Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.4.4

Chain Structure of Intuitionistic Level Subgroups of Groups

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Received: July 20, 2023 Accepted : Feb. 14, 2024

Abstract: A classic result in fuzzy group theory states that level subgroups of any fuzzy subgroup of a finite group form a chain. Here we are trying to generalise this result to the context of intuitionistic fuzzy groups. We have obtained a characterisation of groups in which all the Intuitionistic Level Subgroups (ILSGs) of every Intuitionistic Fuzzy Subgroups (IFSGs) form chains. We have also precisely obtained the finite groups having this property. We throw some insight into the cases of infinite groups also.

Keywords: Fuzzy subgroup; Intuitionistic fuzzy subgroup; Level subgroup; Intuitionistic level subgroup; Finite group; Infinite group.

2020 Mathematics Subject Classification. 03E72; 20N25; 06F99; 20F99.

1 Introduction

In 1965, L A Zadeh[1] put forth the concept of fuzzy sets by making partial membership or belongingness possible in a set. This was accomplished by defining fuzzy subset of a set X using a membership function $\mathscr{M} : X \to I = [0,1]$, instead of the classic characteristic function $\chi : X \to \{0,1\}$. Following this, the fuzzy approach was applied in group theory by A Rosenfeld[2], who defined fuzzy subgroups of a group. In 1983, K T Atanassov[3] came up with the concept of intuitionistic fuzzy sets as a further generalisation of the theory of the fuzzy sets. In 1989, R Biswas[4] applied the intuitionistic fuzzy approach in group theory, and developed the theory of intuitionistic fuzzy subgroups of a group. Researches in this area of fuzzy algebra are still in progress which include study of various aspects of t-Intuitionistic Fuzzy Subgroups by M Gulzar et. al [5] in 2020 and fuzzification of Lie algebras by S Shaqaqha ([6], [7], [8]) in 2023.
In 1981, P S Das[9] did a detailed research on the theory of level subgroups of fuzzy subgroups of a finite group and proved that they form a chain. In our previous works ([10], [11], [12], [13]) we have made an attempt to carry over this result to intuitionistic fuzzy context and have arrived at the following conclusions: (1) ILSGs in all IFSGs of $\mathbb{Z}_{p^n}$ form chains (where p is a prime and $n \in \mathbb{N}$), (2) ILSGs in about 77.77% of IFSGs of $\mathbb{Z}_{pq}$ (where p and q are distinct primes) form chains, (3) ILSGs in 62.5% of IFSGs of the Klein-4 Group form chains and (4) ILSGs in 76% of IFSGs of the Dihedral group D_3 form chains. Having settled these special cases, in this paper we take up the case of an arbitrary group. This paper is organised into four main sections. In the first section we discuss some basic concepts and results which are relevant for the proper understanding of the article. In the second section we propose the main result, which is a characterisation of those groups whose all IFSGs have the "chain property". In the third section we obtain another specific characterization for finite groups and in the fourth section we try to throw some insight into the cases of infinite groups.

2 Preliminaries

In this article we are using some basic concepts of fuzzy subsets (FS) [1], intuitionistic fuzzy subsets (IFS) [3] and intuitionistic level subset (ILS) [14] which are explained in detail in our previous works ([10], [11], [12], [13]).

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The following are some important definitions and results that we use in this article during the construction of IFSGs.

Definition 1.[2] *A fuzzy subset $\mathscr{M}$ of a group G is called a* ***Fuzzy Subgroup*** *(FSG) of G if, for every $x,y \in G$*
(1) $\mathscr{M}(xy) \geq \wedge[\mathscr{M}(x), \mathscr{M}(y)]$
(2) $\mathscr{M}(x^{-1}) = \mathscr{M}(x)$.

Proposition 1.[2] *If $\mathscr{M}$ is a FSG of a group G, then the identity element in G will have the highest degree of membership in $\mathscr{M}$.*

Proposition 2.[9] *A fuzzy subset $\mathscr{M}$ of a group G will be a FSG of G if and only if $\mathscr{M}_t = \{x \in X : \mathscr{M}(x) \geq t\}$ is a subgroup of G for all $0 \leq t \leq \mathscr{M}(e)$, and $\mathscr{M}_t$ will be called the* ***Level Subgroup*** *of $\mathscr{M}$ at t.*

Definition 2.[15] *A relation $\preceq$ on a set X is called a* **partial ordering** *if it is reflexive, antisymmetric and transitive. A set X together with a partial ordering $\preceq$ is called a partially ordered set or* **poset**. *The elements a and b of a poset are called* **comparable** *if either $a \preceq b$ or $b \preceq a$. If every two elements of X are comparable then X is called a totally ordered or* **chain**.
Let A be a subset of a poset X. If u is an element of X such that $a \preceq u$ for all elements $a \in A$, then u is called an **upper bound** *of A. If l is an element of X such that $l \preceq a$ for all elements $a \in A$, then l is called a* **lower bound** *of A. An element x is called the* **least upper bound** *of A if $x \preceq u$ for every upper bound u of A. An element y is called the* **greatest lower bound** *of A if $l \preceq y$ for every lower bound l of A. A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a* **lattice**.

