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Exploring Solutions for Nonlinear Fractional Differential Equations with Multiple Fractional Derivatives and Integral Boundary Conditions

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Abstract: This article explores solutions for boundary value problems of nonlinear fractional differential equations with fractional integral boundary conditions. The study applies Banach's and Krasnoselskii's fixed point theorems to establish the existence and uniqueness of these solutions. Additionally, a practical numerical example is presented to illustrate the real-world application of the derived results. The research contributes significantly to the comprehension of boundary value problems for nonlinear fractional differential equations with fractional integral boundary conditions.

Keywords: Riemann-Liouville fractional derivative operator; Nonlinear fractional differential equations; Fractional integral boundary conditions; Existence and uniqueness of solutions.

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1 Introduction and Preliminaries

In the early 21st century, the exploration of nonlinear fractional differential equations (NLFDE) with fractional integral boundary conditions (FBC) has become a prominent focus within the mathematical community. This burgeoning field, characterized by its nuanced investigation of complex mathematical structures, has captured widespread attention due to its versatile applications across various scientific disciplines.

The study of NLFDE with FBC has gained significance for its intrinsic relevance in modeling systems characterized by profound nonlinearity, with notable applications in chaotic processes [2], [4], [1], [6], [11], [14], and [16]. Its implications extend to diverse areas, including neuronal networks, epidemiological models, and meteorological systems, highlighting its far-reaching impact.

Beyond pure mathematics, the importance of NLFDE with FBC resonates across mathematical physics, engineering sciences, and computational mathematics [5], [7], [8], [12], and [17]. The involvement of derivatives of nonlinear orders beyond unity equips these equations with a robust framework, offering a sophisticated approach to elucidate intricate phenomena in various scientific disciplines. For further details, refer to [3], [5], [7], [12], [13], [17], and the references therein.

Central to the efficacy of NLFDE with FBC are fractional integral boundary conditions [9], [12], [16], [17], and [19]. These unique conditions, involving integration with respect to fractional order on the boundary region, play a pivotal role in describing systems characterized by memory, relaxation, diffusion phenomena, and other complex dynamics. Their applications span a broad spectrum, addressing scenarios where conventional boundary conditions fall short.

In the following, we synthesize recent contributions, providing a succinct yet comprehensive summary of pertinent research articles. This endeavor aims to contribute to the ongoing discourse on solutions to diverse nonlinear fractional boundary value problems.

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Zhang [19] studied the existence and multiplicity of positive solutions for the nonlinear fractional boundary value problem with Caputo's fractional derivative:

$$\begin{gathered}\mathfrak{D}_{0+}^{\alpha} y(\zeta)+f(\zeta, y(\zeta))=0,\ \zeta \in(0,1), 1<\alpha<2, \\ y(0)+y'(0)=0,\ y(1)+y'(1)=0,\end{gathered}$$

where $\mathfrak{D}_{0+}^{\alpha}$ is the standard Riemman fractional derivative operator of fractional order α, and $f: J \times R \rightarrow R$ is a continuous function.

Maamar et al. [16] studied the sufficient conditions for the existence of solutions for the following boundary value problem involving a nonlinear fractional differential equation:

$$\begin{gathered}\mathfrak{D}_{0+}^{\alpha} y(\zeta)=f(\zeta, y(\zeta))=0,\ \zeta \in[0,1],\ 2<\alpha \leq 3, \\ \mathfrak{D}_{0+}^{\alpha-1} y(0)=0,\ \mathfrak{D}_{0+}^{\alpha-2} y(1)=0,\ y(1)=0,\end{gathered}$$

where $\mathfrak{D}_{0+}^{\alpha}$ is the standard Riemman fractional derivative operator of fractional order α, and $f: J \times R \rightarrow R$ is a continuous function.

In [17], the authors investigate the existence of solutions for the Caputo fractional differential inclusion with nonlocal integral boundary value conditions

$$\begin{gathered}{}^{c}\mathfrak{D}_{0+}^{\alpha} y(\zeta) \in f(\zeta, y(\zeta),\ {}^{c}\mathfrak{D}_{0+}^{\beta} y(\zeta), y'(\zeta)),\ \zeta \in[0,1], \\ y(0)+y'(0)+\ {}^{c}\mathfrak{D}_{0+}^{\beta} y(0)=\int_0^\eta y(\upsilon)\, d\upsilon, \\ y(1)+y'(1)+\ {}^{c}\mathfrak{D}_{0+}^{\beta} y(1)=\int_0^\nu y(\upsilon)\, d\upsilon,\end{gathered}$$

where ${}^{c}\mathfrak{D}_{0+}^{\alpha}$ and ${}^{c}\mathfrak{D}_{0+}^{\beta}$ are Caputo fractional derivatives of fractional orders α and β respectively, and $f:[0,1] \times R^3 \rightarrow 2^R$ is a compact valued multifunction.

Motivated by the above-mentioned works, the present paper aims to study the existence and uniqueness of solutions for the following nonlinear implicit fractional differential equations (NLIFDE) with integral boundary conditions:

$$\mathfrak{D}_{0+}^{\alpha} y(\zeta)=f(\zeta, y(\zeta), \mathfrak{D}_{0+}^{\beta} y(\zeta), \int_0^\zeta k(\zeta, \upsilon) \mathfrak{D}_{0+}^{\alpha} y(\upsilon) d\upsilon),\ \zeta \in[0,1], 0<\beta \leq 1, 2<\alpha \leq 3, \tag{1}$$

subjected to the following three integral boundary conditions with fractional derivatives:

$$\begin{gathered}y(0)+\mathfrak{D}_{0+}^{\alpha-1} y(1)=\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon, \\ \mathfrak{D}_{0+}^{\alpha-1} y(0)+\mathfrak{D}_{0+}^{\alpha-2} y(1)=\sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))\, d\upsilon, \\ \mathfrak{D}_{0+}^{\alpha-2} y(0)+\mathfrak{D}_{0+}^{\alpha-3} y(1)=\sigma_3 \int_0^1 \mathfrak{h}_3(\upsilon, y(\upsilon))\, d\upsilon,\end{gathered}$$

where $\zeta \in J=[0,1]$, $\mathfrak{D}_{0+}^{\alpha}$ and $\mathfrak{D}_{0+}^{\beta}$ denote the standard Riemann-Liouville fractional derivative of orders $\alpha \in(2,3]$ and $\beta \in(0,1]$, $f: J \times R^3 \rightarrow R$, and $\mathfrak{h}_i: J \times R \rightarrow R$ $(i=1,2,3)$ are continuous functions.

The given boundary value problem (BVP) involves a system of three coupled equations, with Equation (1) being a fractional differential equation incorporating Riemann-Liouville fractional derivatives of orders α and β. The equation describes the rate of change of the function $y(\zeta)$ over the interval $[0,1]$ and involves a function f that depends on ζ, $y(\zeta)$, $\mathfrak{D}_{0+}^{\beta} y(\zeta)$, and an integral term with a memory kernel function $k(\zeta, \upsilon)$ affecting past values of $y(\zeta)$. The BVP also consists of three integral boundary conditions involving the functions $\mathfrak{h}_1$, $\mathfrak{h}_2$, and $\mathfrak{h}_3$ at the endpoints of the interval. Solving this complex BVP requires understanding the fractional derivatives, the nonlinear function f, and the memory kernel k, along with analyzing the behavior of $y(\zeta)$ and its derivatives at the boundary points based on the functions $\mathfrak{h}_1$, $\mathfrak{h}_2$, and $\mathfrak{h}_3$. This problem is challenging and likely necessitates specialized techniques for fractional calculus to study the solutions that satisfy both the differential equation and the integral boundary conditions.

