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# Central Tendency Measurements Estimation for Skew Normal Distributions Using Taylor Series Expansion and Simpson's Rule

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**Abstract:** This study aims to estimate central tendency measurements, including mode and median of skew normal distributions using Taylor series expansion and Simpson's Rule. Skew normal distributions are characterized as continuous, unimodal, and strictly quasi-concave. Specifically, to compute the mode approximately, the derivative of the sum of the first three terms in the Taylor series expansion is set to be zero and then the equation is solved to find the unknown and to compute the median, the definite integration of skew normal distribution is evaluated using Simpson's Rule. SAS macro programs are developed to verify and assess the accuracy of these computations under different skewness levels.

**Keywords:** Central Tendency; Approximation; Estimation, Skew Normal Distribution; Unimodal; Quasi-Concave; Taylor Series Expansion; Simpson's Rule.

**2010 Mathematics Subject Classification**. 62H10; 62H05.

## 1 Introduction

Central tendency measures, such as the mean $(M)$, mode $(M_o)$, and median $(M_e)$, are essential for describing the statistical characteristics of data distributions. The mode $M_o$ represents the most frequently occurring value in a data set. The median $(M_e)$ is the midpoint, where half the data points are smaller and half are larger. While mean, median and mode are straightforward to calculate for symmetric distributions, whether discrete or continuous, estimating them for skewed distributions can be challenging (Beyer, 2021). Most researchers use the mean to summarize continuous data but prefer the median for the skewed data, as the mean is more affected by extreme values (McClave et al., 2010). Skewed distributions often cause discrepancies between the mean and median, making robust estimators like the median more reliable.

The relationship between mode, median, and mean in skewed distributions is well-documented, but there are exceptions to traditional inequalities. Methods for robustly estimating the mode, including transformation and Taylor's expansion, have been widely discussed. Traditionally, it was shown that for a positively skewed, unimodal distribution, the inequality $M_o < M_e < M$ holds, and the reverse for negatively skewed distributions, $M_o > M_e > M$. However, counterexamples and conditions for these inequalities have been provided by van Zwet (1979), Abadir (2005), von Hippel (2005) and Zheng et al. (2017).

Bickel (2002) discussed methods to estimate both a simple mode, based on the mean and standard deviation, and a more robust mode, based on the median and the standardized median absolute deviation to create a reliable mode estimator less influenced by outliers. Another method for estimating the mode or median of a skewed distribution involves variable

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transformation. If a random variable X follows a continuous, skewed, and unimodal distribution, its mode can be approximated by transforming it to become approximately normal. The mean of this normal distribution can then be used to estimate the mode of X based on the transformation relationship. Additionally, Taylor's formula is used to approximate functions by polynomials, providing an important method for mode estimation through its generalization of the mean value theorem.

For further discussion, we define normal and skew normal distributions. The standard normal probability function is defined as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty. \tag{1}$$

The standard normal cumulative distribution function is given by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt, -\infty < x < \infty. \tag{2}$$

Then the univariate probability density function (pdf) of the skew normal distribution with skew parameter $\lambda$ is defined as

$$f(x,\lambda) = 2\phi(x)\Phi(\lambda x), -\infty < x, \lambda < \infty. \tag{3}$$

The skew normal distribution (3), denoted by $SN(\lambda)$, was first introduced by O'Hagan and Leonard (1976). The univariate normal distribution is a special case of the univariate skew normal distribution when $\lambda = 0$. As the absolute value of the skew parameter $\lambda$ increases, so does the skewness. The distribution is right-skewed if $\lambda > 0$ and left-skewed if $\lambda < 0$.

Various studies have explored alternative forms and applications of the skew normal distribution. Ashour and Abdel-Hameed (2010) and Mudholkar and Hutson (2000) provided alternative forms and quantile functions. Andel et al. (1984) studied a stochastic process underpinning the distribution. Al-Olaimat and Al-Zou'bi (2018) examined the efficiency of adaptive methods using simulated alpha skew normal two-stage data. Shanker and Shukla (2019) introduced and investigated the asymmetric zero-truncated Poisson-Aradhana distribution. Nagarjuna and Chesneau (2020) proposed the sine inverse power Lomax (SIPL) distribution, discussing its range of skewness, peakedness, and plateness. Alencar et al. (2022) developed a robust model for censored and/or missing data based on finite mixtures of multivariate skew-normal distributions for heterogeneous populations with asymmetric behavior. Morales et al. (2022) derived moments of folded and doubly truncated multivariate extended skew-normal distributions. Hasanalipour (2022) studied Fisher information in order statistics related to the skewness parameter. Alhribat and Samuh (2023) introduced a technique for generating fractional continuous probability distributions by solving fractional differential equations associated with well-known symmetric or skew continuous probability distributions.