Proposition 3.[9] *For a FSG $\mathscr{M}$ of a finite group G with $Im(\mathscr{M}) = \{a_n : n = 1,...,p\}$ (where $Im(\mathscr{M})$ denotes the image set of $\mathscr{M}$), the collection $\{\mathscr{M}_{a_n} : n = 1,...,p\}$ will contain all level subgroups of $\mathscr{M}$. Furthermore, if $\{a_1, a_2, a_3, ..., a_p\}$ is strictly decreasing, then all these level subgroups will form a strictly increasing chain given by $\{x \in G : \mathscr{M}(x) = \mathscr{M}(e)\} = \mathscr{M}_{a_1} \subset \mathscr{M}_{a_2} \subset \mathscr{M}_{a_3} \subset ... \subset \mathscr{M}_{a_p} = G$.*

Proposition 3 stated above is the result proved by P S Das [9] in fuzzy context, which we are trying to carry forward to intuitionistic fuzzy context.

Definition 3.[16] *An IFS $\mathscr{M} = \{\langle x, \mathscr{M}^+(x), \mathscr{M}^-(x)\rangle : x \in G\}$ of a group G is called an* ***Intuitionistic Fuzzy Subgroup*** *(IFSG) of G if and only if*
(1) $\mathscr{M}^+(xy) \geq \wedge[\mathscr{M}^+(x), \mathscr{M}^+(y)]$
(2) $\mathscr{M}^+(x^{-1}) = \mathscr{M}^+(x)$
(3) $\mathscr{M}^-(xy) \leq \vee[\mathscr{M}^-(x), \mathscr{M}^-(y)]$, and
(4) $\mathscr{M}^-(x^{-1}) = \mathscr{M}^-(x)$.

Proposition 4.([16], [17]) *If $\mathscr{M}$ is an IFSG of a group G, then the identity element in G will have the highest degree of membership and least degree of non-membership with respect to $\mathscr{M}$. Also, the generators will have the least degree of membership and highest degree of non-membership with respect to $\mathscr{M}$.*

Proposition 5.([14], [18]) *An IFS $\mathscr{M}$ of a group G will be an IFSG of G if and only if $\mathscr{M}_{\alpha,\beta} = \{x \in G : \mathscr{M}^+(x) \geq \alpha$ and $\mathscr{M}^-(x) \leq \beta\}$ is a subgroup of G for all $0 \leq \alpha \leq \mathscr{M}^+(e)$, $\mathscr{M}^-(e) \leq \beta \leq 1$, and $\mathscr{M}_{\alpha,\beta}$ will be called the* ***Intuitionistic Level Subgroup*** *(ILSGs) of $\mathscr{M}$ at (α, β).*

Proposition 6.([10], [11]) *If $\mathscr{M} = \{\langle x, \mathscr{M}^+(x), \mathscr{M}^-(x)\rangle : x \in G\}$ is an IFSG in a group G and if $\langle a\rangle = \langle b\rangle$ for some $a, b \in G$, then a and b will have same degree of membership and same degree of non-membership with respect to $\mathscr{M}$ (where $\langle a\rangle$ denotes the cyclic subgroup of G generated by $a \in G$).*
Similarly, if $\langle a\rangle \subseteq \langle b\rangle$, then $\mathscr{M}^+(a) \geq \mathscr{M}^+(b)$ and $\mathscr{M}^-(a) \leq \mathscr{M}^-(b)$.

Proposition 7.[18] *If $\mathscr{M} = \{\langle x, \mathscr{M}^+(x), \mathscr{M}^-(x)\rangle : x \in G\}$ is an IFSG in a finite group G with $Im(\mathscr{M}^+) = \{a_n : n = 1,...,p\}$ and $Im(\mathscr{M}^-) = \{b_m : m = 1,...,q\}$. Then*

$$\{\mathscr{M}_{a_n,b_m} : n = 1,...,p; m = 1,...,q\}$$

will contain all ILSGs of G.

*Remark.*From the above proposition 7, it can be concluded that the intuitionistic fuzzy analogue of the first part of proposition 3 holds true.