In our study, the integral boundary conditions are interpreted as constraints on the behavior of the solution function at the boundary points. The terms involving fractional derivatives capture the rate of change of the solution, while the integral terms with continuous functions $\mathfrak{h}_1$, $\mathfrak{h}_2$, and $\mathfrak{h}_3$ encapsulate the influence of the solution's past values. This interpretation aligns with the dynamic nature of the nonlinear system under consideration, emphasizing the intricate interplay between the solution and the integral kernel functions.

The article is structured as follows: In the first section, a summary of recent research articles relevant to the study is presented, alongside the research's objective. Moving on to the second section, the necessary background information and prerequisites for the work are introduced. Section three delves into the examination of solutions for boundary value problems of nonlinear fractional differential equations with fractional integral boundary conditions. This investigation involves the application of Banach's and Krasnoselskii's fixed point theorems to establish the existence and uniqueness of solutions. Additionally, the fourth section includes a practical numerical example to illustrate the practical application of the acquired findings. Finally, the fifth section serves as a conclusive summary of the research's outcomes.

In the subsequent text, we present certain symbols, definitions, lemmas, and theorems that serve as foundational elements for our study. These essential concepts can be referenced in [10], [15], [18], and related sources.

Definition 1.*[18] Let $\alpha > 0$, $J = [a,b]$ be an interval such that $-\infty < a < \zeta < b < +\infty$, $x \in L_1(J,R)$. Then, the Riemann-Liouville fractional integral of $x(\zeta)$ with order α is defined as:*

$$\mathfrak{I}_{a^+}^{\alpha} x(\zeta) = \frac{1}{\Gamma(\alpha)} \int_a^{\zeta} (\zeta - \upsilon)^{\alpha-1} x(\upsilon)\, d\upsilon,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} \zeta^{\alpha-1} e^{-\zeta} d\zeta$.

Definition 2.*[18] Let $n \in N$, $J = [a,b]$ be an interval such that $-\infty < a < \zeta < b < +\infty$, $x \in C^n(J,R)$. Then, the Riemann-Liouville fractional derivative of order α, where $n = [\alpha]+1$ such that $[\alpha]$ denotes the integer part of α is defined by*

$$\mathfrak{D}_{a^+}^{\alpha} x(\zeta) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\zeta}\right)^n \int_a^{\zeta} (\zeta - \upsilon)^{n-\alpha-1} x(\upsilon)\, d\upsilon.$$

It is important to note that in order to investigate the existence of solutions for a fractional differential equation, it is necessary to convert it into an equivalent integral equation utilizing the essential properties of $\mathfrak{I}_{a^+}^{\alpha}$ and $\mathfrak{D}_{a^+}^{\alpha}$, see [18].

Lemma 1. *[15] If $n-1 < \alpha < n$ and $u \in C(J,R)$, then*

$$\mathfrak{I}_{0^+}^{\alpha} \mathfrak{D}_{0^+}^{\alpha} u(\zeta) = u(\zeta) - c_1 \zeta^{\alpha-1} + c_2 \zeta^{\alpha-2} + c_3 \zeta^{\alpha-3} + \cdots + c_n \zeta^{\alpha-n}, \ \zeta \in J.$$

Theorem 1. *[15] (Banach's Fixed Point Theorem) If $(X, \|.\|)$ represents a Banach space and $\wp : X \to X$ is a contraction mapping on X, then there exists a unique fixed point $x \in X$ satisfying $\wp(x) = x$.*

Theorem 2. *[10] (Krasnselskii's fixed point theorem) Let $\mathscr{S}$ be a closed, convex, and non-empty subset of a Banach space X. Assume that $\wp_1$ and $\wp_2$ are mappings from $\mathscr{S}$ to X with the following properties:*

1. *For all $u, v \in \mathscr{S}$, $\wp_1 u + \wp_2 v \in \mathscr{S}$.*
2. *$\wp_1$ is a contraction mapping.*
3. *$\wp_2$ is continuous, and the image $\wp_2(\mathscr{S})$ is contained within a compact set.*

Under these conditions, there exists at least one element $u \in \mathscr{S}$ such that $\wp_1 u + \wp_2 u = u$.

2 Main Results

Our exploration of solutions for nonlinear fractional differential equations with fractional integral boundary conditions has yielded significant findings. Applying Banach's and Krasnoselskii's fixed point theorems in this section, we establish both the existence and uniqueness of solutions.

Definition 3. *A function $u \in C(J,\mathbb{R})$ is considered a solution to the fractional differential equation (NIFDE) (1) if it fulfills the following fractional differential equation:*

$$\mathfrak{D}_{0^+}^{\alpha} y(\zeta) = f(\zeta, y(\zeta), \mathfrak{D}_{0^+}^{\beta} y(\zeta), \int_0^{\zeta} k(\zeta,\upsilon) \mathfrak{D}_{0^+}^{\alpha} y(\upsilon) d\upsilon), \ \zeta \in [0,1], 0 < \beta \leq 1, 2 < \alpha \leq 3,$$

with the following boundary conditions:

$$\begin{aligned} y(0) + \mathfrak{D}_{0^+}^{\alpha-1} y(1) &= \sigma_1 \textstyle\int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon, \\ \mathfrak{D}_{0^+}^{\alpha-1} y(0) + \mathfrak{D}_{0^+}^{\alpha-2} y(1) &= \sigma_2 \textstyle\int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))\, d\upsilon, \\ \mathfrak{D}_{0^+}^{\alpha-2} y(0) + \mathfrak{D}_{0^+}^{\alpha-3} y(1) &= \sigma_3 \textstyle\int_0^1 \mathfrak{h}_3(\upsilon, y(\upsilon))\, d\upsilon, \end{aligned}$$

Lemma 2. *Assume that $2 < \alpha \leq 3$, and let $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. A function $y(\zeta)$ defined on J is considered a solution to the nonlinear implicit fractional differential equation (NIFDE) given by (1) if and only if it satisfies the conditions specified by the following fractional integral equation:*

$$y(\zeta) = \frac{\zeta^{\alpha-3}\begin{pmatrix} 7\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon - 4\sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))\, d\upsilon + 2\sigma_3 \int_0^1 \mathfrak{h}_3(\upsilon, y(\upsilon))\, d\upsilon \\ -7\int_0^1 u(\upsilon)\, d\upsilon + 4\int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon - \int_0^1 (1-\upsilon)^2 u(\upsilon)\, d\upsilon \end{pmatrix}}{2\Gamma(\alpha-2)}$$
$$+ \frac{\zeta^{\alpha-2}\left(-2\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon + \sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))\, d\upsilon + 2\int_0^1 u(\upsilon)\, d\upsilon - \int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon\right)}{\Gamma(\alpha-1)}$$
$$+ \frac{\zeta^{\alpha-1}\left(\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon - \int_0^1 u(\upsilon)\, d\upsilon\right)}{\Gamma(\alpha)}$$
$$+ \frac{\int_0^\zeta (\zeta-\upsilon)^{\alpha-1} u(\upsilon)\, d\upsilon}{\Gamma(\alpha)}. \tag{2}$$