Considering location ($\xi$) and scale ($\omega$) parameters, the more general univariate skew normal probability density function is given by

$$f(x,\xi,\omega) = \frac{2}{\omega}\phi\big(\frac{x-\xi}{\omega}\big)\Phi\big(\lambda\big(\frac{x-\xi}{\omega}\big)\big). \tag{4}$$

Without loss generality, we focus on the main properties of the standardized skew normal distribution (3) $SN(\lambda)$, which can be obtained by letting the location parameter $\xi = 0$ and the scale parameter $\omega = 1$) in equation (4). Let Z be a random variable following $SN(\lambda)$ with density function $f(x,\lambda)$ given by equation (3). Additionally, let $\phi(x)$ represent the standard normal density function defined in equation (1). The following properties hold true (Azzalini and Capitaniao, 2014):
(a) $f(x,0) = \phi(x)$ for all x;
(b) $f(0,\lambda) = \phi(0)$ for all $\lambda$;
(c) $-Z \sim SN(-\lambda)$, equivalently $f(-x,\lambda) = f(x,-\lambda)$ for all x;
(d) $Z^2 \sim \chi_1^2$ irrespective of $\lambda$.

Figures 1 and 2 display skew normal density plots for the skew normal parameter $\lambda = \pm 1, \pm 2$. The standard normal curve is included in both figures as a reference. Properties (a), (b), and (c) mentioned above are illustrated by these figures. It is evident from the figures that the distribution $SN(\lambda)$ becomes highly skewed when $\lambda = \pm 2$. However, in practical data analysis, encountering data from highly skewed populations when $|\lambda| > 2$ is uncommon.
Azzalini et al. (2014) provided a complex formula to approximate the mode of the skew normal distribution. However, there is a gap in the current statistical literature regarding the exact computation of the median and mode for most

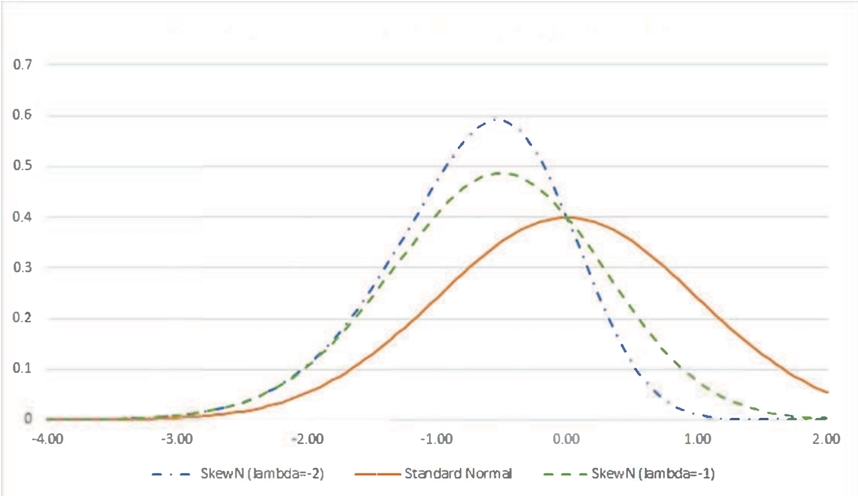


**Fig. 1:** Skew Normal Density Functions When $\lambda = -1, -2$

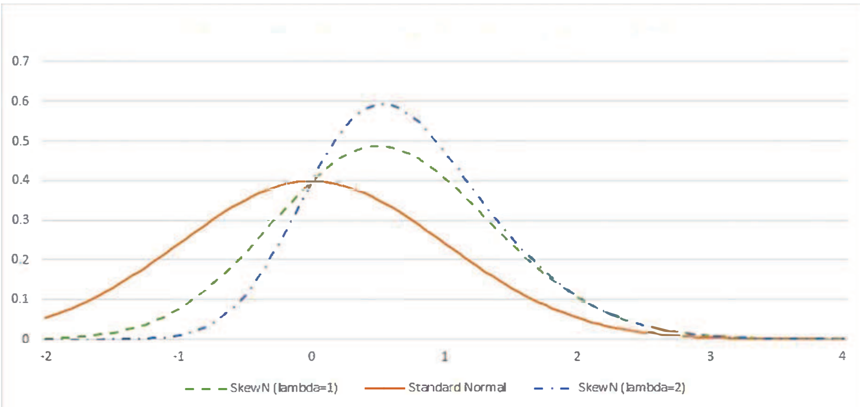


**Fig. 2:** Skew Normal Density Functions When $\lambda = 1, 2$

skewed unimodal continuous distributions. In the standard normal distribution, the mean, median, and mode are all equal to 0. For a more general normal distribution, the mode and median remain equal to the mean. In skewed distributions, whether positively or negatively, extreme values significantly affect the mean, pulling it in the direction of the skew. In contrast, the median and mode are more robust statistics and are not influenced as much by the skew as the mean. Consequently, the traditional approach is to estimate the median and mode, as current methods are limited in determining their exact values once the distribution is skewed. One option for estimating the mode of the skew normal distribution is using the Taylor expansion of $\Phi(\lambda x)$.

In next section,Taylor's expansion of $\Phi(\lambda x)$ is derived first. Then the estimation of the mode and median of skew normal distribution will be done using and without using Taylor's expansion of $\Phi(\lambda x)$ or using Simpson's rule, or using Azzalini and Capitaniao (2014) formulas respectively. To find the true mode, unimodal and continuous property of skew normal distributions was utilized and a SAS macro procedure was developed for this purpose. To find the true median, Simpson's rule was utilized and a SAS macro procedure was also created. Also, comparison of estimation of mode and median of skew normal distribution by different methods will be done. At the end, remarks, discussion and conclusions will be presented.