3 The Main Result

In the next proposition we give a necessary and sufficient condition on an arbitrary group G so that all the ILSGs of every IFSG of G may form a chain.

*Remark.*In the proof of the following proposition, **Intuitionistic Level Representation (ILR)** [11] of IFS $\mathscr{M}$ in a non-empty finite set X refers to the *finite sequence* $\tilde{Ł}(\mathscr{M}) = \{\mathscr{M}_{t_1,s_1}, \mathscr{M}_{t_1,s_2}, ..., \mathscr{M}_{t_1,s_q}, \mathscr{M}_{t_2,s_1}, \mathscr{M}_{t_2,s_2}, ..., \mathscr{M}_{t_2,s_q}, ..., \mathscr{M}_{t_p,s_1}, \mathscr{M}_{t_p,s_2}, ..., \mathscr{M}_{t_p,s_q}\}$ which contains all intuitionistic level subsets of $\mathscr{M}$, where $Im(\mathscr{M}^+) = \{t_i : i = 1,2,3,...,p\}$ and $Im(\mathscr{M}^-) = \{s_j : j = 1,2,3,...,q\}$ with $1 \geq t_1 > t_2 > ... > t_p \geq 0$ and $0 \leq s_1 < s_2 < ... < s_q \leq 1$.

Proposition 8.*The ILSGs of all IFSGs of a group G form chains if, and only if, the lattice of all subgroups of G is a chain.*

*Proof.*Suppose the lattice of all subgroups of G is a chain and let $\mathscr{M}$ be any IFSG of G. Then clearly the ILSGs of $\mathscr{M}$ will form a chain, since all the ILSGs of $\mathscr{M}$ are also subgroups of G. This proves the sufficiency part.
Conversely, suppose the lattice of all subgroups of G is not a chain. Then there exist two subgroups G_1 and G_2 in G such that $G_1 \nsubseteq G_2$ and $G_1 \nsupseteq G_2$. Let $G_3 = G_1 \cap G_2$. Define $\mathscr{M}^+, \mathscr{M}^- : G \to I$ as follows:

$$\mathscr{M}^+(x) = \begin{cases} a_1, & \forall x \in G_3 \\ a_2, & \forall x \in G_1 - G_3 \\ a_3, & \forall x \in G - G_1 \end{cases}$$

$$\mathscr{M}^-(x) = \begin{cases} b_1, & \forall x \in G_3 \\ b_2, & \forall x \in G_2 - G_3 \\ b_3, & \forall x \in G - G_2 \end{cases}$$

where $a_1 > a_2 > a_3$ and $b_1 < b_2 < b_3$ in I. It can be easily verified that $\mathscr{M}$ satisfies all the four axioms for an IFSG and hence $\mathscr{M}$ is an IFSG of G whose geometric representation is given in figure 1.

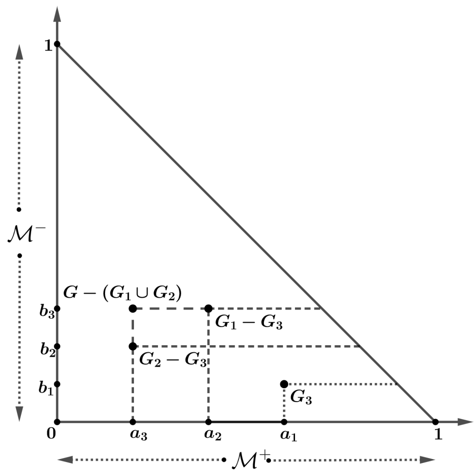

Fig. 1: IFSG $\mathscr{M}$ in proof of proposition 8

The ILR of $\mathscr{M}$ is

$$\begin{aligned}\tilde{Ł}(\mathscr{M}) &= \{\mathscr{M}_{a_1,b_1}, \mathscr{M}_{a_1,b_2}, \mathscr{M}_{a_1,b_3}, \mathscr{M}_{a_2,b_1}, \mathscr{M}_{a_2,b_2}, \mathscr{M}_{a_2,b_3}, \mathscr{M}_{a_3,b_1}, \mathscr{M}_{a_3,b_2}, \mathscr{M}_{a_3,b_3}\} \\ &= \{G_3, G_3, G_3, G_3, G_3, G_1, G_3, G_2, G\}\end{aligned}$$

Hence, the distinct ILSGs of $\mathscr{M}$ are: G_3, G_1, G_2, G which do not form a chain since $G_1 \nsubseteq G_2$ and $G_1 \nsupseteq G_2$. This proves the necessary part.

