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Existence and Uniqueness Theorems of Multi-Dimensional Integro-Differential Equations with Conformable Fractional Differointegrations

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Abstract: The primary goal of this article is to theoretically state and demonstrate the existence of unique solutions to three different types of multi-dimensional conformable fractional partial integro-differential equation problems. The proofs of these theorems are based on a well-known Banach's fixed point theorem. On the basis of the Lipschitz condition, necessary requirements are developed that the infernal function of the integral operators must satisfy.

Keywords: Fractional partial integro-differential equations, existence and uniqueness theorem, Fixed point theorem of Banach, Conformable calculus.

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1 Introduction

Fractional calculus was introduced by several researchers, such as Leibniz and Euler in the 18^{th} century, but it was not studied extensively until the 20^{th} century and began to be studied more systematically [4]. In the 1970s, authors like Samko, Kilbas, and Marichev developed the theory of fractional integrals and derivatives that was based on the "Riemann-Liouville" definition of fractional derivatives [5,1].

Among the important topics in fractional calculus, which grows a great interest by researches is the partial integro differential equations (PIDEs) and thus there has been an increase interest in this topic in the recent years in extending fractional calculus for different topics of pure and applied mathematics and also for higher dimensions, as well as, different types of fractional derivatives. This has led some interested researchers in such branches to develop the theory of the subject to the so called conformable fractional calculus, which is applied in this work for PIDEs that will be called later on as conformable fractional order partial integro-differential equations (CFPIDEs). These equations are considered as a generalization of fractional partial differential equations, and they involve both fractional derivatives and fractional integrals of conformable type. The theorem of the existence of a unique solution for PIDEs were established in 2017 by Hussain et al. with "Caputo" fractional derivatives and "Riemann-Liouville" reactional integrals which has some inconsistencies of the existing fractional derivatives [10,4]. Also, studied by Gambo et al. in 2018 [12,14,2] for differential equations with fractional derivatives based on the generalized "Caputo" fractional derivatives. In their work, they proved that under certain conditions, the CFPIDEs have a unique solution that can be expressed as a series solution. The conditions they imposed were based on a new concept of conformable fractional derivative, which is a modification of the "Riemann-Liouville" fractional order derivative that is better suited for higher-dimensional problems. Since the publication of Jafari et al.s result, there has been a growing interest in the study of CFPIDEs and their applications in several fields like physics, engineering, and finance, which are well employ conformable derivatives and integrals. Researchers have continued to

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refine the theory and develop new techniques for solving CFPIDEs, and it is likely that this area of mathematics will continue to be an active area of research for years to come [6,7,8].

It is important to recall that the fixed-point most results, which are well-known and has its basis on the “contraction mapping principle” or the “Banachs fixed point theorem”. In addition, the principle has a wide range of applications not only on the different branches of mathematical topics, but also in economics, chemistry, biotechnology, computer sciences, engineering, and various disciplines, which are based on the mentioned effect, it was developed extensively by several researchers [3]. This enables the researchers to determine which operator best suits their needs in order to describe the dynamics of actual situations.

The fixed point theorem of Banach will be utilized in this article as the main objectiveto formulate and verify three types ofCFPIDEs of the existence and uniqueness theorems, namely:

- **Problem 1**

$$D_x^{\alpha}u(x,y) = g(x,y) + \int_a^x \int_c^y k(x,y,\xi,\eta,u(\xi,\eta))d\eta d\xi,\, x \in [a,b], y \in [c,d]. \tag{1}$$

- **Problem 2**

$$\frac{\partial u(x,y)}{\partial x} = g(x,y) + I_{\xi}^{\beta} I_{\eta}^{\gamma} k(x,y,\xi,\eta,u(\xi,\eta)),\, \beta,\gamma \in \mathbb{R}^+,\, x \in [a,b], y \in [c,d]. \tag{2}$$

- **Problem 3**

$$D_x^{\alpha}u(x,y) = g(x,y) + I_{\xi}^{\beta} I_{\eta}^{\gamma} k(x,y,\xi,\eta,u(\xi,\eta)),\, \alpha \in \mathbb{R}^+,\, x \in [a,b], y \in [c,d]. \tag{3}$$

where D and I are respectively the conformal differential and integral operators of fractional order.

