Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.4.7

The wrapped Monsef Distribution with Application to Tawaf Data

M.M.E. Abd El-Monsef[1,*], *E.I. Soliman*[2], *M. A. El-Qurashi*[2]

[1] Tanta University, Faculty of Science
[2] Institute of Basic and Applied Science, College of Engineering and Technology, Arab Academy for Science, Technology and Maritime Transport, P.O. Box 1029, Abu Quir Campus, Alexandria, Egypt

Received: Aug. 3, 2023 Accepted : June 24, 2024

Abstract: Various forms of rituals based on circular motions either clockwise or anti-clockwise. Tawaf is one of the most important rituals of the pilgrimage and refers to walk in circles around the Holy Kaabah in an anti-clockwise motion. It's an act of devotion to bring the pilgrim closer to Allah spiritually. In this paper, the wrapped Monsef distribution is proposed to model the direction of the pilgrims at tawaf. The behavior of the density function with changing parameter values is investigated and expressions for its characteristic function, trigonometric moments, and other related circular measures are derived. Maximum likelihood estimation is used to estimate parameters. A simulation analysis is conducted to demonstrate that the resulting ML estimator is accurate. Finally, the proposed model is tested using three real-life datasets. Its performance is compared with some wrapped distributions to examine its flexibility and it's found that the proposed model has the upper hand against the competitors.

Keywords: Monsef distribution, wrapped distribution, trigonometric moments, simulation, Circular data.

2010 Mathematics Subject Classification. 60E05, 62E10, 62E15.

1 Introduction

When data points are dispersed on a circle rather than a straight line (or a portion of one), the resulting phenomenon cannot be represented using a random variable with standard distributions, leading to the formation of so-called circular distributions. Biology, geology, geography, meteorology, physics, political science, and image analysis are just a few of the scientific fields that use directional data. For more applications of directional data see for example [1,2].

Circular or wrapped distributions create lots of flexible probability distributions that can handle the many properties of circular data. [3] introduced the basic idea of creating wrapped distributions. Several studies on wrapped distributions can be found in the literature handling their statistical properties and inference approaches. Since then, considerable effort has been expended on introducing a new trigonometric moment with an associated parameter into a wrapped distribution. For example, wrapped Laplace distribution [4], wrapped exponential distribution [5], wrapped t family circular distribution [6], wrapped gamma distribution [7], wrapped stable family of distributions [8], wrapped chi-square distribution [9], wrapped weighted exponential distribution [10], wrapped generalized Gompertz distribution [11], wrapped geometric distribution [12], wrapped Lindley distribution [13], wrapped quasi Lindley distribution [14], Exponential-wrapped distributions [15] and the wrapped modified Lindley distribution [16]. In addition, any periodic phenomenon with a known period, such as a day, a week, a month, or a year can be represented on a circle whose radius matches the period of the individual being represented.

The one-parameter Monsef distribution is a special case of the mixture Erlang distribution introduced by Abd El-Monsef [17]. Since then, many papers introduced some extensions for Monsef distribution due to its simplicity [18,19]. To determine the effectiveness of the wrapped Monsef distribution as a circular model, its density is wrapped around a unit circle and its properties are evaluated in this paper.

* Corresponding author e-mail: mmezzat@science.tanta.edu.eg

The paper is organized as follows: the probability density function of the wrapped Monsef distribution is introduced in Section 2. The trigonometric moments and some other related measures are derived in Section 3. In Section 4, the maximum likelihood estimator of the distribution parameter is derived and a simulation study is conducted to test its consistency. Three datasets are used in Section 5 to examine the flexibility of the proposed model against some other wrapped distributions.

2 Definition and Derivation

Let x be a random variable follows Monsef distribution with pdf given by:

$$g(x)=\frac{\lambda^3}{2+\lambda(2+\lambda)}(x+1)^2\mathrm{e}^{-x\lambda};\quad x>0,\lambda>0 \tag{1}$$

Then, the pdf of the wrapped Monsef distribution can be derived as in the following theorem.

Theorem 1. *The pdf of the wrapped Monsef distribution (WM) can be written as:*

$$f_{WM}(\theta)=\frac{e^{2\pi\lambda-\theta\lambda\left((1-2\pi+\theta)^2+e^{2\pi\lambda}\left(4\pi^2+(1+\theta)\left(4\pi+\left(-2+e^{2\pi\lambda}\right)(1+\theta)\right)\right)\right)\lambda^3}}{\left(-1+e^{2\pi\lambda}\right)^3(2+\lambda(2+\lambda))};\theta\in[0,2\pi),\lambda>0 \tag{2}$$

Proof. The wrapped Monsef distribution (WM) can be defined by wrapping Monsef distribution around the circumference of a unit circle. In other words, $\theta\equiv X+2k\pi$, for some integer k. That is,

$$\theta=X(\bmod 2\pi),\quad \text{for } \theta\in[0,2\pi)$$

Then the pdf of WM is given by:

$$f_{WM}(\theta)=\sum_{m=0}^{\infty}g(\theta+2m\pi),0<m<\infty$$

where, $g(x)$ is the pdf of Monsef distribution defined in (1). So we have

$$\begin{aligned}
f_{WM}(\theta)&=\frac{\lambda^3}{2+\lambda(2+\lambda)}\sum_{m=0}^{\infty}(\theta+1+2m\pi)^2\mathrm{e}^{-(\theta+2m\pi)\lambda}\\
&=\frac{\lambda^3}{2+\lambda(2+\lambda)}\mathrm{e}^{-\theta\lambda}\sum_{m=0}^{\infty}(\theta+1+2m\pi)^2\mathrm{e}^{-2m\pi\lambda}\\
&=\frac{\lambda^3}{2+\lambda(2+\lambda)}\mathrm{e}^{-\theta\lambda}\left[(\theta+1)^2\sum_{m=0}^{\infty}\mathrm{e}^{-2m\pi\lambda}+4(\theta+1)\pi\sum_{m=0}^{\infty}m\mathrm{e}^{-2m\pi\lambda}\right].
\end{aligned}$$