## 2 Taylor's expansion of $\Phi(\lambda x)$

To derive the Taylor's expansion of $\Phi(\lambda x)$, the first order derivative $\Phi'(\lambda x)$, the second order derivative $\Phi''(\lambda x)$, the third order derivative $\Phi'''(\lambda x)$ and etc. should be determined and evaluated at point 0 respectively, as:

$$\begin{aligned}
\Phi(\lambda x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda x} e^{-\frac{1}{2}x^2}dt, \Phi(0) = \frac{1}{2}, \\
\Phi'(\lambda x) &= \frac{1}{\sqrt{2\pi}}\lambda e^{-\frac{1}{2}\lambda^2 x^2}, \Phi'(0) = \frac{\lambda}{\sqrt{2\pi}}, \\
\Phi''(\lambda x) &= \frac{-\lambda^3 x}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2 x^2}, \Phi''(0) = 0, \\
\Phi'''(\lambda x) &= \frac{\lambda^3}{\sqrt{2\pi}}(-1+\lambda^2 x^2)e^{-\frac{1}{2}\lambda^2 x^2} + \frac{\lambda^5 x^2}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2 x^2}, \Phi'''(0) = -\frac{\lambda^3}{\sqrt{2\pi}}.
\end{aligned}$$

Thus, we have Taylor's expansion of $\Phi(\lambda x)$.

$$\begin{aligned}
\Phi(\lambda x) &= \Phi(0) + x\Phi'(0) + \frac{1}{2!}x^2\Phi''(0) + \frac{1}{3!}x^3\Phi'''(0) + \cdots \\
&= \frac{1}{2} + \frac{\lambda}{\sqrt{2\pi}}x + \frac{1}{2!}x^2(0) - \frac{1}{3!}\frac{\lambda^3}{\sqrt{2\pi}}x^3 + \cdots \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{i=0}^{\infty}\frac{(-i)^i(\lambda x)^{2i+1}}{i!2^i(2i+1)}. \qquad (5)
\end{aligned}$$

Especially, when the skew parameter $\lambda = 1$, we have

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{\infty}\frac{(-1)^i x^{2i+1}}{i!2^i(2i+1)}. \qquad (6)$$

## 3 Estimate the Mode of Skew Normal Distribution Using Taylor's Expansion of $\Phi(\lambda x)$

Azzalini and Capitaniao (2014) provided a formula for computing the mode of the skew normal distribution approximately. But the formula is lengthy and complex. In this article, we provided simple yet relatively accurate formulas to estimate the mode of the skew normal distributions using Taylor's expansion. The partial sum formed by the first $n+1$ terms of a Taylor series is a polynomial of degree $n$ that is called the $n^{th}$ Taylor polynomial of the function. Taylor polynomials are approximations of a function, which become generally more accurate as n increases. To compute the approximation of the mode, we keep the first three terms in the Taylor series first. The derivative of the sum of the first three terms in the Taylor series then becomes a polynomial of degree 2, which is set to be zero so that the equation is solved to find the unknown $M_o$. There are two methods to achieve this purpose. Firstly, the Taylor's expansion of $\Phi(\lambda x)$ will be used directly without deriving the Taylor's expansion of $SN(\lambda)$. Secondly, the Taylor's expansion of $\Phi(\lambda x)$ will not be used. Instead, the Taylor's expansion of $SN(\lambda)$ will be derived and used. Based on (5), we have the following:

$$\begin{aligned}
f(x,\lambda) &= \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\Phi(\lambda x) \\
&= \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\left\{\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{i=0}^{\infty}\frac{(-1)^i(\lambda x)^{2i+1}}{i!2^i(2i+1)}\right\} \\
f'(x,\lambda) &= \sqrt{\frac{2}{\pi}}(-x)e^{-\frac{1}{2}x^2}\left\{\frac{1}{2} + \frac{\lambda x}{\sqrt{2\pi}} - \frac{1}{3!}\frac{\lambda^3 x^2}{\sqrt{2\pi}} + \cdots\right\} \\
&\quad + \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\left\{\frac{\lambda}{\sqrt{2\lambda}} - \frac{1}{3!}\frac{3\lambda^3 x^3}{\sqrt{2\pi}} + \cdots\right\}
\end{aligned}$$

Let $f'(x,\lambda)=0$, then we have

$$
\begin{aligned}
-\frac{1}{2}x-\frac{\lambda x^2}{\sqrt{2\pi}}+\frac{1}{3!}\frac{\lambda^3x^4}{\sqrt{2\pi}}+\cdots+\frac{\lambda}{\sqrt{2\pi}}-\frac{\lambda^3x^2}{2\sqrt{2\pi}}+\cdots=0\\
\frac{\lambda}{\sqrt{2\pi}}-\frac{1}{2}x-\frac{2\lambda+\lambda^3}{2\sqrt{2\pi}}x^2+\cdots=0\\
(2\lambda+\lambda^3)x^2+\sqrt{2\pi}x-2\lambda\approx 0 \qquad (7)\\
x_{1,2}\approx\frac{-\sqrt{2\pi\pm 8\lambda(2\lambda+\lambda^3)}}{2(2\lambda+\lambda^3)}=\frac{-\sqrt{2\pi}\pm\sqrt{2\pi+8\lambda^2(2+\lambda^2)}}{2\lambda(2+\lambda^2)}.
\end{aligned}
$$

**Proposition 1**: Assuming a random variable $Z$ is distributed as skew normal $SN(\lambda)$, then the mode of $Z$ can be estimated approximately by the following formula:

$$
M_o(\lambda)\approx\frac{-\sqrt{2\pi}+\sqrt{2\pi+8\lambda^2(2+\lambda^2)}}{2\lambda(2+\lambda^2)}. \qquad (8)
$$

If $\lambda=0$, then based on (7) we have the standard normal distribution and $M_o(0)=0$. If $\lambda=1$, then we have right skew normal distribution, then based on (8) we have $M_o(1)\approx\frac{-\sqrt{2\pi}+\sqrt{2\pi+24}}{6}=0.49940$.