2 Basic Concepts for Banach Fixed Point Theorem

In addition to identifying the necessary conditions for stating and proving these theorems, this part offers some basic and preliminary concepts needed to prove the theorem on the existence of a unique solution to CFPIDEs. The main objective of this section is to provide some needed definitions and a theorem, which includes the Banach fixed point theorem that will be used in this article. We will now begin with the fundamental ideas behind this work, skipping over some of the more basic ideas from undergraduate research.

Definition 1.[9] *Let $(X, \|\cdot\|)$ be a normed linear space and $A : X \to X$ be a mapping on X. A point $x \in X$ is said to be a fixed point of A if $Ax = x$.*

Definition 2.[9] *On a normed linear space $(X, \|\cdot\|)$, a mapping $A : X \to X$ is said to be contraction map with contractivity factor c, if there is a nonnegative real number $c \in (0,1)$, satisfying:*

$$\|Ax - Ay\| \le c\|x - y\|, for all\ x and\ y \in X.$$

The “Banach fixed point theorem”, which will be outlined below, is the theorem that will be utilized in the next sections.

Theorem 1.[10] *Let $(X, \|\cdot\|)$ be a complete normed linear space and suppose that a mapping $A : X \to X$ is a contraction, then A posses a unique fixed point.*

An additional definition that is also necessary in the present work is the definition of Lipschitzian function.

Definition 3.[11] *Let $f : \Omega \to \mathbb{R}^n$ be a continuously differentiable function over a subset Ω of $\mathbb{R}^n$, then f is said to satisfy Liptischitz condition if there exists a constant $L > 0$ (dependent on both the function and the interval), such that:*

$$\|f(a) - f(b)\| \le L\|a - b\|, for all\ a,b \in \Omega.$$

*Remark.*The space $C_x^{m-1}([a,b]\times[c,d])$ will be used to denote the Banach space of all real valued functions u defined on $[a,b]\times[c,d]$, which are continuous with continuously m^{th} order partial derivatives with respect to x.

3 Conformable Fractional Calculus

3.1 Definition of Conformable Fractional Derivative [12]

Let $a \geq 0$ and $\alpha \in (0,1)$ and if a function $f : [a,b] \to \mathbb{R}$ is given, then the conformable fractional derivative of f of order α is defined by:

$$D_x^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}, \tag{4}$$

for $x > a$. If the function f is differentiable of order α over the open interval $(0,a)$, $a > 0$ and $\lim_{x\to 0^+} f^{\alpha}(x)$ exists. Then, $D_x^{\alpha} f(0) = \lim_{x\to 0^+} f(x)$.

3.2 Properties of Conformable Fractional Derivatives [12]

Suppose that $\alpha \in (0,1]$ and the functions f,g have conformable fractional derivatives of order α at a point $x > 0$, which belong to the domain of f and g. Then:

1.$D_x^{\alpha}(c) = 0$, $c \in \mathbb{R}$.
2.$D_x^{\alpha}(c_1 f + c_2 g)(x) = c_1 D_x^{\alpha} f(x) + c_2 D_x^{\alpha} g(x)$, $c_1, c_2 \in \mathbb{R}$.
3.$D_x^{\alpha}(fg)(x) = g(x) D_x^{\alpha} f(x) + f(x) D_x^{\alpha} g(x)$.
4.$D_x^{\alpha}\left(\frac{f}{g}\right)(x) = \frac{g(x) D_x^{\alpha} f(x) - f(x) D_x^{\alpha} g(x)}{[g(x)]^2}$.
5.$D_x^{\alpha}(f \circ g)(x) = f'(g(x)) D_x^{\alpha} g(x)$ (Chain Rule).
6.If, Moreover, if f is a differentiable function of the first order, then $D_x^{\alpha} f(x) = x^{1-\alpha}\frac{df(x)}{dx}$.