$$\begin{aligned}
&=\frac{e^{2\pi\lambda-\theta\lambda}\left(e^{2\pi\lambda}\left(-2+e^{2\pi\lambda}\right)(1+\theta)^2+(1-2\pi+\theta)^2+4e^{2\pi\lambda}\pi(1+\pi+\theta)\right)\lambda^3}{\left(-1+e^{2\pi\lambda}\right)^3(2+\lambda(2+\lambda))}\\
&=\frac{e^{2\pi\lambda-\theta\lambda}\left((1-2\pi+\theta)^2+e^{2\pi\lambda}\left(4\pi^2+(1+\theta)\left(4\pi+\left(-2+e^{2\pi\lambda}\right)(1+\theta)\right)\right)\right)\lambda^3}{\left(-1+e^{2\pi\lambda}\right)^3(2+\lambda(2+\lambda))}
\end{aligned}$$

Theorem 2.*The cdf of WM distribution can be written as follows:*

$$f_{WM}(\theta) = \frac{1}{1+(1+\lambda)^2}\left[\frac{1}{(-1+A)^3}\left((-1+A)^2\left(2+2\lambda+\lambda^2 - Ae^{-\theta\lambda}\left(2+2(1+\theta)\lambda+(1+\theta)^2\lambda^2\right)\right)\right.\right.$$
$$\left.\left.+4e^{-\theta\lambda}\left(-1+e^{\theta\lambda}\right)\pi^2\lambda^2 A\left[1+e^2\pi\lambda\right] - 4A\left(-1+e^{2\pi\lambda}\right)\pi\left(\lambda e^{-\theta\lambda}[1+\lambda+\theta\lambda]-\lambda[1+4\pi\lambda]\right)\right)\right];$$

$\theta \in [0, 2\pi), \lambda > 0, A = e^{2\pi\lambda - \theta\lambda}$

*Proof.*Based on [1,2] works, the cdf of the wrapped distribution can be derived as follows:

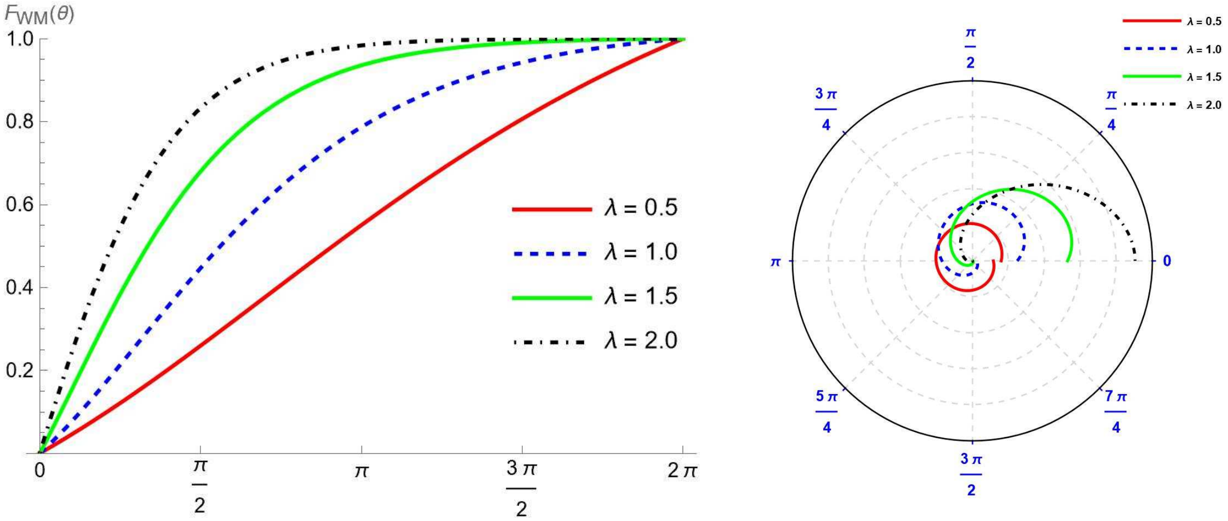
$$F(\theta) = \sum_{m=0}^{\infty} G_x(\theta + 2m\pi) - G_x(2m\pi)$$

where $G(x)$ is the cdf of the original distribution.
Applying the method we get:

$$\begin{aligned}
F_{WM}(\theta) &= \sum_{m=0}^{\infty} \frac{e^{-2m\pi\lambda}\left(1+(1+(1+2m\pi)\lambda)^2\right)}{1+(1+\lambda)^2} - \frac{e^{(-2m\pi-\theta)\lambda}\left(1+(1+(1+2m\pi+\theta)\lambda)^2\right)}{1+(1+\lambda)^2} \\
&= \frac{1}{1+(1+\lambda)^2}\sum_{m=0}^{\infty}\left[e^{-2m\pi\lambda}\left(1+(1+(1+2m\pi)\lambda)^2\right) - e^{(-2m\pi-\theta)\lambda}(1+(1+(1+ \right. \\
&\left. 2m\pi+\theta)\lambda)^2)\right] \\
&= \frac{1}{1+(1+\lambda)^2}\sum_{m=0}^{\infty}\left[e^{-2m\pi\lambda}\left(2+2\lambda+4m\pi\lambda+\lambda^2+4m\pi\lambda^2+4m^2\pi^2\lambda^2\right) - \right. \\
& e^{(-2m\pi-\theta)\lambda}\left(2+2\lambda+4m\pi\lambda+2\theta\lambda+\lambda^2+4m\pi\lambda^2+4m^2\pi^2\lambda^2+2\theta\lambda^2+ \right. \\
&\left.\left. 4m\pi\theta\lambda^2+\theta^2\lambda^2\right)\right] \\
&= \frac{1}{1+(1+\lambda)^2}\sum_{m=0}^{\infty}\left[e^{-2m\pi\lambda}\left((2+2\lambda+\lambda^2)+4\pi\lambda(1+4\pi\lambda)m+4\pi^2\lambda^2m^2\right) - \right. \\
& e^{(-2m\pi-\theta)\lambda}\left((2+2\lambda+2\theta\lambda+\lambda^2+2\theta\lambda^2+\theta^2\lambda^2)+4\pi\lambda(1+\lambda+\theta\lambda)m+ \right. \\
&\left.\left. 4\pi^2\lambda^2m^2\right)\right] \\
&= \frac{1}{1+(1+\lambda)^2}\sum_{m=0}^{\infty}\left[e^{-2m\pi\lambda}\left((2+2\lambda+\lambda^2)-\left(2+2(1+\theta)\lambda+(1+\theta)^2\lambda^2\right)e^{-\theta\lambda}\right)+ \right. \\
&\left. me^{-2m\pi\lambda}\left(4\pi\lambda(1+4\pi\lambda)-4\pi\lambda e^{-\theta\lambda}(1+\lambda+\theta\lambda)\right)+m^2e^{-2m\pi\lambda}\left(4\pi^2\lambda^2-4\pi^2\lambda^2e^{-\theta\lambda}\right)\right]
\end{aligned}$$

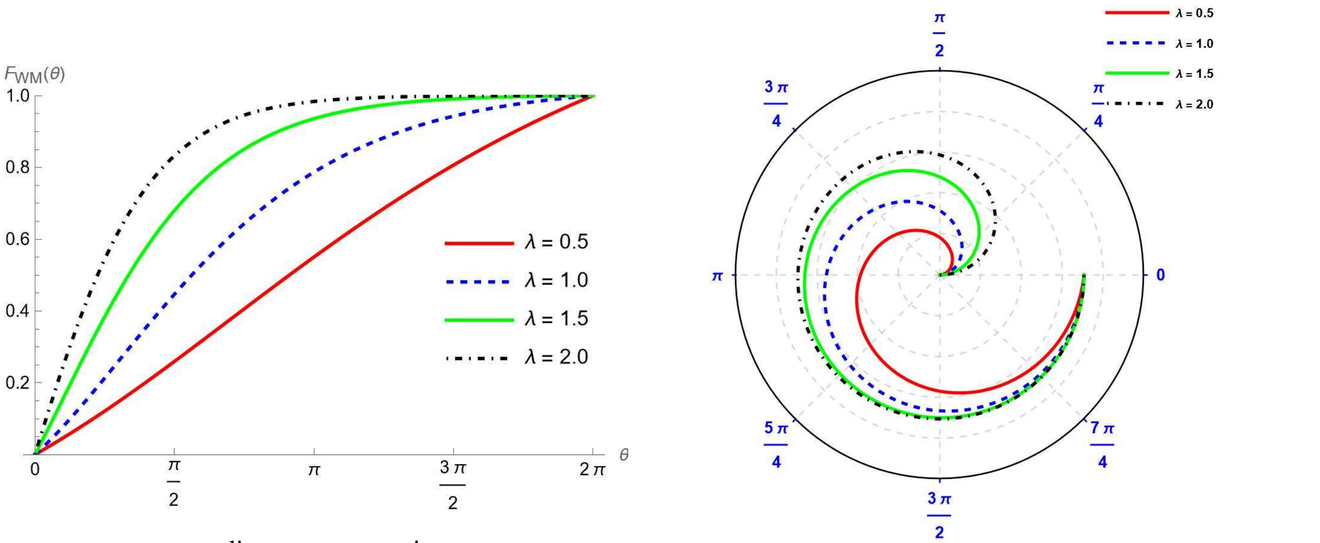
$$= \frac{1}{1+(1+\lambda)^2}\left[\frac{1}{(-1+e^{2\pi\lambda})^3}\left(\left(-1+e^{2\pi\lambda}\right)^2\left(2+2\lambda+\lambda^2-e^{2\pi\lambda-\theta\lambda}\left(2+2(1+\theta)\lambda+(1+\theta)^2\lambda^2\right)\right)+\right.\right.$$
$$\left.\left.4e^{-\theta\lambda}\left(-1+e^{\theta\lambda}\right)\pi^2\lambda^2e^{2\pi\lambda}\left[1+e^{2\pi\lambda}\right]-4e^{2\pi\lambda}\left(-1+e^{2\pi\lambda}\right)\pi\left(\lambda e^{-\theta\lambda}[1+\lambda+\theta\lambda]-\lambda[1+4\pi\lambda]\right)\right)\right] \square$$

Figures 1 and 2 illustrate the behavior of the pdf and cdf of the proposed model in linear and circular representations at different parameter values.



a. linear representation b. circular representation

Fig. 1: The pdf behavior of wrapped Monsef distribution



a. linear representation

Wrapped Monsef Distribution

b. circular representation

Fig. 2: The cdf behavior of wrapped Monsef distribution

3 Statistical Properties

In this section, the characteristic function, trigonometric moments and some other location and dispersion measures were derived.