4 ILSGs in finite groups

With the help of the main result proved in the previous section, next we obtain a necessary and sufficient condition for a finite group G so that all the ILSGs of every IFSG of G may form a chain.

Proposition 9.[19] *The lattice of subgroups of a finite group G is a chain if, and only if, G is isomorphic to* $\mathbb{Z}_{p^n}$ *for some prime number p and some positive integer n.*

Using proposition 9 we can restate proposition 8 for finite groups as follows.

Proposition 10.*The ILSGs of all IFSGs of a finite group G form chains if, and only if, G is isomorphic to* $\mathbb{Z}_{p^n}$ *for some prime number p and some positive integer n.*

Consider the group $G = \{1, -1, i, -i\}$ under the multiplication of complex numbers. Then subgroups of G are $\{1\} = \langle 1 \rangle, \{1, -1\} = \langle -1 \rangle, G = \langle i \rangle = \langle -i \rangle$ and hence the lattice of subgroups of G is a chain as shown in figure 2.

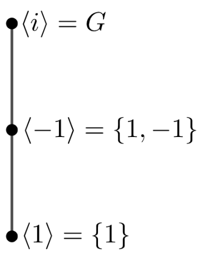

Fig. 2: Lattice of subgroups of G in illustration 4

Let $\mathscr{M} = \{\langle x, \mathscr{M}^+(x), \mathscr{M}^-(x) \rangle : x \in G\}$ be an IFSG of G. Since $\langle 1 \rangle \subset \langle -1 \rangle \subset \langle i \rangle = \langle -i \rangle$ and since $-i$ is the inverse of i, by proposition 6 and axioms of IFSG, we get:

$$\mathscr{M}^+(1) \geq \mathscr{M}^+(-1) \geq \mathscr{M}^+(i) = \mathscr{M}^+(-i)$$
$$\mathscr{M}^-(1) \leq \mathscr{M}^-(-1) \leq \mathscr{M}^-(i) = \mathscr{M}^-(-i)$$

Hence $\mathscr{M}^+, \mathscr{M}^- : G \to I$ should be defined as follows:

$$\mathscr{M}^+(x) = \begin{cases} a_1, & if\ x = 1 \\ a_2, & if\ x = -1 \\ a_3, & if\ x = i, -i \end{cases}$$

$$\mathscr{M}^-(x) = \begin{cases} b_1, & if\ x = 1 \\ b_2, & if\ x = -1 \\ b_3, & if\ x = i, -i \end{cases}$$

where $a_1 > a_2 > a_3$ and $b_1 < b_2 < b_3$ in I. Then ILR of $\mathscr{M}$ is:

$$\begin{aligned} \tilde{Ł}(\mathscr{M}) &= \{\mathscr{M}_{a_1,b_1}, \mathscr{M}_{a_1,b_2}, \mathscr{M}_{a_1,b_3}, \mathscr{M}_{a_2,b_1}, \mathscr{M}_{a_2,b_2}, \mathscr{M}_{a_2,b_3}, \mathscr{M}_{a_3,b_1}, \mathscr{M}_{a_3,b_2}, \mathscr{M}_{a_3,b_3}\} \\ &= \{\{1\}, \{1\}, \{1\}, \{1\}, \{1, -1\}, \{1, -1\}, \{1\}, \{1, -1\}, G\} \end{aligned}$$

Hence the distinct ILSGs of $\mathscr{M}$ are: $\{1\}, \{1, -1\}, G$ which form a chain.

*Remark.*It may be noted that the group G in illustration 4 is in fact isomorphic to $\mathbb{Z}_{2^2} = \mathbb{Z}_4$ and hence supports proposition 10.

Consider the group $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ under the addition modulo 6. Then subgroups of $\mathbb{Z}_6$ are $\{0\} = \langle 0 \rangle, \{0,2,4\} = \langle 2 \rangle = \langle 4 \rangle, \{0,3\} = \langle 3 \rangle, \mathbb{Z}_6 = \langle 1 \rangle = \langle 5 \rangle$ and hence the lattice of subgroups of G is not a chain as shown in figure 3.