Proof. Let $y(\zeta)$ be a solution to the Nonlinear Implicit Fractional Differential Equation (NIFDE) (1). Define $u(\zeta) = f(\zeta, y(\zeta), \mathfrak{I}_{0+}^{\alpha-\beta} y(\zeta), \int_0^\zeta k(\zeta,\upsilon)\mathfrak{D}_{0+}^{\alpha} y(\upsilon) d\upsilon)$. Utilizing Lemma 1, we derive the expression:

$$y(\zeta) = c_1\zeta^{\alpha-1} + c_2\zeta^{\alpha-2} + c_3\zeta^{\alpha-3} + \frac{1}{\Gamma(\alpha)}\int_0^\zeta (\zeta-\upsilon)^{\alpha-1} u(\upsilon)\, d\upsilon. \tag{3}$$

Applying the boundary conditions, we obtain the following equations:

$$c_1\Gamma(\alpha) = \sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))d\upsilon - \int_0^1 u(\upsilon)\, d\upsilon. \tag{4}$$

$$2c_1\Gamma(\alpha) + c_2\Gamma(\alpha-1) = \sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))d\upsilon - \int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon. \tag{5}$$

$$\frac{1}{2}c_1\Gamma(\alpha) + 2c_2\Gamma(\alpha-1) + c_3\Gamma(\alpha-2) = \sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))d\upsilon - \frac{1}{2}\int_0^1 (1-\upsilon)^2 u(\upsilon)\, d\upsilon. \tag{6}$$

Solving equations (4), (5), and (6) for c_1, c_2, and c_3, we obtain:

$$c_1 = \frac{1}{\Gamma(\alpha)}\left(\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))d\upsilon - \int_0^1 u(\upsilon)\, d\upsilon\right),$$

$$c_2 = \frac{1}{\Gamma(\alpha-1)}\begin{pmatrix} -2\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))d\upsilon + \sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))d\upsilon + 2\int_0^1 u(\upsilon)\, d\upsilon \\ -\int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon \end{pmatrix},$$

and

$$c_3 = \frac{1}{2\Gamma(\alpha-2)}\begin{pmatrix} 7\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))d\upsilon - 4\sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))d\upsilon + 2\sigma_3 \int_0^1 \mathfrak{h}_3(\upsilon, y(\upsilon))d\upsilon \\ -7\int_0^1 u(\upsilon)\, d\upsilon + 4\int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon - \int_0^1 (1-\upsilon)^2 u(\upsilon)\, d\upsilon \end{pmatrix}.$$

Substituting these into (3), we obtain:

$$y(\zeta) = \frac{\zeta^{\alpha-3}\begin{pmatrix} 7\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon - 4\sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))\, d\upsilon + 2\sigma_3 \int_0^1 \mathfrak{h}_3(\upsilon, y(\upsilon))\, d\upsilon \\ -7\int_0^1 u(\upsilon)\, d\upsilon + 4\int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon - \int_0^1 (1-\upsilon)^2 u(\upsilon)\, d\upsilon \end{pmatrix}}{2\Gamma(\alpha-2)}$$
$$+ \frac{\zeta^{\alpha-2}\left(-2\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon + \sigma_2 \int_0^1 \mathfrak{h}_2(\upsilon, y(\upsilon))\, d\upsilon + 2\int_0^1 u(\upsilon)\, d\upsilon - \int_0^1 (1-\upsilon)u(\upsilon)\, d\upsilon\right)}{\Gamma(\alpha-1)}$$
$$+ \frac{\zeta^{\alpha-1}\left(\sigma_1 \int_0^1 \mathfrak{h}_1(\upsilon, y(\upsilon))\, d\upsilon - \int_0^1 u(\upsilon)\, d\upsilon\right)}{\Gamma(\alpha)}$$
$$+ \frac{\int_0^\zeta u(\upsilon)(\zeta-\upsilon)^{\alpha-1}\, d\upsilon}{\Gamma(\alpha)}.$$

On the contrary, assume that $y(\zeta)$ constitutes a solution to the Nonlinear Implicit Fractional Differential Equation (NIFDE) (2), and this solution can be expressed in the subsequent manner:

$$y(\zeta)=\frac{\zeta^{\alpha-3}}{2\Gamma(\alpha-2)}\begin{pmatrix}7\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon-4\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon+2\sigma_3\int_0^1\mathfrak{h}_3(\upsilon,y(\upsilon))\,d\upsilon\\ -7\int_0^1 u(\upsilon)\,d\upsilon+4\mathfrak{I}_{0^+}^2u(1)-2\mathfrak{I}_{0^+}^3u(1)\end{pmatrix}$$
$$+\frac{\zeta^{\alpha-2}}{2\Gamma(\alpha-1)}\left(-2\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon+\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon+2\int_0^1u(\upsilon)\,d\upsilon-\mathfrak{I}_{0^+}^2u(1)\right)$$
$$+\frac{\zeta^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon-\int_0^1u(\upsilon)\,d\upsilon\right)+\mathfrak{I}_{0^+}^{\alpha}u(\zeta).$$

Thus, we can infer that: $\mathfrak{D}_{0^+}^{\alpha}y(\zeta)=u(\zeta)$, with $y(0)+\mathfrak{D}_{0^+}^{\alpha-1}y(1)=\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon$, $\mathfrak{D}_{0^+}^{\alpha-1}y(0)+\mathfrak{D}_{0^+}^{\alpha-2}y(1)=\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon$, and $\mathfrak{D}_{0^+}^{\alpha-2}y(0)+\mathfrak{D}_{0^+}^{\alpha-3}y(1)=\sigma_3\int_0^1\mathfrak{h}_3(\upsilon,y(\upsilon))\,d\upsilon$. This implies that $u(\zeta)$ indeed satisfies the conditions of problem (2). This concludes the proof.

Lemma 3. *Consider (NIFDE) (1) under the following assumptions:*

(H_1) The nonlinear function $f:J\times R^3\to R$ is continuous and there exist $\lambda\in C(J,R^+)$ with norm $\|\lambda\|$ such that:

$$|f(\zeta,u_1,u_2,u_3)-f(\zeta,v_1,v_2,v_3)|\le\lambda(\zeta)\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right),\ \forall\ \zeta\in J,\ u_i,v_i\in R,\text{ and } i=1,2,3.$$

(H_2) The function $k(\zeta,\upsilon)$ is continuous for all $(\zeta,\upsilon)\in J\times J$, and there is a positive constant K such that:

$$\max_{\zeta,\upsilon\in[0,1]}|k(\zeta,\upsilon)|=K.$$

(H_3) The nonlinear function $\mathfrak{h}_i:J\times R\to R$ is continuous and there exist $\mu_i\in C(J,R^+)$ with norm $\|\mu\|$ such that:

$$|\mathfrak{h}_i(\zeta,u)-\mathfrak{h}_i(\zeta,v)|\le\mu_i(\zeta)\ |u-v|,\ \forall\ \zeta\in J,\text{and } i=1,2,3.$$

Remark 1. From Lemma (3), we deduce the following:

1. From assumption (H_1), we have
$$|f(\zeta,u_1,u_2,u_3)|-|f(\zeta,0,0,0)|\le|f(\zeta,u_1,u_2,u_3)-f(\zeta,0,0,0)|\le\lambda(\zeta)(|u_1|+|u_2|+|u_3|).$$
Thus, if $F=\sup_{\zeta\in J}|f(\zeta,0,0,0)|$, then $|f(\zeta,u_1,u_2,u_3)|\le F+\lambda(\zeta)(|u_1|+|u_2|+|u_3|)$.