3.1 Characteristic function

The characteristic function of circular random variables is defined only at integer values. The characteristic function of the wrapped Monsef distribution can be derived as in the following theorem.

Theorem 3. *The characteristic function of the WM distribution can be written as follows:*

$$\varphi(p) = \rho_p e^{i\mu_p}; \quad p \text{ is an integer}$$

where, $\rho_p = \frac{\lambda^3\left(\left(2+2\lambda+\lambda^2\right)^2+\left(2(1+\lambda)p+p^2\right)^2\right)^{\frac{1}{2}}}{(2+\lambda(2+\lambda))\left(\lambda^2+p^2\right)^{1.5}}$ *and* $\mu_p = 3\tan^{-1}[\mathrm{p}/\lambda] - \tan^{-1}\left(\frac{\left(2(1+\lambda)p+p^2\right)}{2+2\lambda+\lambda^2}\right)$ *are the length and direction vectors for the p th trigonometric moment.*

Proof. For any integer p, the characteristic function of the WM distribution can be written as:

$$\varphi_X(p) = E\left(e^{ipx}\right) = \frac{\lambda^3(\lambda - ip)^{-3}\left(2+2\lambda+\lambda^2 - 2(1+\lambda)ip + ip^2\right)}{(2+\lambda(2+\lambda))}$$

Since for $a,b,r \in \mathbb{R}^+$, we have $(a-ib)^{-r} = \left(a^2+b^2\right)^{-\frac{r}{2}} \mathrm{e}^{ir\tan^{-1}(b/a)}$ [15] .
Therefore, we can write

$$(\lambda - ip)^{-3} = \left(\lambda^2+p^2\right)^{-1.5} \mathrm{e}^{3i\tan^{-1}(p/\lambda)}, \text{ and}$$
$$\left(2+\lambda(2+\lambda) - \left(2(1+\lambda)p+p^2\right)i\right)$$
$$= \left((2+\lambda(2+\lambda))^2 + \left(2(1+\lambda)p+p^2\right)^2\right)^{\frac{1}{2}} \mathrm{e}^{-i\tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+\lambda(2+\lambda)}\right)}$$

Then, $\varphi_X(p) = \frac{\lambda^3}{2+\lambda(2+\lambda)}\left(\lambda^2+p^2\right)^{-1.5} \mathrm{e}^{3i\tan^{-1}(p/\lambda)}\left((2+\lambda(2+\lambda))^2 + (2(1+\lambda)p+\right.$

$$\left.p^2\right)^2\Big)^{0.5} \mathrm{e}^{-i\tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+\lambda(2+\lambda)}\right)} \frac{e^{3i\tan^{-1}(p/\lambda)-i\tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+\lambda(2+\lambda)}\right)}\lambda^3\left(\left(2(1+\lambda)p+p^2\right)^2+(2+\lambda(2+\lambda))^2\right)^{0.5}}{\left(p^2+\lambda^2\right)^{1.5}(2+\lambda(2+\lambda))}$$
$$= \frac{\lambda^3\left((2+\lambda(2+\lambda))^2+\left(2(1+\lambda)p+p^2\right)^2\right)^{0.5}}{\left(\lambda^2+p^2\right)^{1.5}(2+\lambda(2+\lambda))} e^{i\left(3\tan^{-1}(p/\lambda)-\tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+\lambda(2+\lambda)}\right)\right)}$$

Let

$$\rho_p = \frac{\lambda^3\left((2+\lambda(2+\lambda))^2+\left(2(1+\lambda)p+p^2\right)^2\right)^{0.5}}{\left(\lambda^2+p^2\right)^{1.5}(2+\lambda(2+\lambda))}$$

and

$$\mu_p = 3\tan^{-1}(p/\lambda) - \tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+\lambda(2+\lambda)}\right)$$

then we have $\varphi(p) = \rho_p e^{i\mu_p}$.

3.2 Trigonometric Moments and Related Measures

The p th trigonometric moment of the wrapped Monsef distribution is the value of the characteristic function at an integer p. It can be written in terms of the p th cosine α_p and sine β_p moments as follows:

$$\varphi_p = \mathrm{E}\left(e^{ip\theta}\right) = \alpha_p + i\beta_p; \quad p = \pm 1, \pm 2, \ldots$$

where,

$$\alpha_p = \mathrm{E}(\cos p\theta) = \rho_p \cos(\mu_p)$$
$$\text{and } \beta_p = \mathrm{E}(\sin p\theta) = \rho_p \sin(\mu_p).$$

So, we have

$$\alpha_\mathrm{p} = \frac{\lambda^3\left(\left(2+2\lambda+\lambda^2\right)^2 + \left(2(1+\lambda)p+p^2\right)^2\right)^{\frac{1}{2}}}{(2+\lambda(2+\lambda))\left(\lambda^2+p^2\right)^{1.5}} \cos\left(3\tan^{-1}(\mathrm{p}/\lambda) - \tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+2\lambda+\lambda^2}\right)\right)$$

and

$$\beta_p = \frac{\lambda^3\left(\left(2+2\lambda+\lambda^2\right)^2 + \left(2(1+\lambda)p+p^2\right)^2\right)^{\frac{1}{2}}}{(2+\lambda(2+\lambda))\left(\lambda^2+p^2\right)^{1.5}} \sin\left(3\tan^{-1}(\mathrm{p}/\lambda) - \tan^{-1}\left(\frac{2(1+\lambda)p+p^2}{2+2\lambda+\lambda^2}\right)\right)$$

