{n} \lambda^k \left[\prod_{j=1}^{k} \frac{\Gamma\left(\left(2^j-1\right)(2\alpha-1)\right)}{\Gamma\left(\left(2^j-1\right)(2\alpha-1)+\alpha\right)}\right] x^{\left(-1+2^k\right)(-2\alpha+1)},$$

and then the unique solution to problem (79) is

$$y(x) = \frac{L_\infty x^{1-\alpha}}{\Gamma(\alpha)} + L_\infty x^{1-\alpha} \sum_{k=1}^{\infty} \lambda^k \left[\prod_{j=1}^{k} \frac{\Gamma\left(\left(2^j-1\right)(2\alpha-1)\right)}{\Gamma\left(\left(2^j-1\right)(2\alpha-1)+\alpha\right)} \right] x^{\left(-1+2^k\right)(-2\alpha+1)}. \tag{80}$$

*Example 15.*Consider the problem

$$\begin{cases} \left(D_-^\alpha y\right)(x) = \mu x^{-\alpha} y(qx),\ x \in [1,+\infty), \\ \lim\limits_{x\to+\infty} x^{\alpha-1} y(x) = \dfrac{L_\infty}{\Gamma(\alpha)}, \end{cases} \tag{81}$$

where $0<\alpha<1$, $\mu \in \mathbb{R}$, $q>1$ and $L_\infty \in \mathbb{R}$.

First, we note that the function $(x,y,z) \mapsto \mu.x^{-\alpha} z$ satisfy the assumption (55) and then from Remark 3 the problem (81) admits a unique solution and using the technique of successive approximations, we show that the unique solution of (81) is given by

$$y(x) = \frac{L_\infty}{\Gamma(\alpha)} x^{1-\alpha} + L_\infty x^{1-\alpha} \sum_{n=1}^{+\infty} \left[\prod_{j=1}^{n} \frac{\Gamma(2j\alpha)}{\Gamma((2j+1)\alpha)} \right] \mu^n q^{n(1-n\alpha)} x^{-2n\alpha}. \tag{82}$$

5 Conclusion

In this paper using Banach's fixed point theorem, we have shown the existence and uniqueness of solutions for a class of fractional-order differential equations with advanced arguments. Several interesting examples are given illustrating the application of our result. On the other hand, it might be interesting to investigate the existence of solutions for modified fractional differential equations modified with deviating arguments on infinite intervals.

Declarations

Competing interests: The author declare no competing of interest.
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