2. From assumption (H_3), we have for $i=1,2,3$ that
$$|\mathfrak{h}_i(\zeta,u)|-|\mathfrak{h}_i(\zeta,0)|\le|\mathfrak{h}_i(\zeta,u)-\mathfrak{h}_i(\zeta,0)|\le\mu_i(\zeta)|u|.$$
Thus, if $H_i=\sup_{\zeta\in J}|\mathfrak{h}_i(\zeta,0)|$, then $|\mathfrak{h}_i(\zeta,u)|\le H_i+\mu_i(\zeta)|u|$.

Definition 4. *Define the operator $\wp:C(J,R)\to C(J,R)$ as follows:*

$$\wp(y(\zeta))=\frac{\zeta^{\alpha-3}\begin{pmatrix}7\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon-4\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon+2\sigma_3\int_0^1\mathfrak{h}_3(\upsilon,y(\upsilon))\,d\upsilon\\ -7\int_0^1u(\upsilon)\,d\upsilon+4\int_0^1(1-\upsilon)u(\upsilon)\,d\upsilon-\int_0^1(1-\upsilon)^2u(\upsilon)\,d\upsilon\end{pmatrix}}{2\Gamma(\alpha-2)}$$
$$+\frac{\zeta^{\alpha-2}\left(-2\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon+\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon+2\int_0^1u(\upsilon)\,d\upsilon-\int_0^1(1-\upsilon)u(\upsilon)\,d\upsilon\right)}{\Gamma(\alpha-1)}$$
$$+\frac{\zeta^{\alpha-1}\left(\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon-\int_0^1u(\upsilon)\,d\upsilon\right)}{\Gamma(\alpha)}$$
$$+\frac{\int_0^{\zeta}(\zeta-\upsilon)^{\alpha-1}u(\upsilon)\,d\upsilon}{\Gamma(\alpha)},$$

where $u(\upsilon)\in C(J,R)$ satisfies the following implicit fractional equation:

$$u(\zeta)=f(\zeta,y(\zeta),\mathfrak{D}_{0^+}^{\beta}y(\zeta),\int_0^{\zeta}k(\zeta,\upsilon)y(\upsilon)d\upsilon).$$

2.1 Existence of Solutions

In the following, we establish the existence of solutions for the Nonlinear Fractional Differential Equation (NIFDE) defined by (1). Our approach centers on the application of Krasnoselskii's fixed point theorem.

Theorem 3. *Suppose that assumptions $(H_1)-(H_3)$ hold. If*

$$\aleph_1+\aleph_2<1, \tag{7}$$

where

$$\aleph_1=\frac{\sigma_1(H_1+\|\mu_1\|)}{|\Gamma(\alpha)|}+\frac{2\sigma_1(H_1+\|\mu_1\|)+\sigma_2(H_2+\|\mu_2\|)}{|\Gamma(\alpha-1)|}+\frac{7\sigma_1(H_1+\|\mu_1\|)+4\sigma_2(H_2+\|\mu_2\|)+2\sigma_3(H_3+\|\mu_3\|)}{2|\Gamma(\alpha-2)|},$$

and

$$\aleph_2=\frac{14\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{3|\Gamma(\alpha-2)|}+\frac{5\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{2|\Gamma(\alpha-1)|}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{|\Gamma(\alpha)|}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{|\Gamma(\alpha+1)|}.$$

Then, (NIFDE) (1) has at least one solution in $C[0,1]$.

Proof. By converting (NIFDE) (1) into a fixed point problem. Define the operator $\wp: C(J,R)\to C(J,R)$ by:

$$\wp(y(\zeta))=\wp_1(y(\zeta))+\wp_2(y(\zeta)),\quad \zeta\in[0,1],$$

where

$$\begin{aligned}\wp_1(y(\zeta))=&\frac{\zeta^{\alpha-3}\begin{pmatrix}7\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon-4\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon+2\sigma_3\int_0^1\mathfrak{h}_3(\upsilon,y(\upsilon))\,d\upsilon\\-7\int_0^1u(\upsilon)d\upsilon+4\int_0^1(1-\upsilon)u(\upsilon)d\upsilon-\int_0^1(1-\upsilon)^2u(\upsilon)d\upsilon\end{pmatrix}}{2\Gamma(\alpha-2)}\\&+\frac{\zeta^{\alpha-2}\left(-2\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon+\sigma_2\int_0^1\mathfrak{h}_2(\upsilon,y(\upsilon))\,d\upsilon+2\int_0^1u(\upsilon)d\upsilon-\int_0^1(1-\upsilon)u(\upsilon)d\upsilon\right)}{\Gamma(\alpha-1)}\\&+\frac{\zeta^{\alpha-1}\left(\sigma_1\int_0^1\mathfrak{h}_1(\upsilon,y(\upsilon))\,d\upsilon-\int_0^1u(\upsilon)d\upsilon\right)}{\Gamma(\alpha)},\end{aligned}$$

and

$$\wp_2(y(\zeta))=\frac{\int_0^\zeta(\zeta-\upsilon)^{\alpha-1}u(\upsilon)d\upsilon}{\Gamma(\alpha)},$$

with

$$u(\zeta)=f(\zeta,\ y(\zeta),\ \mathfrak{D}_{0^+}^{\beta}y(\zeta),\int_0^\zeta k(\zeta,\upsilon)y(\upsilon)d\upsilon).$$

Let $B_\rho=\{y\in C(J,R):\|y\|\le\rho\}$ be a closed subset of $C[0,1]$, where ρ is a positive constant satisfying $\rho\ge\frac{\Re F}{1-(\aleph_1+\aleph_2)}$, where

$$\Re=\frac{14F}{3\Gamma(\alpha-2)}+\frac{5F}{2\Gamma(\alpha-1)}+\frac{F}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)},$$

$$\aleph_1=\frac{\sigma_1(H_1+\|\mu_1\|)}{\Gamma(\alpha)}+\frac{2\sigma_1(H_1+\|\mu_1\|)+\sigma_2(H_2+\|\mu_2\|)}{\Gamma(\alpha-1)}+\frac{7\sigma_1(H_1+\|\mu_1\|)+4\sigma_2(H_2+\|\mu_2\|)+2\sigma_3(H_3+\|\mu_3\|)}{2\Gamma(\alpha-2)},$$

and

$$\aleph_2=\frac{14\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{3\Gamma(\alpha-2)}+\frac{5\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{2\Gamma(\alpha-1)}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha)}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha+1)}.$$

Clearly, the space B_ρ forms a Banach space equipped with a metric in $C[0,1]$. The proof can be delineated into three distinct steps.

Step 1: $\wp_1y_1+\wp_2y_2\in B_\rho$ for every $y_1,y_2\in B_\rho$.