It is necessary to note that the above conformable fractional order derivative and its properties can be generalized for and order $\alpha > 1$. Moreover, conformable fractional derivative for a function of several variables are defined similarly as it is given for ordinary conformable fractional derivative (see equation (4)), as in the next definition:

Definition 4.*Let f be a function with m variables $x_1, x_2, \ldots, x_m$, then the conformable partial derivative of f of order $\alpha \in (0,1)$ with respect to x_i, for some $i \in 1,2,\ldots,m$ is given by:*

$$D_{x_i}^{\alpha} f(x_1, x_2, \ldots, x_m) = \lim_{\varepsilon \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i e^{\varepsilon x_i^{-a}}, x_{i+1}, \ldots, x_m) - f(x_1, x_2, \ldots, x_m)}{\varepsilon}, \tag{5}$$

and using similar property of (6) given above, getting:

$$D_{x_i}^{\alpha} f(x_1, x_2, \ldots, x_m) = x_i^{1-\alpha} \frac{\partial}{\partial x_i} f(x_1, x_2, \ldots, x_m)$$

as well as the other properties (1-5) are also satisfied and generalized for fractional conformable partial derivatives.

3.3 Definition of Conformal Fractional Integral [12]

Let $a \geq 0$, and $\alpha \in (0,1)$ and if $f : [a,b] \to \mathbb{R}$ is a given continuous function, then the conformable fractional integral of f of order α is given by:

$$I_x^{\alpha} f(x) = \int_a^x \frac{f(t)}{(t-a)^{1-\alpha}} dt.$$

whenever the last improper "Riemann" integral exists.

3.4 Properties of Conformable Fractional Integrals [[12]]

If $\alpha \in (0,1)$ and $f,g : [0,b] \to \mathbb{R}$ be continuous functions. Then:

1.$I_x^\alpha \lambda = \lambda (x-a)^\alpha$.
2.$I_x^\alpha (f+g)(x) = I_x^\alpha f(x) + I_x^\alpha g(x)$.
3.$I_x^\alpha (\lambda f)(x) = \lambda I_x^\alpha f(x)$.
4.$I_x^\alpha f(x) = \int_a^x \frac{f(t)}{(t-a)^{1-\alpha}} dt = -\int_x^a \frac{f(t)}{(t-a)^{1-\alpha}} dt$.
5.$I_\alpha^x f(x) = I_a^\alpha c - \int_b^x \frac{f(t)}{(t-b)^{1-\alpha}} dt$.
6.$I_x^\alpha f(a) = 0$.

Also, as in the conformable fractional derivatives, the conformable fractional integrals an its properties can be generalized for and order $\alpha > 1$.

4 Existence and Uniqueness of CFPIDEs Via Problem 1

We will use in this section the fixed point principle, which has its basis on the "Banach fixed point theorem" (or the "contraction mapping theorem") to establish and prove the existence and uniquenesstheorem of problem 1, which is given by equation (1) as:

$$D_x^\alpha u(x,y) = g(x,y) + \int_a^x \int_c^y k(x,y,\xi,\eta,u(\xi,\eta))\, d\eta d\xi,\ x \in [a,b],\ y \in [c,d]$$

where k is the "kernel function", g is any given continuous function and u is the unknown real valued function that need to be determined when solving the considered problem, a,b,c and d are real numbers, $m-1 < \alpha \leq m,\ m \in \mathbb{N}$.

Theorem 2.*Consider the multi-dimensional integro-differential equation given by equation (1) over $x \in [a,b],\ y \in [c,d]$ and when that the kernel k is Lipschitzian with respect to u and having a Lipschitz constant L, such that $L < \dfrac{\alpha+1}{(b-a)^{\alpha+1}(d-c)}$. Then equation (1) has a unique solution.*

*Proof.*Apply $I_x^\alpha,\ \alpha > 0$ to the both sides of the CFPIDE (1), which will implies to:

$$I_x^\alpha D_x^\alpha u(x,y) = I_x^\alpha g(x,y) + I_x^\alpha \int_a^x \int_c^y k(x,y,\xi,\eta,u(\xi,\eta))\, d\eta d\xi \tag{6}$$

and hence:

$$\begin{aligned} u(x,y) &= u_0(x,y) + I_x^\alpha g(x,y) + \int_a^x \int_a^\zeta \int_c^y (\zeta-a)^{\alpha-1} k(x,y,\xi,\eta,u(\xi,\eta))\, d\eta d\xi d\zeta \\ &= Au(x,y) \end{aligned} \tag{7}$$

where A is an integral operator defined by:

$$Au(x,y) = u_0(x,y) + I_x^\alpha g(x,y) + \int_a^x \int_a^\zeta \int_c^y (\zeta-a)^{\alpha-1} k(x,y,\xi,\eta,u(\xi,\eta))\, d\eta d\xi d\zeta$$