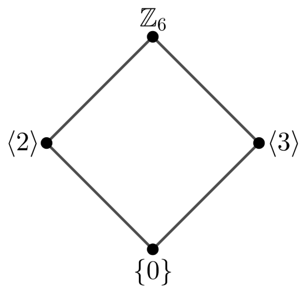

Fig. 3: Lattice of subgroups of G in illustration 4

Suppose $\mathscr{M}$ is an IFSG of $\mathbb{Z}_6$. As stated in proposition 4 the identity element 0 should be having highest degree of membership and least degree of non-membership. Also the generators 1 and 5 should have least degree of membership and highest degree of non-membership. Using proposition 6, since $\langle 2 \rangle = \langle 4 \rangle$, 2 and 4 should have same degrees of membership and non-membership. Now since 2 and 3 are independent of each other (in the perspective of cyclic subgroups generated by them), they can be independently assigned degrees of membership and non-membership. Taking all these into account $\mathscr{M} = \{\langle 0,0.7,0.1 \rangle, \langle 1,0.4,0.4 \rangle, \langle 2,0.4,0.3 \rangle, \langle 3,0.5,0.4 \rangle, \langle 4,0.4,0.3 \rangle, \langle 5,0.4,0.4 \rangle\}$ is an IFSG of $\mathbb{Z}_6$, whose ILR is given as:

$$\begin{aligned}\tilde{\text{Ł}}(\mathscr{M}) &= \left\{\mathscr{M}_{0.7,0.1}, \mathscr{M}_{0.7,0.3}, \mathscr{M}_{0.7,0.4}, \mathscr{M}_{0.5,0.1}, \mathscr{M}_{0.5,0.3}, \mathscr{M}_{0.5,0.4}, \mathscr{M}_{0.4,0.1}, \mathscr{M}_{0.4,0.3}, \mathscr{M}_{0.4,0.4}\right\} \\ &= \{\{0\},\{0\},\{0\},\{0\},\{0\},\{0,3\},\{0\},\{0,2,4\},\mathbb{Z}_6\}\end{aligned}$$

Hence the distinct ILSGs of $\mathscr{M}$ are: $\{0\},\{0,3\},\{0,2,4\},\mathbb{Z}_6$ which do not form a chain.

*Remark.*It may be noted that the group $\mathbb{Z}_6$ in illustration 4 is not isomorphic to $\mathbb{Z}_{p^n}$ for any p and n and hence supports proposition 10.

5 ILSGs in infinite groups

Next we will consider the case of an infinite cyclic group.

Remark.[11] In the following example we have used a geometric representation of IFSG $\mathscr{M}$, which is described as follows: Let X be any set and $\mathscr{M} = \{\langle x, \mathscr{M}^+(x), \mathscr{M}^-(x) \rangle : x \in X\}$ be an IFS of X. To represent $\mathscr{M}$ and its ILS geometrically, we take $\mathscr{M}^+$ along x-axis and $\mathscr{M}^-$ along y-axis. Then an element x in $\mathscr{M}$ is represented by the point $(\mathscr{M}^+(x), \mathscr{M}^-(x))$ in the xy-plane. In this representation all elements of $\mathscr{M}$ will lie inside the triangle bounded by $x = 0, y = 0$ and the line $x + y = 1$. Also, an ILS $\mathscr{M}_{a,b}$ of $\mathscr{M}$ will consist of all elements of X represented by points in the region $\{(x,y) : x \geq a, y \leq b, x + y \leq 1\}$.

*Example 1.*Consider the group of integers $\mathbb{Z}$ under addition. Then lattice of subgroups of $\mathbb{Z}$ is not a chain since the subgroups $2\mathbb{Z}$ and $3\mathbb{Z}$ of $\mathbb{Z}$ are such that $2\mathbb{Z} \nsubseteq 3\mathbb{Z}$ and $2\mathbb{Z} \nsupseteq 3\mathbb{Z}$. Let $2\mathbb{Z} \cap 3\mathbb{Z} = H$ and define $\mathscr{M}^+, \mathscr{M}^- : \mathbb{Z} \to I$ as follows:

$$\mathscr{M}^+(x) = \begin{cases} a_1, & \forall x \in H \\ a_2, & \forall x \in 2\mathbb{Z} - H \\ a_3, & \forall x \in G - 2\mathbb{Z} \end{cases}$$

$$\mathscr{M}^-(x) = \begin{cases} b_1, & \forall x \in H \\ b_2, & \forall x \in 3\mathbb{Z} - H \\ b_3, & \forall x \in G - 3\mathbb{Z} \end{cases}$$

where $a_1 > a_2 > a_3$ and $b_1 < b_2 < b_3$ in I. It can be easily verified that $\mathscr{M}$ satisfies all the four axioms for an IFSG and hence $\mathscr{M}$ is an IFSG of $\mathbb{Z}$ whose geometric representation is given in figure 4.