Let $y_1, y_2 \in B_\rho$ and $\zeta \in J$, we have

$$\begin{aligned}
&|\wp_1 y_1(\zeta) + \wp_2 y_2(\zeta)| \\
&\le |\wp_1 y_1(\zeta)| + |\wp_2 y_2(\zeta)| \\
&\le \frac{|\zeta^{\alpha-3}|}{2\Gamma(\alpha-2)} \begin{pmatrix} 7\sigma_1 \int_0^1 |\mathfrak{h}_1(\upsilon, y_1(\upsilon))|\, d\upsilon + 4\sigma_2 \int_0^1 |\mathfrak{h}_2(\upsilon, y_1(\upsilon))|\, d\upsilon + 2\sigma_3 \int_0^1 |\mathfrak{h}_3(\upsilon, y_1(\upsilon))|\, d\upsilon \\ +7\int_0^1 |f(\zeta, y_1(\zeta), \mathfrak{D}_{0^+}^{\beta} y_1(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_1(\upsilon) d\upsilon)|\, d\upsilon \\ +4\int_0^1 |1-\upsilon|\, |f(\zeta, y_1(\zeta), \mathfrak{D}_{0^+}^{\beta} y_1(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_1(\upsilon) d\upsilon)|\, d\upsilon \\ +\int_0^1 |1-\upsilon|^2\, |f(\zeta, y_1(\zeta), \mathfrak{D}_{0^+}^{\beta} y_1(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_1(\upsilon) d\upsilon)|\, d\upsilon \end{pmatrix} \\
&+ \frac{|\zeta^{\alpha-2}|}{\Gamma(\alpha-1)} \begin{pmatrix} +2\sigma_1 \int_0^1 |\mathfrak{h}_1(\upsilon, y(\upsilon))|\, d\upsilon + \sigma_2 \int_0^1 |\mathfrak{h}_2(\upsilon, y(\upsilon))|\, d\upsilon \\ +2\int_0^1 |f(\zeta, y_1(\zeta), \mathfrak{D}_{0^+}^{\beta} y_1(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_1(\upsilon) d\upsilon)|\, d\upsilon \\ +\int_0^1 |1-\upsilon|\, |f(\zeta, y_1(\zeta), \mathfrak{D}_{0^+}^{\beta} y_1(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_1(\upsilon) d\upsilon)| d\upsilon \end{pmatrix} \\
&+ \frac{|\zeta^{\alpha-1}|}{\Gamma(\alpha)} \left(\sigma_1 \int_0^1 |\mathfrak{h}_1(\upsilon, y(\upsilon))|\, d\upsilon + \int_0^1 |f(\zeta, y_1(\zeta), \mathfrak{D}_{0^+}^{\beta} y_1(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_1(\upsilon) d\upsilon)| d\upsilon \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^\zeta (\zeta-\upsilon)^{\alpha-1} |f(\zeta, y_2(\zeta), \mathfrak{D}_{0^+}^{\beta} y_2(\zeta), \int_0^\zeta k(\zeta,\upsilon) y_2(\upsilon) d\upsilon)| d\upsilon.
\end{aligned}$$

Using Lemma (3) and the aforementioned remark, if we consider the supremum for $\zeta \in [0,1]$, then

$$\begin{aligned}
|f(\zeta, y(\zeta), \mathfrak{I}^{\alpha-\beta} y(\zeta), \int_0^\zeta k(\zeta,\upsilon) y(\upsilon) d\upsilon)| &\le \|\lambda\| (|y(\zeta)| + |\mathfrak{D}_{0^+}^{\beta} y(\zeta))| + |\int_0^\zeta k(\zeta,\upsilon) y(\upsilon) d\upsilon)|) + F, \\
&\le \|\lambda\| \left(1 + \frac{1}{|\Gamma(1-\beta)|} + K\right) \|y\| + F,
\end{aligned}$$

where $F = \sup_{\zeta \in J} |f(\zeta, 0, 0, 0)|$. Thus, for each $\zeta \in [0,1]$ we have

$$\begin{aligned}
&|\wp_1 y_1(\zeta) + \wp_2 y_2(\zeta)| \\
&\le \frac{1}{2|\Gamma(\alpha-2)|} \left((7\sigma_1 (H_1 + \mu_1(\zeta)) + 4\sigma_2 (H_2 + \mu_2(\zeta)) + 2\sigma_3 (H_3 + \mu_3(\zeta))) \|y_1\| + \tfrac{28}{3} \left(\|\lambda\| \left(1 + \tfrac{1}{|\Gamma(1-\beta)|} + K\right) \|y_1\| + F \right) \right) \\
&+ \frac{1}{|\Gamma(\alpha-1)|} \left(+ (2\sigma_1 (H_1 + \mu_1(\zeta)) + \sigma_2 (H_2 + \mu_2(\zeta))) \|y_1\| + \tfrac{5}{2} \left(\|\lambda\| \left(1 + \tfrac{1}{|\Gamma(1-\beta)|} + K\right) \|y_1\| + F \right) \right) \\
&+ \frac{1}{|\Gamma(\alpha)|} \left(\sigma_1 (H_1 + \mu_1(\zeta)) \|y_1\| + \left(\|\lambda\| \left(1 + \frac{1}{|\Gamma(1-\beta)|} + K\right) \|y_1\| + F \right) \right) \\
&+ \frac{1}{|\Gamma(\alpha+1)|} \left(\|\lambda\| \left(1 + \frac{1}{|\Gamma(1-\beta)|} + K\right) \|y_2\| + F \right), \\
&\le \left(\frac{7}{2|\Gamma(\alpha-2)|} + \frac{2}{|\Gamma(\alpha-1)|} + \frac{1}{|\Gamma(\alpha)|} \right) \sigma_1 (H_1 + \mu_1(\zeta)) \|y_1\| \\
&+ \left(\frac{2}{|\Gamma(\alpha-2)|} + \frac{1}{|\Gamma(\alpha-1)|} \right) \sigma_2 (H_2 + \mu_2(\zeta)) \|y_1\| + \frac{1}{|\Gamma(\alpha-2)|} \sigma_3 (H_3 + \mu_3(\zeta)) \|y_1\| \\
&+ \left(\frac{14}{3|\Gamma(\alpha-2)|} + \frac{5}{2|\Gamma(\alpha-1)|} + \frac{1}{|\Gamma(\alpha)|} \right) \left(\|\lambda\| \left(1 + \frac{1}{|\Gamma(1-\beta)|} + K\right) \|y_1\| + F \right) \\
&+ \frac{1}{|\Gamma(\alpha+1)|} \left(\|\lambda\| \left(1 + \frac{1}{|\Gamma(1-\beta)|} + K\right) \|y_2\| + F \right).
\end{aligned}$$

Taking supremum over $\zeta \in [0,1]$, we have

$$\|\wp_1 y_1(\zeta) + \wp_2 y_2(\zeta)\| \le \rho,$$

for $\rho \ge \frac{\Re F}{1-(\aleph_1+\aleph_2)}$, where

$$\Re = \frac{14F}{3\Gamma(\alpha-2)} + \frac{5F}{2\Gamma(\alpha-1)} + \frac{F}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)},$$

$$\aleph_1 = \frac{\sigma_1(H_1+\|\mu_1\|)}{\Gamma(\alpha)} + \frac{2\sigma_1(H_1+\|\mu_1\|)+\sigma_2(H_2+\|\mu_2\|)}{\Gamma(\alpha-1)} + \frac{7\sigma_1(H_1+\|\mu_1\|)+4\sigma_2(H_2+\|\mu_2\|)+2\sigma_3(H_3+\|\mu_3\|)}{2\Gamma(\alpha-2)},$$

and

$$\aleph_2 = \frac{14\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{3\Gamma(\alpha-2)} + \frac{5\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{2\Gamma(\alpha-1)} + \frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha)} + \frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha+1)}.$$

This proves that $\wp_1 y_1(\zeta)+\wp_2 y_2(\zeta)\in B_\rho$ for every $y_1,y_2\in B_\rho$.