In circular forms, the arithmetic mean can't be used as a measure of the center of a set of directions as it depends on the choice of the origin and the rotation. The mean direction vector introduces information about the distribution mean and it's preferable to use more than the arithmetic mean in wrapped distributions. The mean direction of the WM distribution is given by

$$\mu = \mu_1 = 3\tan^{-1}(1/\lambda) - \tan^{-1}\left[\frac{1+2(1+\lambda)}{2+\lambda(2+\lambda)}\right]$$

The circular concentration towards a direction ρ_1 can be considered as a measure of dispersion. The larger values of ρ_1, i.e. close to 1 , indicates that the observations are more concentrated around the mean direction μ. The circular concentration of the WM distribution is

$$\rho_1 = \frac{\lambda^3\sqrt{(3+2\lambda)^2 + (2+\lambda(2+\lambda))^2}}{\left(1+\lambda^2\right)^{3/2}(2+\lambda(2+\lambda))}$$

The circular variance of the WM distribution can be calculated as

$$\mathrm{V} = 1-\rho_1 = 1 - \frac{\lambda^3\sqrt{(3+2\lambda)^2 + (2+\lambda(2+\lambda))^2}}{\left(1+\lambda^2\right)^{3/2}(2+\lambda(2+\lambda))}$$

The circular deviation of the WM distribution is given by

$$\sigma = \sqrt{-2\ln\rho_1} = \sqrt{-2\ln\left(\frac{\lambda^3\sqrt{(1+2(1+\lambda))^2 + (2+\lambda(2+\lambda))^2}}{\left(1+\lambda^2\right)^{3/2}(2+\lambda(2+\lambda))}\right)}$$

The central trigonometric moments are presented by $\bar{\alpha}_p = \rho_p \cos(\mu_p - p\mu_1)$ and $\bar{\beta}_p = \rho_p \sin(\mu_p - p\mu_1)$. As a result, the proposed distribution's central trigonometric moments are as follows:

$$\begin{aligned}\bar{\alpha}_\mathrm{p} = &\frac{1}{\left(p^2+\lambda^2\right)^{1.5}(2+\lambda(2+\lambda))}\left(\lambda^3\left((p(p+2(1+\lambda)))^2\right.\right.\\ &\left.+(2+\lambda(2+\lambda))^2\right)^{0.5}\cos\left(3\tan^{-1}[p/\lambda]\right.\\ &\left.\left.-p\left(3\tan^{-1}[1/\lambda] - \tan^{-1}\left[\frac{1+2(1+\lambda)}{2+\lambda(2+\lambda)}\right]\right) - \tan^{-1}\left[\frac{p(p+2(1+\lambda))}{2+\lambda(2+\lambda)}\right]\right)\right)\end{aligned}$$

and

$$\bar{\beta}_p = \frac{1}{(p^2+\lambda^2)^{1.5}(2+\lambda(2+\lambda))}\left(\lambda^3\left((p(p+2(1+\lambda)))^2\right.\right.$$
$$\left.+(2+\lambda(2+\lambda))^2\right)^{0.5}\sin\left(3\tan^{-1}[p/\lambda]\right.$$
$$\left.\left.-p\left(3\tan^{-1}[1/\lambda]-\tan^{-1}\left[\frac{1+2(1+\lambda)}{2+\lambda(2+\lambda)}\right]\right)-\tan^{-1}\left[\frac{p(p+2(1+\lambda))}{2+\lambda(2+\lambda)}\right]\right)\right)$$

The coefficient of skewness is $\xi_1^0 = \frac{\bar{\beta}_2}{V^{1.5}}$ given by

$$\xi_1^0 = \frac{1}{(4+\lambda^2)^{1.5}(2+\lambda(2+\lambda))\left(1-\frac{\lambda^3\sqrt{13+\lambda(20+\lambda(12+\lambda(4+\lambda)))}}{(1+\lambda^2)^{1.5}(2+\lambda(2+\lambda))}\right)^{1.5}}$$
$$\times\left(\lambda^3\sqrt{68+\lambda(72+\lambda(24+\lambda(4+\lambda)))}\mathrm{Sin}[A]\right)$$

where $A = 3\tan^{-1}[\lambda/2]-6\cot^{-1}[\lambda]-\tan^{-1}\left[\frac{4(2+\lambda)}{2+\lambda(2+\lambda)}\right]+2\tan^{-1}\left[\frac{3+2\lambda}{2+\lambda(2+\lambda)}\right]$

The coefficient of kurtosis $\xi_2^0 = \frac{\bar{\alpha}_2-(1-V)^4}{V^2}$ is given by

$$\xi_2^0 = \frac{1}{\left(1-\frac{\lambda^3\sqrt{(1+2(1+\lambda))^2+(2+\lambda(2+\lambda))^2}}{(1+\lambda^2)^{1.5}(2+\lambda(2+\lambda))}\right)^2}$$
$$\times\left(-\frac{\lambda^{12}\left((1+2(1+\lambda))^2+(2+\lambda(2+\lambda))^2\right)^2}{(1+\lambda^2)^6(2+\lambda(2+\lambda))^4}+\frac{\lambda^3\sqrt{(4+4(1+\lambda))^2+(2+\lambda(2+\lambda))^2}\sin[B]}{(4+\lambda^2)^{1.5}(2+\lambda(2+\lambda))}\right)$$

where $B = 3\tan^{-1}[2/\lambda]-2\left(3\tan^{-1}[1/\lambda]-\tan^{-1}\left[\frac{1+2(1+\lambda)}{2+\lambda(2+\lambda)}\right]\right)-\tan^{-1}\left[\frac{4+4(1+\lambda)}{2+\lambda(2+\lambda)}\right]$

Table 1 illustrated the values of these characteristics for various values of λ.