Take $u_1,u_2 \in C_x^{m-1}([a,b]\times[c,d])$, then to prove that A is a contraction, we have:

$$\begin{aligned} \|Au_1(x,y) - Au_2(x,y)\| &= \Bigg\| u_0(x,y) + I_x^\alpha g(x,y) + \int_a^x \int_a^\zeta \int_c^y (\zeta-a)^{\alpha-1} k(x,y,\xi,\eta,u_1(\xi, \\ &\eta))\, d\eta d\xi d\zeta - u_0(x,y) - I_x^\alpha g(x,y) - \int_a^x \int_a^\zeta \int_c^y (\zeta-a)^{\alpha-1} k(x,y,\xi,\eta,u_2(\xi,\eta))\, d\eta d\xi d\zeta \Bigg\| \\ &= \left\| \int_a^x \int_a^\zeta \int_c^y (\zeta-a)^{\alpha-1} \left[k(x,y,\xi,\eta,u_1(\xi,\eta)) - k(x,y,\xi,\eta,u_2(\xi,\eta))\right] d\eta d\xi d\zeta \right\| \\ &\leq \int_a^x \int_a^\zeta \int_c^y (\zeta-a)^{\alpha-1} \left\| k(x,y,\xi,\eta,u_1(\xi,\eta)) - k(x,y,\xi,\eta,u_2(\xi,\eta)) \right\| d\eta d\xi d\zeta \end{aligned} \tag{8}$$

$$\leq L\int_a^x\int_a^\zeta\int_c^y(\zeta-a)^{\alpha-1}\|u_1(\xi,\eta)-u_2(\xi,\eta)\|\,d\eta d\xi d\zeta$$

$$\leq L\|u_1-u_2\|\int_a^x\int_a^\zeta\int_c^y(\zeta-a)^{\alpha-1}\,d\eta d\xi dz$$

$$\leq L\|u_1-u_2\|\frac{(x-a)^{\alpha+1}(y-c)}{\alpha+1}$$

Hence:

$$\|Au_1-Au_2\|\leq L\|u_1-u_2\|\frac{(x-a)^{\alpha+1}(y-c)}{\alpha+1},\ x\in[a,b],\ y\in[c,d] \tag{9}$$

Now, upon imposing the supremum value over between brackets x and y, implies:

$$\|Au_1-Au_2\|\leq\frac{L(b-a)^{\alpha+1}(d-c)}{\alpha+1}\|u_1-u_2\|$$

Since $\frac{L(b-a)^{\alpha+1}(d-c)}{\alpha+1}<1$, because $L<\frac{\alpha+1}{(b-a)^{\alpha+1}(d-c)}$

Therefore, as a result, A is a contractive operator, and so, according to the "Banach fixed point theorem", a unique fixed point of A exists, implying that the integro-differential equation (1) has a unique solution.

*Example 1.*Consider the nonlinear integro-differential equation of fractional order:

$$T_x^\alpha u(x,y)=g(x,y)+\int_a^x\int_c^y ste^{u(s,t)}\,dtds$$

where $g(x,y)=2y^2x^{2-\alpha}-\frac{x^2y^2}{4}-\frac{x^4y^4}{16}$ and the exact solution $u(x,y)=x^2y^2$.