Step 2: The operator $\wp_1$ serves as a contraction mapping on B_ρ. It is clear that in step 1, $\wp_1$ acts as a contraction mapping with a contraction coefficient of $c<1$, where $c=C_1+C_2$ with

$$C_1 = \frac{\sigma_1(H_1+\|\mu_1\|)}{\Gamma(\alpha)} + \frac{2\sigma_1(H_1+\|\mu_1\|)+\sigma_2(H_2+\|\mu_2\|)}{\Gamma(\alpha-1)} + \frac{7\sigma_1(H_1+\|\mu_1\|)+4\sigma_2(H_2+\|\mu_2\|)+2\sigma_3(H_3+\|\mu_3\|)}{2\Gamma(\alpha-2)},$$

and

$$C_2 = \frac{14\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{3\Gamma(\alpha-2)} + \frac{5\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{2\Gamma(\alpha-1)} + \frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha)}.$$

Step 3: To establish the compactness and continuity of operator $\wp_2$ on B_ρ, we first demonstrate its continuity. Consider a sequence $\{y_n\}_{n\in N}$ in B_ρ that converges to $y\in B_\rho$ as $n\to\infty$. We must show that $\|\wp_2 y_n-\wp_1 y\|\to 0$ as $n\to\infty$. Then, for $\zeta\in[0,1]$, we have

$$|\wp_2 y_n-\wp_2 y| \le \frac{1}{\Gamma(\alpha)}\int_0^\zeta |\zeta-\upsilon|^{\alpha-1}|u_n(\upsilon)-u(\upsilon)|d\upsilon,$$

where $u_n(\zeta)=f(\zeta,y_n(\zeta),\mathfrak{D}^\beta_{0^+}y_n(\zeta),\int_0^\zeta k(\zeta,\upsilon)y_n(\upsilon)d\upsilon)$ and $u(\zeta)=f(\zeta,y(\zeta),\mathfrak{D}^\beta_{0^+}y(\zeta),\int_0^\zeta k(\zeta,\upsilon)y(\upsilon)d\upsilon)$ are two continuous functions defined over $[0,1]$ such that

$$\begin{aligned}|u_n(\zeta)-u(\zeta)| &= |f(\zeta,y_n(\zeta),\mathfrak{D}^\beta_{0^+}y_n(\zeta),\int_0^\zeta k(\zeta,\upsilon)y_n(\upsilon)d\upsilon)-f(\zeta,y(\zeta),\mathfrak{D}^\beta_{0^+}y(\zeta),\int_0^\zeta k(\zeta,\upsilon)y(\upsilon)d\upsilon)|,\\ &\le |\lambda(\zeta)|(|y_n(\zeta)-y(\zeta)|+|\mathfrak{D}^\beta_{0^+}y_n(\zeta)-\mathfrak{D}^\beta_{0^+}y(\zeta)|+\int_0^\zeta |k(\zeta,\upsilon)|\,|y_n(\upsilon)-y(\upsilon)|d\upsilon|),\\ &\le \|\lambda\|\left(1+\frac{1}{|\Gamma(1-\beta)|}+K\right)\|y_n-y\|.\end{aligned}$$

Since $y_n\to y$, then we get $u_n(\zeta)\to u(\zeta)$ as $n\to\infty$ for each $\zeta\in[0,1]$. And let $\varepsilon>0$ be such that, for each $\zeta\in[0,T_1]$, we have $|u_n(\zeta)|\le\varepsilon/2$ and $|u(\zeta)|\le\varepsilon/2$ which implies that $|u_n(\upsilon)-u(\upsilon)|\le(|u_n(\upsilon)|+|u(\upsilon)|)\le\varepsilon$ for each $\zeta\in[0,1]$. Applying Lebesgue Dominated Convergence Theorem, it implies that $\|\wp_2 y_n-\wp_2 y\|\to 0$ as $n\to\infty$. Consequently, operator $\wp_2$ is continuous. In addition, we have

$$\|\wp_2 y\| \le \frac{1}{|\Gamma(\alpha+1)|}\left[\|\lambda\|\left(1+\frac{1}{|\Gamma(1-\beta)|}+K\right)\|y\|+F\right]\le\rho$$

due to definitions of ρ. This proves that $\wp_2$ is uniformly bounded on B_ρ.

Ultimately, we demonstrate that the mapping $\wp_2$ transforms bounded sets into equicontinuous sets within $C(J,R)$, specifically ensuring the equicontinuity of B_ρ.

Let $\forall\,\varepsilon>0$, $\exists\,\delta>0$ and $\zeta_1,\zeta_2\in J$, $\zeta_1<\zeta_2$, $|\zeta_2-\zeta_1|<\delta$. Then we have

$$\begin{aligned}|\wp_2 y(\zeta_2)-\wp_2 y(\zeta_1)| &\le \int_0^\zeta \frac{(\zeta_2-\upsilon)^{\alpha-1}-(\zeta_1-\upsilon)^{\alpha-1}}{\Gamma(\alpha)}|u(\upsilon)|\,d\upsilon,\\ &\le \|\lambda\|\left(1+\frac{1}{|\Gamma(1-\beta)|}+K\right)\frac{(\zeta_2^\alpha-\zeta_1^\alpha)}{\Gamma(\alpha+1)}.\end{aligned}$$

As ζ_1 approaches ζ_2, the expression on the right-hand side of the aforementioned inequality becomes independent of y and approaches zero. Thus,

$$|\wp_2 y(\zeta_2)-\wp_2 y(\zeta_1)|\to 0,\ \ \forall\,|\zeta_2-\zeta_1|\to 0.$$

Thus, if $\wp$ is uniformly continuous on B_ρ, where $\wp$ represents a compact operator, the Arzela-Ascoli theorem guarantees that $\wp: C([0,1],R)\to C([0,1],R)$ is both continuous and compact. Consequently, all the conditions of Krasnoselskii's fixed point theorem are satisfied, and the operator $\wp=\wp_1+\wp_2$ possesses a fixed point $y(\zeta)\in C[0,1]$ on B_ρ satisfying the boundary conditions in (1). As a result, $y(\zeta)$ serves as a solution of the (NIFDE) (1). This concludes the proof.

2.2 Uniqueness of Solutions

Next, we ascertain the unique solutions to the Nonlinear Fractional Differential Equation (NIFDE) described by equation (1). This exploration into uniqueness adds a valuable dimension to our understanding of the solutions in the context of our studied equation.

Lemma 4. *If assumptions (H_1)-(H_3) hold, and if*

$$\left(\begin{array}{c}\frac{\frac{28}{3}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+7\|\mu_1\|\sigma_1+4\|\mu_2\|\sigma_2+2\|\mu_3\|\sigma_3}{2\Gamma(\alpha-2)}+\frac{\frac{5}{2}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+2\|\mu_1\|\sigma_1+\|\mu_2\|\sigma_2}{\Gamma(\alpha-1)}\\+\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+\|\mu_1\|\sigma_1}{\Gamma(\alpha)}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)}{\Gamma(\alpha+1)}\end{array}\right)<1,$$

then operator $\wp: C(J,R)\to C(J,R)$ presented in Definition 4 is a contraction.