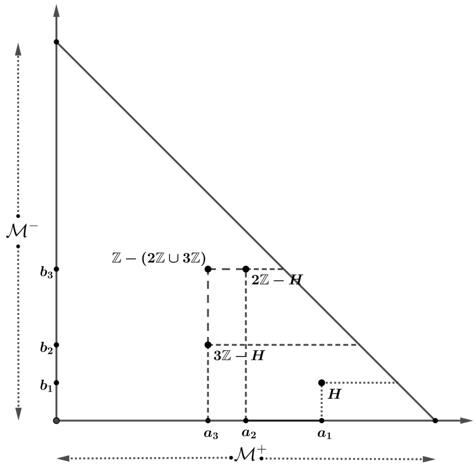

Fig. 4: IFSG $\mathscr{M}$ in example 1

The ILR of $\mathscr{M}$ is:

$$\begin{aligned}\tilde{Ł}(\mathscr{M}) &= \{\mathscr{M}_{a_1,b_1},\mathscr{M}_{a_1,b_2},\mathscr{M}_{a_1,b_3},\mathscr{M}_{a_2,b_1},\mathscr{M}_{a_2,b_2},\mathscr{M}_{a_2,b_3},\mathscr{M}_{a_3,b_1},\mathscr{M}_{a_3,b_2},\mathscr{M}_{a_3,b_3}\}\\ &= \{H,H,H,H,H,2\mathbb{Z},H,3\mathbb{Z},G\}\end{aligned}$$

Hence the distinct ILSGs of $\mathscr{M}$ are: $H,2\mathbb{Z},3\mathbb{Z},G$ which do not form a chain since $2\mathbb{Z}\not\subseteq 3\mathbb{Z}$ and $2\mathbb{Z}\not\supseteq 3\mathbb{Z}$.

*Remark.*Every infinite cyclic group is isomorphic to $\mathbb{Z}$ and we know that the lattice of subgroups of $\mathbb{Z}$ is not a chain. So there does not exist an infinite cyclic group whose lattice of subgroups is a chain. And hence, *there does not exist an infinite cyclic group whose ILSGs of all IFSGs form chains*.

In the next two examples we will see cases of infinite non-cyclic groups. First one is an infinite non-cyclic group in which ILSGs of all IFSGs form chains and second is an infinite non-cyclic group in which ILSGs of some IFSGs do not form chain.

*Example 2.*Let G be the well known $Prüfer\ p-group$, $\mathbb{Z}(p^\infty)=\{z\in\mathbb{C}:z^{p^n}=1,n\in\mathbb{N}\}$ (p being any prime number) with respect to multiplication of complex numbers. Here G consists of the set of all p^{th} roots of unity, $(p^2)^{th}$ roots of unity, $(p^3)^{th}$ roots of unity, and so on. Then:
(i) G is an infinite non-cyclic group.
(ii) Every proper subgroup of G is finite and cyclic.
(iii) If A and B are any two subgroups of G, then either $A\subseteq B$ or $B\subseteq A$ (i.e., subgroups of G form a chain).
(iv) For every $n\geq 0$, G has a unique subgroup with p^n elements given by $G_n=\{z\in\mathbb{C}:z^{p^n}=1\}$.
(v) $G=\bigcup_{n\in\mathbb{N}}G_n$.

The lattice of subgroups of G which is a chain is shown in figure 5 and figure 6 depicts the case when $p=2$.