**Proposition 2**: Assuming a random variable $Z$ is distributed as skew normal $SN(\lambda)$, let $Y=\xi+\omega Z,(\xi\varepsilon R,\omega\varepsilon R^+)$, then we have $Y=SN(\xi,\omega^2,\lambda)$ and the $M_o$ of $Y$ is

$$
M_o\approx\xi+\frac{\omega(-\sqrt{2\pi}+\sqrt{2\pi+8\lambda^2(2+\lambda^2)})}{2\lambda(2+\lambda^2)}. \qquad (9)
$$

*Proof.* Based on Proposition 1, the mode of random variable $Z$ is as (8) and based on Azzalini and Capitaniao (2014) we have (9).

## 4 Alternative Approximate of the Skew Normal Distribution Mode Using Taylor's Expansion of $SN(\lambda)$

On the other hand, there is another way to estimate the mode of skew normal distribution $SN(\lambda)$. The mode can also be estimated approximately using the Taylor's expansion of $SN(\lambda)$. The different orders of derivatives of the skew normal $SN(\lambda)$ are derived first then evaluated at point 0. Based on the definition of $SN(\lambda)$ (3), we have the following:

$$
\begin{aligned}
f(x,\lambda)&=2\phi(x)\Phi(\lambda x)=\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\int_{-\infty}^{\lambda x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt, f(0,\lambda)=\frac{1}{\sqrt{2\pi}},\\
f'(x,\lambda)&=\sqrt{\frac{2}{\pi}}(-x)e^{-\frac{1}{2}x^2}\int_{-\infty}^{\lambda x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt+\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}(\lambda)\left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}(\lambda x)^2}\\
&=-\sqrt{\frac{2}{\pi}}xe^{-\frac{1}{2}x^2}\int_{-\infty}^{\lambda x}e^{-\frac{1}{2}t^2}dt+\frac{\lambda}{\pi}e^{-\frac{1}{2}(1+\lambda^2)x^2}, f'(0,\lambda)=\frac{\lambda}{\pi},\\
f''(x,\lambda)&=-\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\int_{-\infty}^{\lambda x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt+\sqrt{\frac{2}{\pi}}x^2e^{-\frac{1}{2}x^2}\Phi(\lambda x)\\
&\quad-\sqrt{\frac{2}{\pi}}xe^{-\frac{1}{2}x^2}(\lambda)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2x^2}-\frac{\lambda}{\pi}(1+\lambda^2)xe^{-\frac{1}{2}(1+\lambda^2)x^2}\\
&=-\sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\Phi(\lambda x)+\sqrt{\frac{2}{\pi}}x^2e^{-\frac{1}{2}x^2}\Phi(\lambda x)-\frac{\lambda}{\pi}xe^{-\frac{1}{2}}(1+\lambda^2)x^2\\
&\quad-\frac{\lambda(1+\lambda^2)}{\pi}xe^{-\frac{1}{2}(1+\lambda^2)x^2}, f''(0,\lambda)=-\frac{1}{\sqrt{2\pi}},
\end{aligned}
$$