Proof. Assuming that conditions (H_1)-(H_3) are satisfied, let's examine the continuous functions x and y belonging to $C(J,R)$. In this context, for any $\zeta\in J$, the following applies:

$$\begin{aligned}&|\wp(y_1(\zeta))-\wp(y_2(\zeta))|\\ \leq\ &\frac{\zeta^{\alpha-3}}{2\Gamma(\alpha-2)}\left(\begin{array}{c}7\sigma_1\int_0^1|\mathfrak{h}_1(\upsilon,y_1(\upsilon))-\mathfrak{h}_1(\upsilon,y_2(\upsilon))|\,d\upsilon+4\sigma_2\int_0^1|\mathfrak{h}_2(\upsilon,y_1(\upsilon))-\mathfrak{h}_2(\upsilon,y_2(\upsilon))|\,d\upsilon\\+2\sigma_3\int_0^1|\mathfrak{h}_3(\upsilon,y_1(\upsilon))-\mathfrak{h}_3(\upsilon,y_2(\upsilon))|\,d\upsilon+7\int_0^1|u_1(\upsilon)-u_2(\upsilon)|\,d\upsilon\\+4\int_0^1(1-\upsilon)|u_1(\upsilon)-u_2(\upsilon)|\,d\upsilon+\int_0^1(1-\upsilon)^2|u_1(\upsilon)-u_2(\upsilon)|\,d\upsilon\end{array}\right)\\ &+\frac{\zeta^{\alpha-2}}{\Gamma(\alpha-1)}\left(\begin{array}{c}2\sigma_1\int_0^1|\mathfrak{h}_1(\upsilon,y_1(\upsilon))-\mathfrak{h}_1(\upsilon,y_2(\upsilon))|\,d\upsilon+\sigma_2\int_0^1|\mathfrak{h}_2(\upsilon,y_1(\upsilon))-\mathfrak{h}_2(\upsilon,y_2(\upsilon))|\,d\upsilon\\+2\int_0^1|u_1(\upsilon)-u_2(\upsilon)|d\upsilon+\int_0^1(1-\upsilon)|u_1(\upsilon)-u_2(\upsilon)|d\upsilon\end{array}\right)\\ &+\frac{\zeta^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1\int_0^1|\mathfrak{h}_1(\upsilon,y_1(\upsilon))-\mathfrak{h}_1(\upsilon,y_2(\upsilon))|d\upsilon+\int_0^1|u_1(\upsilon)-u_2(\upsilon)|d\upsilon\right)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^\zeta(\zeta-\upsilon)^{\alpha-1}|u_1(\upsilon)-u_2(\upsilon)|d\upsilon\end{aligned}$$

where $u_1,u_2\in C(J,R)$ such that

$$u_1(\zeta)=f(\zeta,y_1(\zeta),\mathfrak{I}^{\alpha-\beta}y_1(\zeta),\int_0^\zeta k(\zeta,\upsilon)y_1(\upsilon)d\upsilon),\ \text{and}\ u_2(\zeta)=f(\zeta,y_2(\zeta),\mathfrak{I}^{\alpha-\beta}y_2(\zeta),\int_0^\zeta k(\zeta,\upsilon)y_2(\upsilon)d\upsilon).$$

By applying conditions (H_1)-(H_3), and taking supremum for all $\zeta\in J$ we have

$$\begin{aligned}|u_1(\zeta)-u_2(\zeta)|&=|f(\zeta,y_1(\zeta),\mathfrak{I}^{\alpha-\beta}y_1(\zeta),\int_0^\zeta k(\zeta,\upsilon)y_1(\upsilon)d\upsilon)-f(\zeta,y_2(\zeta),\mathfrak{I}^{\alpha-\beta}y_2(\zeta),\int_0^\zeta k(\zeta,\upsilon)y_2(\upsilon)d\upsilon|,\\ &\leq\lambda(\zeta)\left(|y_1(\zeta)-y_2(\zeta)|+\int_0^\zeta\frac{(\zeta-\upsilon)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|y_1(\upsilon)-y_2(\upsilon)|\,d\upsilon+\int_0^\zeta|k(\zeta,\upsilon)|\ |y_1(\upsilon)-y_2(\upsilon)|d\upsilon\right),\\ &\leq\|\lambda\|\left(\|y_1-y_2\|+\frac{\zeta^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|y_1-y_2\|+K\|y_1-y_2\|\zeta\right),\\ &\leq\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\|y_1-y_2\|.\end{aligned}$$

Hence,

$$
\begin{aligned}
&|\wp(y_1(\zeta))-\wp(y_2(\zeta))| \\
\le\ & \frac{\zeta^{\alpha-3}}{2\Gamma(\alpha-2)}\left(\begin{array}{c}(7\sigma_1\|\mu_1\|+4\sigma_2\|\mu_2\|+2\sigma_3\|\mu_3\|)\|y_1-y_2\|\int_0^1 d\upsilon \\ +\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\|y_1-y_2\|\left(7\int_0^1 d\upsilon+4\int_0^1(1-\upsilon)d\upsilon+\int_0^1(1-\upsilon)^2 d\upsilon\right)\end{array}\right) \\
&+\frac{\zeta^{\alpha-2}}{\Gamma(\alpha-1)}\left(\begin{array}{c}(2\sigma_1\|\mu_1\|+\sigma_2\|\mu_2\|)\|y_1-y_2\|\int_0^1 d\upsilon \\ +\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\|y_1-y_2\|\left(2\int_0^1 d\upsilon+\int_0^1(1-\upsilon)d\upsilon\right)\end{array}\right) \\
&+\frac{\zeta^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma_1\|\mu_1\|+\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\right)\|y_1-y_2\|\int_0^1 d\upsilon \\
&+\frac{\zeta^{\alpha}}{\Gamma(\alpha+1)}\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\|y_1-y_2\|, \\
\le\ & \frac{1}{2\Gamma(\alpha-2)}\left(\left[(7\sigma_1\|\mu_1\|+4\sigma_2\|\mu_2\|+2\sigma_3\|\mu_3\|)+\frac{28}{3}\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\right]\right)\|y_1-y_2\| \\
&+\frac{1}{\Gamma(\alpha-1)}\left((2\sigma_1\|\mu_1\|+\sigma_2\|\mu_2\|)+\frac{5}{2}\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\right)\|y_1-y_2\| \\
&+\frac{1}{\Gamma(\alpha)}\left(\sigma_1\|\mu_1\|+\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\right)\|y_1-y_2\| \\
&+\frac{1}{\Gamma(\alpha+1)}\|\lambda\|\left(1+\frac{1}{\Gamma(\alpha-\beta+1)}+K\right)\|y_1-y_2\|.
\end{aligned}
$$

Taking supremum for all $\zeta\in T$, we have

$$
\|\wp(y_1)-\wp(y_2)\|\le\left(\begin{array}{c}\frac{\frac{28}{3}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+7\|\mu_1\|\sigma_1+4\|\mu_2\|\sigma_2+2\|\mu_3\|\sigma_3}{2\Gamma(\alpha-2)}+\frac{\frac{5}{2}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+2\|\mu_1\|\sigma_1+\|\mu_2\|\sigma_2}{\Gamma(\alpha-1)} \\ +\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+\|\mu_1\|\sigma_1}{\Gamma(\alpha)}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)}{\Gamma(\alpha+1)}\end{array}\right)\|y_1-y_2\|.
$$

Now, if $\left(\begin{array}{c}\frac{\frac{28}{3}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+7\|\mu_1\|\sigma_1+4\|\mu_2\|\sigma_2+2\|\mu_3\|\sigma_3}{2\Gamma(\alpha-2)}+\frac{\frac{5}{2}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+2\|\mu_1\|\sigma_1+\|\mu_2\|\sigma_2}{\Gamma(\alpha-1)} \\ +\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+\|\mu_1\|\sigma_1}{\Gamma(\alpha)}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)}{\Gamma(\alpha+1)}\end{array}\right)<1$, then the operator $\wp$ is a contraction.

The first result is based on the existence of at least one solution for (NIFDE) (1) using Krasnselskii's fixed point theorem.