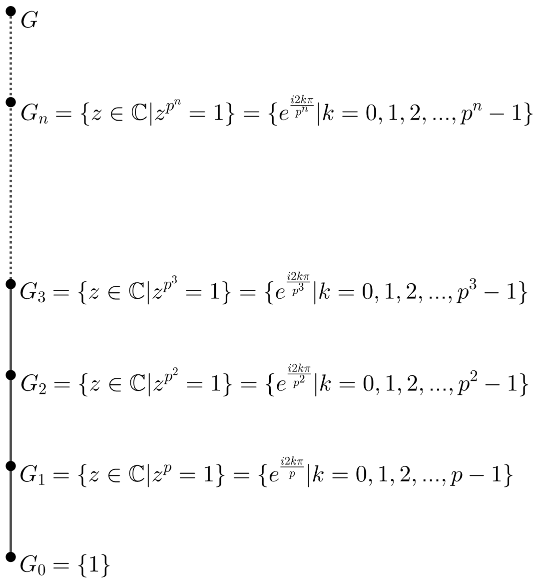

Fig. 5: Lattice of subgroups of G in example 2

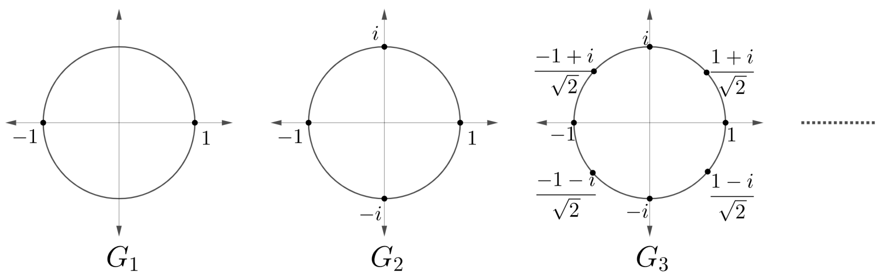

Fig. 6: G_1, G_2, G_3 as in example 2, for $p = 2$

Let $\mathscr{M} = \{\langle x, \mathscr{M}^+(x), \mathscr{M}^-(x)\rangle : x \in G\}$ be any IFSG of G. Since $G_0 = \{1\} \subset G_1 = \left\langle e^{\frac{i2k\pi}{p}} \right\rangle \subset G_2 = \left\langle e^{\frac{i2k\pi}{p^2}} \right\rangle \subset G_3 = \left\langle e^{\frac{i2k\pi}{p^3}} \right\rangle \subset ... \subset G_n = \left\langle e^{\frac{i12k\pi}{p^n}} \right\rangle \subset ... \subset G$, by properties and axioms of IFSG, one can establish that $\mathscr{M}^+, \mathscr{M}^- : G \to I$ must have the following forms:

$$\mathscr{M}^+(x) = \begin{cases} a_1, & x = 1 \\ a_2, & x \in G_1 - \{1\} \\ a_3, & x \in G_2 - G_1 \\ ... & ... \\ a_{n+1}, & x \in G_n - G_{n-1} \\ ... & ... \\ ... & ... \end{cases}$$

$$\mathscr{M}^-(x) = \begin{cases} b_1, & x = 1 \\ b_2, & x \in G_1 - \{1\} \\ b_3, & x \in G_2 - G_1 \\ ... & ... \\ b_{n+1}, & x \in G_n - G_{n-1} \\ ... & ... \\ ... & ... \end{cases}$$

where $a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \geq ...$ and $b_1 \leq b_2 \leq b_3 \leq ... \leq b_n \leq ...$ in I. The geometric representation of $\mathscr{M}$ is given in figure 7 and from the figure it is clear that the distinct ILSGs of $\mathscr{M}$ form a chain.

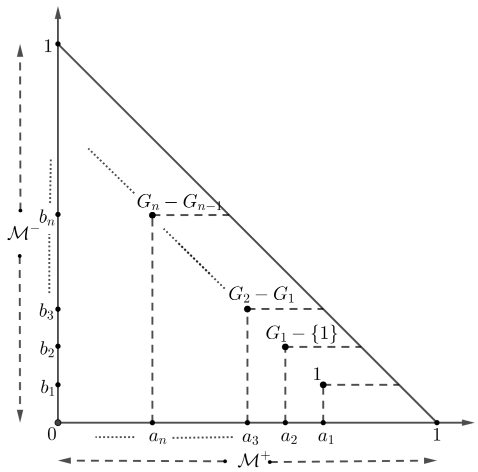

Fig. 7: IFSG $\mathscr{M}$ in example 2

*Example 3.*Next we will see another infinite non-cyclic group. Consider the additive group $G = \{\frac{a}{p^k} | a \in \mathbb{Z}, k \in \mathbb{N}\}$ for any fixed prime number p, which is a subgroup of rationals under addtion. Then G is not cyclic, since for any $\frac{a}{p^k} \in G$, $\langle \frac{a}{p^k} \rangle$ will not contain the elements of G with denominator greater than p^k. Also, the subgroups of G will not form a chain. In particular, we can see that the subgroups $G_1 = \langle \frac{2}{p} \rangle$ and $G_2 = \langle \frac{3}{p} \rangle$ are such that $G_1 \nsubseteq G_2$ and $G_1 \nsupseteq G_2$. So if we define an IFSG $\mathscr{M}$ of G as in the proof of theorem 8, then the ILSGs of $\mathscr{M}$ will not form a chain.