$$
\begin{aligned}
f'''(x,\lambda) &= -\sqrt{\frac{2}{\pi}}(-x)e^{-\frac{1}{2}x^2}\Phi(\lambda x) - \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}(\lambda)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2x^2} + 2x(\sqrt{\frac{2}{\pi}})e^{-\frac{1}{2}x^2}\Phi(\lambda x) \\
&+ \sqrt{\frac{2}{\pi}x^2}(-x)e^{-\frac{1}{2}x^2}\Phi(\lambda x) + \sqrt{\frac{2}{\pi}}x^2e^{-\frac{1}{2}x^2}(\lambda)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2x^2} \\
&- \frac{\lambda}{\pi}e^{-\frac{1}{2}(1+\lambda^2)x^2} - \frac{\lambda}{\pi}x(-(1+\lambda^2)x)e^{-\frac{1}{2}(1+\lambda^2)x^2} \\
&- \frac{\lambda(1+\lambda^2)}{\pi}e^{-\frac{1}{2}(1+\lambda^2)x^2} - \frac{\lambda(1+\lambda^2)}{\pi}x(-(1+\lambda^2)x)e^{-\frac{1}{2}(1+\lambda^2)x^2} \\
&= \sqrt{\frac{2}{\pi}}xe^{-\frac{1}{2}x^2}\Phi(\lambda x) - \sqrt{\frac{2}{\pi}}\lambda e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2x^2} + \frac{2\sqrt{2}x}{\sqrt{\pi}}e^{-\frac{1}{2}x^2}\Phi(\lambda x) \\
&- \sqrt{\frac{2}{\pi}}x^3e^{-\frac{1}{2}x^2}\Phi(\lambda x) + \frac{\lambda}{\pi}x^2e^{-\frac{1}{2}x^2}e^{-\frac{1}{2}\lambda^2x^2} \\
&- \frac{\lambda}{\pi}e^{-\frac{1}{2}(1+\lambda^2)x^2} + \frac{\lambda(1+\lambda^2)}{\pi}x^2e^{-\frac{1}{2}(1+\lambda^2)x^2} \\
&- \frac{\lambda(1+\lambda^2)}{\pi}e^{-\frac{1}{2}(1+\lambda^2)x^2} + \frac{1}{\pi}\lambda(1+\lambda^2)^2x^2e^{-\frac{1}{2}(1+\lambda^2)x^2}, \\
f'''(0,\lambda) &= -\sqrt{\frac{2}{\pi}}\lambda\frac{1}{\sqrt{2\pi}} - \frac{\lambda}{\pi} - \frac{\lambda(1+\lambda^2)}{\pi} = -\frac{\lambda(3+\lambda^2)}{\pi}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
f(x,\lambda) &= f(0,\lambda) + f'(0,\lambda)x + \frac{f''(0,\lambda)x^2}{2!} + \frac{f'''(0,\lambda)x^3}{3!} + \cdots \\
&= \frac{1}{\sqrt{2\pi}} + \frac{\lambda}{\pi}x - \frac{1}{\sqrt{2\pi}}(\frac{1}{2})x^2 - \frac{1}{6}\frac{\lambda(3+\lambda^2)}{\pi}x^3 + \cdots \\
\text{Let } f'(x,\lambda) &= 0 \\
&=> \frac{\lambda}{\pi} - \frac{x}{\sqrt{2\pi}} - \frac{1}{2}\frac{\lambda(3+\lambda^2)}{\pi}x^2 + \cdots = 0 \\
&=> -2\lambda + \sqrt{2\pi}x + \lambda(3+\lambda^2)x^2 \approx 0 \qquad (10) \\
&=> x_{1,2} \approx \frac{-\sqrt{2\pi} \pm \sqrt{2\pi + 8\lambda^2(3+\lambda^2)}}{2\lambda(3+\lambda^2)}. \qquad (11)
\end{aligned}
$$

**Proposition 3**: Assuming a random variable $Z$ is distributed as skew normal $SN(\lambda)$, then the mode of $Z$ can be estimated approximately by the following formula:

$$M_o(\lambda) \approx \frac{-\sqrt{2\pi} + \sqrt{2\pi + 8\lambda^2(3+\lambda^2)}}{2\lambda(3+\lambda^2)}. \qquad (12)$$

From (10), we can see that if $\lambda = 0$ then $M_o = 0$, which is the mode of the standard normal distribution. From (12), we can see that if $\lambda = 1$ then

$$M_0 \approx \frac{-\sqrt{2\pi} + \sqrt{2\pi + 32}}{8} = 0.46009.$$

**Proposition 4**: Assuming a random variable $Z$ is distributed as skew normal $SN(\lambda)$, let $Y = \xi + \omega Z, (\xi \varepsilon R, \omega \varepsilon R^+)$, then we have $Y = SN(\xi, \omega^2, \lambda)$ and the $M_o$ of $Y$ is

$$M_o \approx \xi + \frac{\omega(-\sqrt{2\pi} + \sqrt{2\pi + 8\lambda^2(3+\lambda^2)})}{2\lambda(3+\lambda^2)}. \qquad (13)$$

*Proof.* Based on proposition 3, the mode of random variable $Z$ is as (12) and based on Azzalini and Capitaniao (2014) we have (13).

## 5 Estimate the Median of the Skew Normal Distribution Using Simpson's Rule

The median of asymmetric unimodal continuous distributions has been discussed intensively. For example, based on Wilson-Hilferty transformation, we know that if $X \sim \chi^2(k)$ then $\sqrt[3]{\frac{X}{k}}$ approximately normally distributed with mean $1-\frac{2}{9k}$ and variance $\frac{2}{9k}$. Then, we have median approximation of $\chi^2(k)$ distribution: $k(1-\frac{2}{9k})^3$ (Johnson et al., 1995). For another example, we consider the standard Weibull distributions. The pdf of a standard Weibull random variable is given by

$$f(x) = kx^{k-1}e^{-x^k}, x \geq 0; k > 0.$$

The median of a standard Weibull random variable is $\sqrt[k]{ln2}$ (Johnson et al., 1995). In addition, we have median approximation of F-distribution $F(m,n)$: $\frac{n(3m-2)}{m(3n-2)}$, (Walck, 2007 and Kerman, 2011). However, there is a gap in the current statistical literature on assessing median of skew normal distributions. We used Simpson's rule to estimate the median of the skew normal distribution. Simpson's rules are several approximations for definite integrals, named after Thomas Simpson $(1710-1761)$. The most basic of these rules, called Simpson's $1/3$ rule, or just Simpson's rule, reads

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right].$$

To compute the median approximately, we evaluated the integration of the skew normal distribution from $-\infty$ to the point x using Simpson's rule. The median can be found when the value of the definite integral

$$\int_{-\infty}^{x} 2\phi(t)\Phi(\lambda t)dt = \frac{1}{2}.$$

For this purpose, a SAS macro program was developed and a median was estimated when the skew parameter $\lambda = 0.1, 0.2, \cdots, 2.0$ and results were presented in Table 1.