For simplicity of calculations using the VIM, as well as, using the properties of conformable differentiation and integration, replace the exponential function by its first degree Taylor series expansion:

$$e^u=1+u$$

The initial approximate solution which will be used in the VIM is assumed to be:

$$u_0(x,y)=2y^2x^{2-\alpha}-\frac{x^2y^2}{4}-\frac{x^4y^4}{16}$$

Now, using the VIM the first few iterations are found to be:

$$u_1(x,y)=0.8x^{2.5}y^{1.5}-0.0000255311x^{8.5}y^8-0.0173611x^{4.5}y^4+0.031746x^{4.5}y^{3.5}-$$

$$0.00240385x^{6.5}y^6-5.55519e-9e^{-9}x^6y^6$$

$$u_2(x,y)=0.8x^{2.5}y^{1.5}-4.7059e^{-10}x^{8.5}y^8-0.013889x^{4.5}y^4-6.6667e^{-7}x^{4.5}y^{3.5}-$$

$$0.1x^{2.5}y^2-0.125x^4+0.01282x^{6.5}y^2-0.0021368x^{6.5}y^6+0.25x^2y^2-3.638e^{-12}x^6y^6-$$

$$0.000063593x^7y^6-0.0000039278x^9y^8-2.2105e^{-8}x^{11}y^{10}+0.010159x^5y^{3.5}+0.00012686x^7y^{5.5}$$

$$u_3(x,y)=1.6x^{2.5}y^{1.5}+2.8868e^{-15}x^{8.5}y^8-0.010417x^{4.5}y^4-1.4552e^{-11}x^{4.5}y^{3.5}-$$

$$9.2973e-8e^{-8}x^{9.5}y^8+1.9783e^{-7}x^{9.5}y^{7.5}-x^{2.5}y^2-0.125x^4-0.8x^{2.5}y^{1.5}-$$

$$3.105e-9x^{11.5}y^{10}+0.011218x^{6.5}y^2-0.0021367e^{-9}x^{6.5}y^6-1.0496e^{-11}x^{13.5}y^{12}+$$

$$0.000035183x^{7.5}y^{5.5}+0.25x^2y^2-3.638e-12x^6y^6-0.000063593x^7y^6+0.0000052372x^9y^4-$$