6 Conclusion

In earlier days a detailed research on the theory of level subgroups of fuzzy subgroups of a finite group had been done and was proved that the level subgroups form a chain. Following this in some of our previous works we have made an attempt to extend this result to intuitionistic fuzzy context and have arrived at the several conclusions regarding some particular groups like $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{pq}$, Klein-4 Group and Dihedral group D_3. In this article we have generalised our previous findings to an arbitrary group. For an arbitrary group G (finite or infinite) we have obtained that, the ILSGs of all IFSGs of G form a chain if, and only if, the lattice of subgroups of G is a chain. In particular, if G is a finite group isomorphic to $\mathbb{Z}_{p^n}$ for some prime number p and some $n \in \mathbb{N}$, then for any IFSG $\mathscr{M}$ of G, all the ILSGs of $\mathscr{M}$ form a chain. All other finite groups have some IFSGs whose ILSGs form a chain, and some whose ILSGs do not form chains. Now coming to the case of infinite groups we have shown that, there does not exist infinite cyclic groups whose ILSGs of all IFSGs form chains. Infinite non cyclic groups can be constructed in which ILSGs of all IFSGs form chains and in which ILSGs of some IFSGs do not form chains. Eventhough the study about ILSGs has come to somewhat an end along this pathway, in our future works we will be trying to get generalisations about ILSGs of Dihedral groups and finite cyclic groups of arbitrary orders.

Declarations

Competing interests: No competing interest declared.
Authors' contributions: The work was designed and executed by the first author, Divya Mary Daise S, and the second author, Deepthi Mary Tresa S. They also prepared the manuscript. The corresponding author, Shery Fernandez, contributed to the design and proofreading of the manuscript.
Funding: No external funding was utilized for this research.
Availability of data and materials: The work presented in this manuscript builds upon our previous research endeavors.

Acknowledgments:We would like to acknowledge that there are no individuals or organizations to acknowledge for their contributions to this research.

References

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8**(1965), 338–353.
[2] A. Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications*, **35**(1971), 512–517.
[3] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20(1)**(1986), 87–96.
[4] R. Biswas, Intuitionistic fuzzy subgroups, *Mathematical Forum*, **X**(1996), 39–44.
[5] M. Gulzar,D. Alghazzawi, M. H. Mateen, and N. Kausar, A certain class of t-intuitionistic fuzzy subgroups, *IEEE Access*, **8**(2020), 163260–163268.
[6] S. Shaqaqha, On fuzzification of n-Lie algebras, *Jordan Journal of Mathematics and Statistics*, **15(3A)**(2022), 523–540.
[7] S. Shaqaqha, Fuzzy HomLie ideals of HomLie algebras, *Axioms 2023*, **12**(2023), 630.
[8] S. Shaqaqha, Complex Fuzzy Lie Algebras, *Jordan Journal of Mathematics and Statistics*, **13(2)**(2020), 231–247.
[9] P. S. Das, Fuzzy groups and level subgroups, *Journal of Mathematical Analysis and Applications*, **84**(1981), 264–269.
[10] Divya Mary Daise S, Deepthi Mary Tresa S and S. Fernandez, The chain structure of intuitionistic level subgroups in cyclic groups of order pq, *Malaya Journal of Mathematik*, **8(4)**(2020), 1818–1823.
[11] Divya Mary Daise S, Deepthi Mary Tresa S and S. Fernandez, Intuitionistic level subgroups in cyclic groups of order p^n, *South East Asian Journal of Mathematics and Mathematical Sciences*, **17(1)**(2021), 125–138.
[12] Divya Mary Daise S, Deepthi Mary Tresa S and S. Fernandez, Intuitionistic level subgroups in the klein-4 group, *Creative Mathematics and Informatics*, **30(2)**(2021), 164–173.
[13] Divya Mary Daise S, Deepthi Mary Tresa S and S. Fernandez, Intuitionistic level subgroups in the dihedral group D_3, *Creative Mathematics and Informatics*, **31(2)**(2022), 185–194.
[14] P. K. Sharma, Intuitionistic fuzzy groups, *(IFRSA) International Journal of Data Warehousing and Mining*. **1(1)** (2011).
[15] K. H. Rosen, *Discrete Mathematics and its Application (6th edition)*, McGraw-Hill, (2007).
[16] N. Palaniappan, S. Naganathan S and K Arjunan, A study on intuitionistic L-Fuzzy subgroups, *Applied Mathematical Sciences*, **3(53)**(2009), 2619–2624.
[17] Divya Mary Daise S, Deepthi Mary Tresa S and S. Fernandez, Intuitionistic level subgroups of intuitionistic fuzzy groups, *Proceedings of International Conference of Mathematics (ICM2018)*, (2018), 78–82.
[18] Divya Mary Daise S and Deepthi Mary Tresa S, On level subgroups of intuitionistic fuzzy groups, *Journal of Computer and Mathematical Sciences*, **7(11)**(2016), 606–612.
[19] A. Jez, Subgroup lattices that are chains, *Rose-Hulman Undergraduate Mathematics Journal*, **7(2)**(2006), Article 4.