Table 1: The value of some measures of the WM distribution

	λ	0.5	1	2	4
Trigonometric moments	α_1	−0.1077	0.0000	0.6208	0.9364
	α_2	−0.0421	−0.1264	0.1500	0.7778
	β_1	0.0923	0.5000	0.6144	0.3257
	β_2	0.0189	0.1952	0.6500	0.5858
Mean direction	μ	2.4330	1.5708	0.7802	0.3347
Circular concentration	ρ_1	0.1418	0.5000	0.8734	0.9914
Circular variance	V	0.8582	0.5000	0.1266	0.0086
Circular standard deviation	σ	1.9764	1.1774	0.5202	0.1312
Central trigonometric moments	$\bar{\alpha}_1$	0.1418	0.5000	0.8734	0.9914
	$\bar{\alpha}_2$	−0.0251	0.1264	0.6515	0.9735
	$\bar{\beta}_1$	0.0000	0.0000	0.0000	0.0000
	$\bar{\beta}_2$	−0.0388	−0.1952	−0.1433	−0.0232
Coefficient of skewness	ξ_1^0	−0.0487	−0.5521	−3.1814	−29.2965
Coefficient of kurtosis	ξ_2^0	−0.0346	0.2556	4.3404	100.0664

4 Estimation and Simulation

If $(\theta_1, \theta_2, \ldots, \theta_n)$ are n independent and identically distributed random variables having the wrapped Monsef distribution with parameter λ; then, the corresponding log-likelihood function can be written as

$$L(\theta) = 3n\log(\lambda) - n\log(2+\lambda(2+\lambda)) - \lambda\sum_{i=1}^{n}\theta_i + \sum_{i=1}^{n}\log\left[\frac{(1+\theta_i)^2}{1-e^{-2\pi\lambda}} + 4\pi(1+\theta_i)\frac{e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2} + 4\pi^2\frac{e^{-2\pi\lambda}(1+e^{-2\pi\lambda})}{(1-e^{-2\pi\lambda})^3}\right] \quad (4)$$

Letting the partial derivative of (4) with respect to λ is equal to zero, we get

$$\frac{n(6+\lambda(4+\lambda))}{\lambda(2+\lambda(2+\lambda))} - \sum_{i=1}^{n}\theta_i + \sum_{i=1}^{n}\frac{2\pi\left((1-2\pi+\theta_i)^2 + e^{4\pi\lambda} + (1+2\pi+\theta_i)^2 + 2e^{2\pi\lambda}(-1+8\pi^2-\theta_i(2+\theta_i))\right)}{e^{6\pi\lambda}(1+\theta_i)^2 - (1-2\pi+\theta_i)^2 + e^{2\pi\lambda}(1+\theta_i)(3-8\pi+3\theta_i) + e^{4\pi\lambda}(-1+2\pi-\theta)(3+2\pi+3\theta_i)} = 0 \quad (5)$$

The solution of (5) can't be obtained directly due to nonlinearity. The Newton-Rapshon techniques can be used as a numerical method to find the solution of the nonlinear equation.

A simulation analysis is conducted to study the performance of the ML estimator. For various values of λ, we used samples of sizes (10, 20, 30, 40, 50, 60, 70, 80, 90 and 100). The simulation was carried 1000 times with different sample sizes and different parameter values. The results are summarized in Table (2). From Table 2; it can be concluded

Table 2: Values of MSE and bias at different values of λ

	$\lambda = 0.1$		$\lambda = 0.5$		$\lambda = 1.5$		$\lambda = 1.75$	
n	MSE	Bais	MSE	Bais	MSE	Bais	MSE	Bais
10	0.0457	0.0859	0.0600	-0.1434	0.0036	-0.7376	0.0006	-0.8883
20	0.0368	0.0664	0.0361	-0.1606	0.0020	-0.7394	0.0221	-0.9059
30	0.0271	0.0538	0.0306	-0.1733	0.0014	-0.7395	0.0332	-0.9130
40	0.0246	0.0501	0.0299	-0.1735	0.0010	-0.7408	0.0241	-0.9127
50	0.0232	0.0486	0.0373	-0.1739	0.0009	-0.7396	0.0131	-0.9116
60	0.0219	0.0464	0.0163	-0.1685	0.0007	-0.7394	0.0133	-0.9126
70	0.0212	0.0425	0.0157	-0.1713	0.0006	-0.7399	0.0136	-0.9139
80	0.0195	0.0375	0.0138	-0.1686	0.0005	-0.7396	0.0274	-0.9188
90	0.0192	0.0369	0.0119	-0.1652	0.0005	-0.7395	0.0601	-0.9277
100	0.0186	0.0340	0.0223	-0.1671	0.0004	-0.7393	0.0474	-0.9249

that the MSE and the Bias decrease as the sample size increases.In addition, at fixed n when the value of λ increased, the bias is also increased and the MSE decreased.