## 6 Mean and Mode Estimation of Skew Normal Distributions by Azzalini and Capitaniao

Most often, in a perfectly symmetrical, non-skewed unimodal distribution the mean, median and mode are equal. However, generally speaking, for a skewed distribution these three central tendency measurements are not equal. The mean of the skew normal distribution (4) can be calculated using the formula $\xi + \omega\delta\sqrt{\frac{2}{\pi}}$, where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$. The exact value of mode of (4) cannot be computed, but can be approximated by the formula $M_o \approx \xi + \omega M_o(\lambda)$, where $M_o(\lambda) \approx \mu_z - \frac{\gamma_1\sigma_z}{2} - \frac{sgn(\lambda)}{2}\exp\{-\frac{2\pi}{|\lambda|}\}$, where $\mu_z = \sqrt{\frac{2}{\pi}}\delta$ and $\sigma_z = \sqrt{1-\mu_z^2}, \gamma_1 = \frac{4-\pi}{2}\frac{\left(\delta\sqrt{\frac{2}{\pi}}\right)^3}{\left(1-2\frac{\delta^2}{\pi}\right)^{\frac{3}{2}}}$ based on Azzalini and Capitaniao (2014). The mean and mode of (4) were estimated when the skew parameter $\lambda = 0, 0.1, 0.2, \cdots, 2.0$ and results were presented in Table 1.

## 7 Remarks and Discussion

Two figures were created to compare the central tendency estimation of skew normal distribution using various methods. According to property (c) in Section 1, if $Z \sim SN(\lambda)$ then $-Z \sim SN(-\lambda)$. Thus, we can assume $\lambda \geq 0$. Additionally, as the skew parameter $\lambda$ increases in absolute value, the skewness of the skew normal distribution increases. In practical data analysis, it is uncommon to encounter data from highly skewed populations when $|\lambda| > 2$. Hence, we assume $0 \leq \lambda \leq 2$.

Figure 3 illustrates the mode estimation of the skew normal distribution using different estimation methods for skew parameters $\lambda = 0.0, 0.1, \cdots, 2.0$. The true mode was determined using the SAS macro procedure. Another mode estimation is based on Azzalini and Capitaniao (2014). Mode 1, estimated using Taylor's expansion of $\Phi(\lambda x)$, is presented in Section 3, and Mode 2, estimated using Taylor's expansion of $SN(\lambda)$, is presented in Section 4.

Figure 4 displays the mode, median, and mean estimations of the skew normal distribution (3) for skew parameters $\lambda = 0.0, 0.1, \cdots, 2.0$. In this figure, the mode is estimated using the method presented in Section 3, the median is

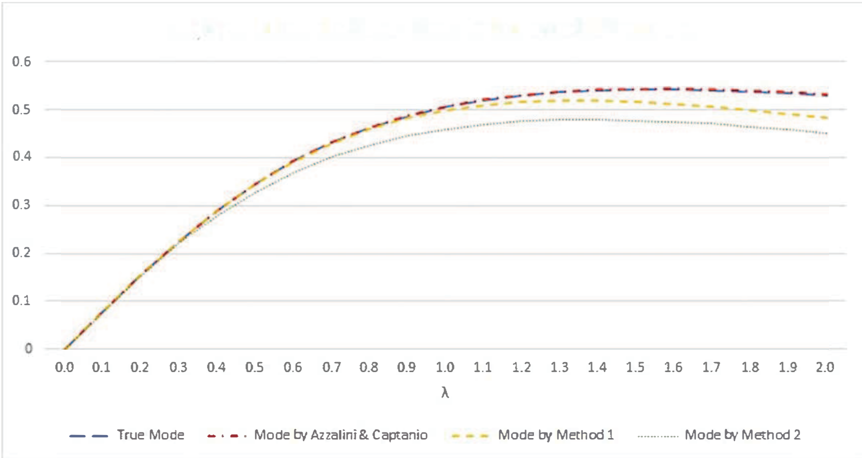


**Fig. 3:** Mode Estimation of Skew Normal Distribution

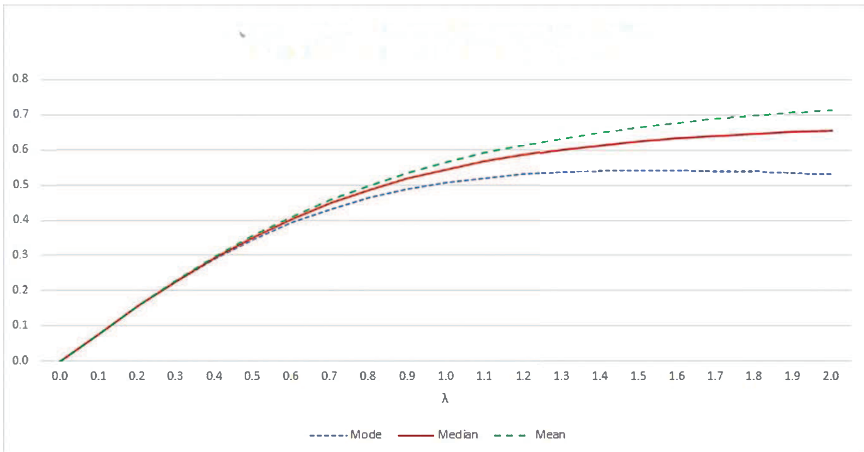


**Fig. 4:** Central Tendency Measurements of Skew Normal Distributions with Different Skewness

estimated using Simpson's Rule, presented in Section 5, and the mean is estimated using the formula in Azzalini and Capitanio (2014).