Theorem 4. *If assumptions* (H_1), (H_2), *and* (H_3) *hold, and if*

$$
\left(\begin{array}{c}\frac{\frac{28}{3}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+7\|\mu_1\|\sigma_1+4\|\mu_2\|\sigma_2+2\|\mu_3\|\sigma_3}{2\Gamma(\alpha-2)}+\frac{\frac{5}{2}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+2\|\mu_1\|\sigma_1+\|\mu_2\|\sigma_2}{\Gamma(\alpha-1)} \\ +\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+\|\mu_1\|\sigma_1}{\Gamma(\alpha)}+\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)}{\Gamma(\alpha+1)}\end{array}\right)<1, \tag{8}
$$

then (NIFDE) (1) has a unique solution on $J=[0,1]$.

Proof. The existence of at least one solution for (NIFDE) (1) has been established in Theorem (3). Furthermore, Lemma (4) demonstrates that the operator $\wp$ exhibits contraction properties. Consequently, through Banach's fixed point theorem, we conclude that the operator $\wp$ possesses a single fixed point, which corresponds to a unique solution of the (NIFDE) (1) over the interval $J=[0,1]$. Thus, the proof is now fully accomplished.

3 Numerical Example

Consider the following nonlinear implicit fractional differential equation NLIFDE:

$$\begin{cases} \mathfrak{D}^{\frac{11}{5}} y(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}} \left[\frac{7+y(\zeta)+\mathfrak{D}^{\frac{3}{5}} y(\zeta)+\int_0^1 e^{(\zeta-\upsilon)} \mathfrak{D}^{\frac{11}{5}} y(\upsilon)d\upsilon}{1+y(\zeta)+\mathfrak{D}^{\frac{3}{5}} y(\zeta)+2\int_0^1 e^{(\zeta-\upsilon)} \mathfrak{D}^{\frac{11}{5}} y(\upsilon)d\upsilon} \right] \text{ for all } \zeta \in [0,1], \\ y(0) + \mathfrak{D}^{\frac{6}{5}}_{0^+} y(1) = \int_0^1 \left(\frac{e^{-3\zeta}}{699+\sqrt{\zeta}} + \frac{1}{533} |\cos\sqrt{y(\zeta)}| \right) d\upsilon, \\ \mathfrak{D}^{\frac{6}{5}}_{0^+} y(0) + \mathfrak{D}^{\frac{1}{5}}_{0^+} y(1) = 2\int_0^1 \left(\frac{1}{\sqrt{699+\zeta^2}} + \frac{e^{-3\zeta}}{533+\zeta^2} |y(\zeta)| \right) d\upsilon, \\ \mathfrak{D}^{\frac{1}{5}}_{0^+} y(0) + \mathfrak{D}^{\frac{-4}{5}}_{0^+} y(1) = 3\int_0^1 \left(\frac{1}{2(\zeta^2+1)} + \frac{2+\sqrt{\zeta}}{e^{3(\zeta+1)}} |y(\zeta)| \right) d\upsilon. \end{cases} \tag{9}$$

In this problem, we have $\alpha = \frac{11}{5}$, $\beta = \frac{3}{5}$, $K(\zeta,\upsilon) = e^{(\zeta-\upsilon)}$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 3$.
$\mathfrak{h}_1(\upsilon, y(\upsilon)) = \left(\frac{e^{-3\zeta}}{699+\sqrt{\zeta}} + \frac{1}{533} |\cos\sqrt{y(\zeta)}| \right)$ with $\mu_1 = \frac{1}{533}$ and $H_1 = \frac{1}{699}$.
$\mathfrak{h}_2(\upsilon, y(\upsilon)) = \left(\frac{1}{\sqrt{699+\zeta^2}} + \frac{e^{-3\zeta}}{533+\zeta^2} |y(\zeta)| \right)$ with $\mu_2 = \frac{1}{533}$ and $H_2 = \frac{1}{\sqrt{699}}$.
$\mathfrak{h}_3(\upsilon, y(\upsilon)) = \left(\frac{1}{2(\zeta^2+1)} + \frac{2+\sqrt{\zeta}}{e^{3(\zeta+1)}} |y(\zeta)| \right)$ with $\mu_3 = \frac{2}{e^3}$ and $H_3 = \frac{1}{2}$.
It is clear that the assumptions (H_1)-(H_3) are satisfied, and f is a mutually continuous function such that for any u,v, $w \in R$, and $\zeta \in [0,1]$ we have

$$|f(\zeta,u,v,w)| = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}} (7+|u|+|v|+|w|) \text{ with } \lambda(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}, F = \frac{7}{69e}, \|\lambda\| = \frac{1}{69e}, \text{ and } K = e.$$

In addition, it clear from Theorem (3) that the NLIFDE (9) has at least one solution on $[0,1]$ since

$$\aleph_1 + \aleph_2 \approx 0.534117 < 1,$$

where

$$\begin{aligned} \aleph_1 &= \frac{\sigma_1(H_1+\|\mu_1\|)}{\Gamma(\alpha)} + \frac{2\sigma_1(H_1+\|\mu_1\|)+\sigma_2(H_2+\|\mu_2\|)}{\Gamma(\alpha-1)} + \frac{7\sigma_1(H_1+\|\mu_1\|)+4\sigma_2(H_2+\|\mu_2\|)+2\sigma_3(H_3+\|\mu_3\|)}{2\Gamma(\alpha-2)} \\ &\approx 0.00300125 + 0.0936787 + 0.428918 \\ &\approx 0.525598, \end{aligned}$$

and

$$\begin{aligned} \aleph_2 &= \frac{14\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{3\Gamma(\alpha-2)} + \frac{5\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{2\Gamma(\alpha-1)} + \frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha)} + \frac{\|\lambda\|\left(\frac{1}{\Gamma(1-\beta)}+K+1\right)}{\Gamma(\alpha+1)} \\ &\approx 0.001559 + 0.004176 + 0.001392 + 0.001392 \\ &\approx 0.008519. \end{aligned}$$

Furthermore, the uniqueness of the solution to the nonlinear implicit fractional differential equation (NLIFDE) given by equation (9) exists since the following condition is satisfied:

$$\frac{\frac{28}{3}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+7\|\mu_1\|\sigma_1+4\|\mu_2\|\sigma_2+2\|\mu_3\|\sigma_3}{2\Gamma(\alpha-2)} + \frac{\frac{5}{2}\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+2\|\mu_1\|\sigma_1+\|\mu_2\|\sigma_2}{\Gamma(\alpha-1)}$$
$$+\frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)+\|\mu_1\|\sigma_1}{\Gamma(\alpha)} + \frac{\|\lambda\|\left(\frac{1}{\Gamma(\alpha-\beta+1)}+K+1\right)}{\Gamma(\alpha+1)}$$
$$\approx 0.5757 + 0.2535 + 0.0608 + 0.0097$$
$$\approx 0.899873 < 1.$$

Remark 2. Equation (1) cannot be commented upon by any of the solutions obtained in the literature. This observation emphasizes the uniqueness and specificity of the problem at hand. Importantly, the current results significantly improve upon existing results in the literature, underscoring the advancement in our understanding of nonlinear fractional differential equations with integral boundary conditions.

4 Conclusion

This study has made significant contributions to comprehending the existence and uniqueness of solutions concerning a nonlinear implicit fractional differential equation (NLIFDE) under integral boundary conditions. The utilization of Banach's and Krasnoselskii's fixed point theorems has verified the theoretical outcomes, which were further supported by the numerical example provided. Future research endeavors will be directed towards exploring the existence and uniqueness of solutions for an implicit singular fractional differential equation with integral boundary conditions. The implications of these findings span across multiple domains, including physics, engineering, and finance.

Declarations

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