5 Applications

This section aimed to evaluate the effectiveness of several wrapped distributions in accurately fitting three separate data sets. In order to compare these results directly, we employed various statistical measures, including the log-likelihood, the Akaike information criterion AIC, the consistent Akaike information criterion CAIC, the Bayesian information criterion BIC, the Hannan information criterion HQIC, the Anderson-Darling test statistic AD, and the Kolmogorov-Smirnov K-S tests. The results showed that the WM distribution offers a more accurate and reliable fit for the three data sets compared to the other distributions examined. The proposed distribution was compared with the wrapped exponential distribution WE(λ). wrapped Lindley exponential distribution WRLE(λ, α), the transmuted wrapped exponential TWE(λ, α), a wrapped Laplace distribution WL(λ, α) and a wrapped weighted exponential distribution WWE(λ, α)

Data 1 (The Tawaf Data)
Tawaf refers to circling the Holy Kaabah seven times in an anti-clockwise direction starting from the black stone. This data set consists of the direction of 63 pilgrims measured in a clockwise direction from the starting line of Tawaf, which is the line connecting the black stone and the green light on the wall opposite to it.

29.44°, 30.61°, 30.62°, 31.10°, 31.42°, 31.70°, 34.08°, 34.97°, 35.11°, 36.14°,
37.71°, 37.72°, 38.30°, 39.18°, 39.39°, 39.46°, 39.96°, 40.48°, 41.66°, 42.57°,
42.79°, 43.14°, 43.59°, 44.27°, 44.55°, 44.91°, 45.63°, 46.29°, 47.43°, 49.37°,
49.50°, 49.98°, 50.12°, 50.72°, 51.00°, 51.32°, 53.40°, 53.52°, 54.17°, 54.17°,
56.46°, 57.63°, 58.12°, 59.30°, 59.85°, 60.19°, 60.34°, 60.63°, 60.83°, 61.11°,
62.22°, 62.41°, 62.72°, 62.87°, 63.69°, 64.57°, 66.02°, 66.07°, 66.10°, 66.47°,
66.94°, 67.41°, 67.60°

Data 2 (Turtle Data Set)
The following data set was given in [13]. It represents the seasonal migratory orientation of 76 turtles after laying eggs.

8°, 9°, 13°, 13°, 14°, 18°, 22°, 27°, 30°, 34°, 38°,
38°, 40°, 44°, 45°, 47°, 48°, 48°, 48°, 48°, 50°, 53°,
56°, 57°, 58°, 58°, 61°, 63°, 64°, 64°, 64°, 65°, 65°,
68°, 70°, 73°, 78°, 78°, 78°, 83°, 83°, 88°, 88°, 88°,
90°, 92°, 92°, 93°, 95°, 96°, 98°, 100°, 103°, 106°, 113°,
118°, 138°, 153°, 153°, 155°, 204°, 215°, 223°, 226°, 237°, 238°,
243°, 244°, 250°, 251°, 257°, 268°, 285°, 319°, 343°, 350°,

Data 3 (The Fisher-B5 Data)
This dataset was introduced firstly by Fisher [20]. It contains the measurements of long-axis orientation of 164 feldspar laths in basalt rock. It contains 60 orientated observations (recorded in degrees).

1°, 1°, 2°, 2°, 3°, 8°, 9°, 12°, 16°, 17°,
19°, 23°, 28°, 28°, 34°, 34°, 35°, 36°, 36°, 37°,
41°, 45°, 49°, 50°, 51°, 53°, 58°, 68°, 69°, 70°,
72°, 72°, 76°, 78°, 80°, 85°, 97°, 97°, 99°, 101°,
105°, 121°, 125°, 126°, 133°, 141°, 143°, 149°, 152°, 156°,
160°, 163°, 167°, 168°, 170°, 171°, 172°, 174°, 175°, 176°,

Table 3: Summary of fits for data 1

Model	MLE	-Log	AIC	CAIC	BIC	HQIC	AD	K-S(stat)
WM	$\hat{\alpha} = 2.000$	45.855	93.711	93.776	95.854	94.554	13.599	0.387
WRLE	$\hat{\alpha} = 0.8187$, $\hat{\beta} = 2.0000$	46.448	96.896	97.096	101.182	98.582	13.142	0.373
WL	$\hat{\alpha} = 1.5829$	49.937	101.873	101.939	104.016	102.716	13.403	0.414
WE	$\hat{\alpha} = 1.1363$	54.542	111.084	111.150	113.227	111.927	26.271	0.604
TWE	$\hat{\alpha} = 1.9234$, $\hat{\beta} = 2.0000$	46.967	97.934	98.134	102.221	99.620	—	1.097

Table 4: Summary of fits for data 2

Model	MLE	-Log	AIC	CAIC	BIC	HQIC	AD	K-S(stat)
WM	$\hat{\alpha}=0.907$	117.498	236.995	237.049	239.326	237.927	4.1556	0.2338
WRLE	$\hat{\alpha}=0.5$ $\hat{\beta}=1.5288$	120.026	244.052	244.217	248.714	245.915	1.7763	0.1543
WL	$\hat{\alpha}=0.7482$	119.647	241.295	241.349	243.626	242.226	8.9406	0.3123
WE	$\hat{\alpha}=0.4228$	120.647	243.2948	243.349	245.626	244.226	18.7436	0.4082
TWE	$\hat{\alpha}=0.7475$ $\hat{\beta}=-0.9513$	117.95	239.89	240.064	244.561	244.226	18.7436	0.1293