Table 1 contains the true mode from the SAS macro procedure in the second column, the true median from Simpson's Rule in the third column, and Modes 1 and 2 as shown in Figure 3. The AC mode from Azzalini and Capitanio (2014) (Figure 3 and column 5 of Table 1) appears to be precise, slightly surpassing the true mode (Figure 3 and column 2 of Table 1) in magnitude. The difference between the AC mode and the true mode increases as the skew parameter $\lambda$ becomes larger. Additionally, the mode approximated using the first method (column 6) is more accurate and larger than the mode using the second method (column 7). Overall, the AC mode is overestimated, while the modes from the first and second methods are underestimated compared to the true mode.

Figure 4 shows that for the skew normal distribution of (3), whether right/positive skewed or left/negative skewed, the median is always between the mean and mode.

From Figure 4 and Table 1, it is evident that the skew normal distribution with $0 \leq \lambda \leq 2.0$ is right (positively) skewed. The mode (column 2 of Table 1) is less than the median (column 3 of Table 1), which in turn is less than the mean (column 4 of Table 1). The median is closer to the mean than to the mode.

Another method to find the mode of the skew normal distribution involves its strictly quasi-concave nature, which implies a strictly unimodal density function. To find the mode, we solve $f'(x) = 0$, where:

$$\begin{aligned} f'(x) &= 2[\phi'(x)\Phi(\lambda x) + \phi(x)\Phi'(\lambda x)(\lambda x)'] \\ &= 2[-x\phi(x)\Phi(\lambda x) + \lambda\phi(x)\phi(\lambda x)] \\ &= -2\phi(x)[x\Phi(\lambda x) - \lambda\phi(\lambda x)] = 0. \\ &\Rightarrow x\Phi(\lambda x) - \lambda\phi(\lambda x) = 0. \end{aligned}$$

This equation can be solved numerically using methods such as the Newton-Raphson method, quasi-Newton method, or bisection method.

## 8 Conclusions

In this article, we explored various methods for estimating central tendency measurements in skew normal distributions, focusing on $\lambda$ values ranging from 0.0 to 2.0. We found that the AC mode slightly overestimates the true mode, while modes estimated using Taylor's expansion of $\Phi(\lambda x)$ or $SN(\lambda)$ slightly underestimate it, with the former being more accurate. Despite the accuracy of AC modes, their calculation involves a lengthy and complex formula. Both the AC method and Taylor's expansion method provide relatively accurate mode approximations when $|\lambda| < 2$, but discrepancies increase for $|\lambda| > 2$. However, in practical data analysis, highly skewed populations with $|\lambda| > 2$ are uncommon.

There remains a gap in the literature, as no formula, even approximate, exists for calculating the median of a skew normal distribution. In this paper, we accurately estimated the true median of skew normal distributions when $|\lambda| < 2$ using Simpson's rule.

We observed that skew normal distributions are right/positively skewed when $\lambda > 0$ and left/negatively skewed when $\lambda < 0$, with the median always between the mean and mode. For future research, the methods used in this article for estimating central tendency measurements can also be applied to skew t-distributions, skew logistic distributions, skew Cauchy distributions, $\chi^2$-distributions, and others. Additionally, further exploration of the variable transformation method for finding the median and mode of skew normal distributions is suggested.

| $\lambda$ | Mode | Median | AC Mean | AC Mode | Mode 1 | Mode 2 |
|---|---|---|---|---|---|---|
| 0.0 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 0.1 | 0.07928 | 0.07900 | 0.07939 | 0.07928 | 0.07928 | 0.07904 |
| 0.2 | 0.15563 | 0.15600 | 0.15648 | 0.15564 | 0.15563 | 0.15384 |
| 0.3 | 0.22654 | 0.22800 | 0.22927 | 0.22654 | 0.22653 | 0.22126 |
| 0.4 | 0.29019 | 0.29400 | 0.29633 | 0.29021 | 0.29014 | 0.27970 |
| 0.5 | 0.34561 | 0.35300 | 0.35682 | 0.34565 | 0.34540 | 0.32884 |
| 0.6 | 0.39254 | 0.40400 | 0.41051 | 0.39264 | 0.39195 | 0.36914 |
| 0.7 | 0.43130 | 0.44800 | 0.45756 | 0.43149 | 0.42997 | 0.40145 |
| 0.8 | 0.46255 | 0.48600 | 0.49843 | 0.46288 | 0.46001 | 0.42675 |
| 0.9 | 0.48717 | 0.51800 | 0.53376 | 0.48766 | 0.48286 | 0.44599 |
| 1.0 | 0.50605 | 0.54400 | 0.56419 | 0.50672 | 0.49940 | 0.46009 |
| 1.1 | 0.52010 | 0.56700 | 0.59039 | 0.52094 | 0.51052 | 0.46984 |
| 1.2 | 0.53011 | 0.58500 | 0.61295 | 0.53113 | 0.51710 | 0.47595 |
| 1.3 | 0.53679 | 0.60100 | 0.63242 | 0.53798 | 0.51993 | 0.47905 |
| 1.4 | 0.54076 | 0.61300 | 0.64927 | 0.54211 | 0.51969 | 0.47966 |
| 1.5 | 0.54252 | 0.62400 | 0.66388 | 0.54402 | 0.51701 | 0.47825 |
| 1.6 | 0.54249 | 0.63300 | 0.67660 | 0.54416 | 0.51240 | 0.47520 |
| 1.7 | 0.54104 | 0.64000 | 0.68772 | 0.54287 | 0.50629 | 0.47084 |
| 1.8 | 0.53845 | 0.64600 | 0.69748 | 0.54044 | 0.49905 | 0.46545 |
| 1.9 | 0.53496 | 0.65100 | 0.70606 | 0.53712 | 0.49097 | 0.45925 |
| 2.0 | 0.53076 | 0.65500 | 0.71365 | 0.53309 | 0.48228 | 0.45244 |

**Table 1:** Mode Estimation of Skew Normal Distribution

## Declarations

**Competing interests**: The authors declare that they have no competing interests.
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## References

[1] K.M.Abadir, *Econometric Theory*, The Mean-median-mode Inequality: Counter Examples, **21(2)**(2005), 477–482.
[2] I.Alhribat,M.H.Samuh, *Jordan Journal of Mathematics and Statistics*, Generating Statistical Distributions Using Fractional Differential Equations, **16(2)**(2023), 379–396.
[3] N.M.Al-Olaimat,L.M.A.Al-Zou'bi, *Jordan Journal of Mathematics and Statistics*, Efficiency of Adaptive Methods Using Simulated Alpha Skew Normal Two-Stage Data, **11(1)**(2018), 69–91.
[4] J.Andel,I.Netuka,K.Zvara, *Kybernetika*, On Threshold Autoregressive Processes, **20(2)**(1984), 89–106.
[5] S.K.Ashour,M.A.Abdel-hameed, *Journal of Advanced Research*, Approximate skew normal distribution, **1(4)**(2010), 341–350.
[6] A.Azzalini,A.Capitanio, *The skew-normal and related families*, University Printing House, Cambridge University Press (Vol.3), United Kingdom, (2014).
[7] A.Beyer, *Introduction to Statistics for Psychology*, Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted, (2021).
[8] D.R.Bickel, *Computational Statistics and Data Analysis*, Robust Estimators of the Mode and Skewness of Continuous data, **39(2)**(2002), 153–163.
[9] F.H.de Alencar,C.E.Galarza,L.A.Matos,V.H.Lachos, *Advances in Data Analysis and Classification*, Finite Mixture Modeling of Censored and Missing Data Using the Multivariate Skew-normal Distribution, **16(3)**(2022), 521–557.
[10] P.Hasanalipour,M.Razmkhah, *Communications in Statistics-Simulation and Computation*, Inference on Skew-normal Distribution Based on Fisher Information in Order Statistics, **51(4)**(2022), 1525–1541.
[11] N.L.Johnson,S.Kotz,N.Balakrishnan, *Continuous Univariate Distributions, Second Edition*, New York, NY: John Wiley and Sons, (1995).
[12] J.Kerman, A Closed-form Approximation for the Median of the Beta Distribution (2011), https://arxiv.org/pdf/1111.0433.
[13] J.T.McClave,P.G.enson,T.Sincich, *Statistics For Business And Economics, 11th Edition*, Pearson, United Kingdom, (2010).
[14] C.E.G.Morales,L.A.Matos,D.K.Dey,V.H.Lachos, *Journal of Computational and Graphical Statistics*, On Moments of Folded and Doubly Truncated Multivariate Extended Skew-normal Distributions, **31(2)**(2022), 455–465.
[15] G.S.Mudholkar,A.D.Hutson, *Journal of Statistical Planning and Inference*, The Epsilon-skew-normal Distribution for Analyzing Near-normal Data, **83(2)**(2000), 291–309.
[16] V.B.V.Nagarjuna,C.Chesneau, *Jordan Journal of Mathematics and Statistics*, A New Extension of the Inverse Power Lomax Distribution, **15(3B)**(2020), 717–740.
[17] A.O'Hagan,T.Leonard, *Biometrika*, Bayes Estimation Subject to Uncertainty About Parameter Constraints, **63(1)**(1976), 201–203.
[18] R.Shanker,K.K.Shuklaa, *Jordan Journal of Mathematics and Statistics*, Zero-Truncated Poisson-Aradhana Distribution with Applications, **12(2)**(2019), 253–263.
[19] P.T.Von Hippel, *Journal of Statistics Education*, Mean, median, and skew: Correcting a textbook rule, **13(2)**(2005), 1–13.
[20] C.Walck, *Particle Physics Group, Fysikum University of Stockholm*, Hand-book on statistical distributions for experimentalists, https://www.stat.rice.edu/~dobelman/textfiles/Distributions/Handbook.pdf, (2007).
[21] W.V.Zwet, *Statistica Neerlandica*, Mean, median, mode II, **33(1)**(1979), 1–5.