Table 5: Summary of fits for data 3

Model	MLE	-Log	AIC	CAIC	BIC	HQIC	AD	K-S(stat)
WM	$\hat{\alpha}=1.3955$	156.512	158.512	158.581	160.606	159.331	0.8924	0.0929
WRLE	$\hat{\alpha}=0.7127$ $\hat{\beta}=1.5442$	156.635	160.635	160.845	164.824	162.273	0.9363	0.0964
WL	$\hat{\alpha}=1.0309$	156.853	158.853	158.922	160.948	159.672	1.0092	0.1112
WE	$\hat{\alpha}=0.6640$	159.430	161.430	161.499	163.524	162.249	1.1247	0.1165
TWE	$\hat{\alpha}=0.8322$ $\hat{\beta}=-0.4359$	157.447	161.447	161.658	165.636	163.086	0.9868	0.1020

In order to see how closely observed data mirrors expected data, some goodness-of-fit tests were used. From Tables 3, 4 and 5 we can notice that all the values of the goodness-of-fit tests of the proposed distribution is lower than the values of the other competitors for the three data. It can be concluded that the proposed model fits the three data better than the other wrapped distributions.

Conclusion

In this paper, the wrapped Monsef distribution was proposed. The behavior of its pdf and cdf is discussed. Most of the circular characteristics of the proposed model were derived. The method of maximum likelihood estimation is used to estimate the model parameter. Three data sets were used to examine the flexibility of the proposed model and it's concluded that it introduces a better fits compared with some other wrapped distributions.

Declarations

Competing interests: The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.
Authors' contributions: Abd El-Monsef conceived of the presented idea. All authors developed the theoretical formalism, performed the analytic calculations and performed the numerical simulations. Both Abd El-Monsef and El-Qurashi contributed to the final version of the manuscript. Soliman contributed to the analysis of the results and to the writing of the manuscript.
Funding: The authors declare that there was no financial support for this work.
Availability of data and materials: The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.
Acknowledgments: The authors are thankful to the reviewers for the very constructive and valuable comments that enhancing the article.

References

[1] Jammalamadaka, S. R., & Sengupta, A. (2001). Topics in circular statistics (Vol. 5). world scientific. USA.
[2] Mardia, K. V., Jupp, P. E., & Mardia, K. (2000). Directional statistics (Vol. 2). Wiley Online Library. UK.
[3] Lévy, P. (1939). L'addition des variables aléatoires définies sur une circonférence. Bulletin de la Société mathématique de France. 67, 1-41.
[4] Jammalamadaka, S. R., & Kozubowski, T. (2003). A new family of circular models: The wrapped Laplace distributions. Advances and applications in statistics, 3(1), 77-103.
[5] Jammalamadaka, S. R., & Kozubowski, T. J. (2004). New families of wrapped distributions for modeling skew circular data. Communications in Statistics-Theory and Methods. 33(9), 2059-2074.
[6] Pewsey, A., Lewis, T., & Jones, M. (2007). The wrapped t family of circular distributions. Australian & New Zealand Journal of Statistics, 49(1), 79-91.
[7] Coelho, C. A. (2007). The wrapped Gamma distribution and wrapped sums and linear combinations of independent Gamma and Laplace distributions. Journal of Statistical Theory and Practice. 1(1), 1-29.
[8] Pewsey, A. (2008). The wrapped stable family of distributions as a flexible model for circular data. Computational Statistics & Data Analysis. 52(3), 1516-1523.
[9] Adnan, M. A. S. (2011). Wrapped Chi-square distribution. Journal of Applied Statistical Science. 18(3), 10
[10] Roy, S., & Adnan, M. A. S. (2012). Wrapped weighted exponential distributions. Statistics & probability letters. 82(1), 77-83.
[11] Roy, S.,& Adnan, M. (2012). Wrapped generalized Gompertz distribution: An application to ornithology. Journal of Biometrics & Biostatistics. 3(6), 153-156.
[12] Jacob, S., & Jayakumar, K. (2013). Wrapped Geometric Distribution: A new Probability Model for Circular Data. Journal of Statistical Theory and Applications. 12(4), 348-355.
[13] Joshi, S., & Jose, K. (2018). Wrapped Lindley distribution. Communications in Statistics-Theory and Methods. 47(5), 1013-1021.
[14] Al-khazaleh, A. M., & Alkhazaleh, S. (2019). On wrapping of quasi Lindley distribution. Mathematics. 7(10), 930.
[15] Chevallier, E. (2021). Exponential-Wrapped Distributions on SL(2,C) and the Möbius Group. International Conference on Geometric Science of Information, Springer, pp. 265-272.
[16] Chesneau, C., Tomy, L., & Jose, M. (2021). Wrapped modified Lindley distribution. Journal of Statistics and Management Systems, 24(5), 1025-1040.
[17] Abd El-Monsef, M. (2021). Erlang mixture distribution with application on COVID-19 cases in Egypt. International Journal of Biomathematics, 14(03), 2150015
[18] Abd El-Monsef, M. M. E., Sohsah, N. M. and Hassanein, W. A. (2021) Unit Monsef Distribution with Regression Model. Asian Journal of Probability and Statistics. 15(4) pp. 330-340.
[19] Abd El-Monsef, M.M.E. and Alshrani, M. (2020) The weighted Monsef distribution. African Journal of Mathematics and Computer Science Research. 13(2), 74-84.
[20] Fisher, N. I. (1995). Statistical analysis of circular data. Cambridge university press. UK.