$$0.0000034915x^9y^8+0.010159x^5y^{3.5}-5.7142e^{-9}x^7y^{5.5}$$

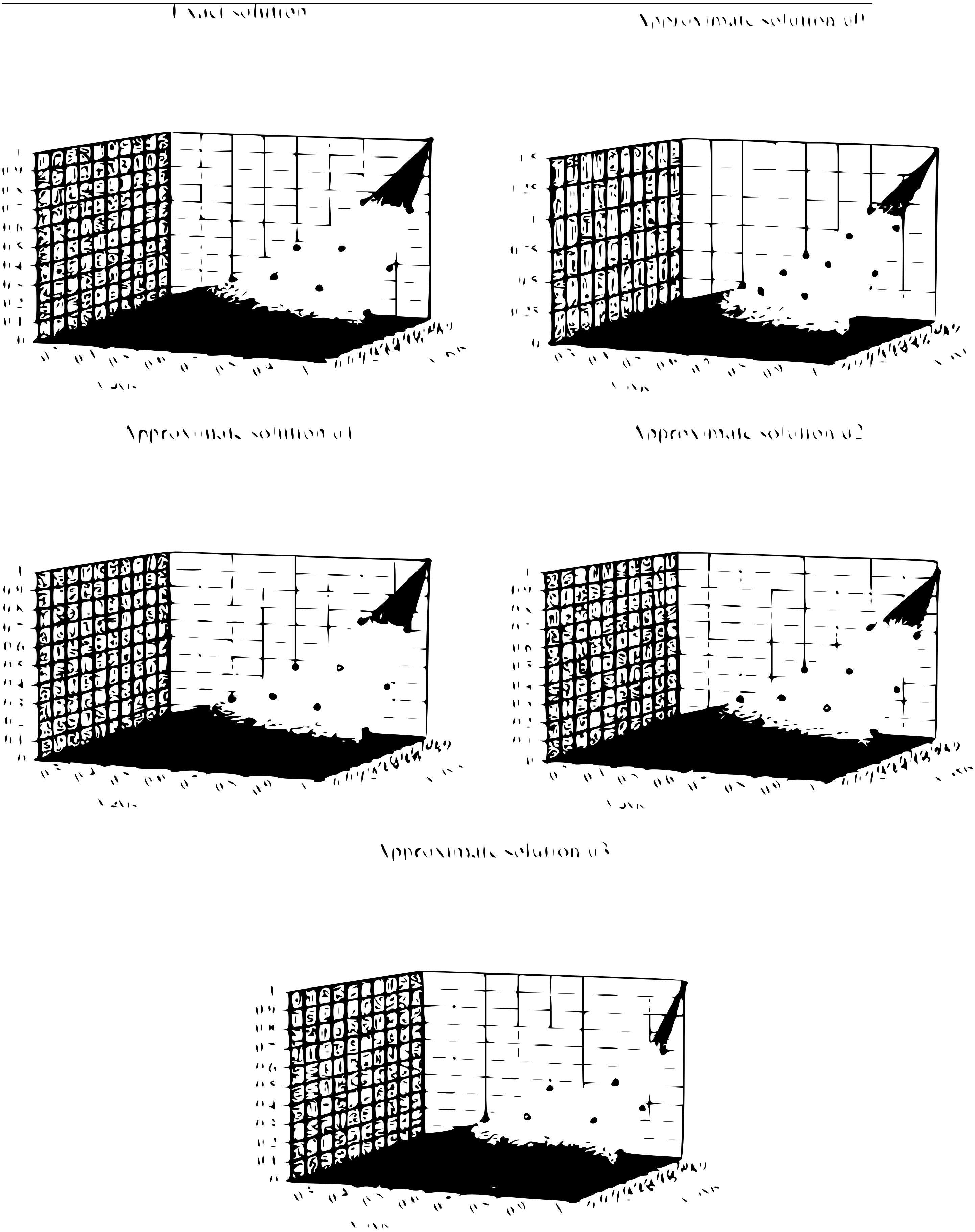

Fig. 1: Exact and approximate solutions of Example 1.

5 Existence and Uniqueness Theorem of Problem 2

In the present section, the existence and uniqueness theorem of equation (2) will be stated and proved as a generalization of the statement and the proof presented in Section 4 above. The CFPIDE of problem 2 is given by:

$$\frac{\partial u(x,y)}{\partial x} = g(x,y) + I_\xi^\beta I_\eta^\gamma k(x,y,\xi,\eta,u(\xi,\eta)),\ x\in[a,b],\ y\in[c,d]$$

where k is the "kernel function", g is any given continuous function and u is the unknown real valued function that need to be determined when solving the considered problem, a,b,c and d are real numbers, $m-1<\beta,\gamma\le m$, $m\in\mathbb{N}$ and I_ξ^β, I_η^γ represents the conformable fractional integral operators of order β and γ, respectively.

Theorem 3.*Consider the multidimensional integro-differential equation of fractional order given by equation (2) and when the kernel k is Lipschitzian with respect to u and having Lipschitz constant L, so that $L<\frac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^\gamma}$, then equation (2) has a unique solution.*

*Proof.*Integrating both sides of the CFPIDE (2), getting:

$$u(x,y) = u_0(x,y) + \int_a^x g(\zeta,y)\,d\zeta + \int_a^x I_\xi^\beta I_\eta^\gamma k(\zeta,y,\xi,\eta,u(\xi,\eta))\,d\eta\,d\zeta$$

which is expanded to:

$$u(x,y) = u_0(x,y) + \int_a^x g(\zeta,y)\,d\zeta + \int_a^x\int_a^\zeta (\xi-a)^{\beta-1}\int_c^y (\eta-c)^{\gamma-1} k(\zeta,y,\xi,\eta,u(\xi,\eta))\,d\eta\,d\xi\,d\zeta \tag{10}$$

Assume the integral operator

$$Au(x,y) = u_0(x,y) + \int_a^x g(\zeta,y)\,d\zeta + \int_a^x\int_a^\zeta (\xi-a)^{\beta-1}\int_c^y (\eta-c)^{\gamma-1} k(\zeta,y,\xi,\eta,u(\xi,\eta))\,d\eta\,d\xi\,d\zeta$$

then the integral equation (10) in operator form $u(x,y)=Au(x,y)$ will have a unique solution if the integral operator A is a contraction.
Let $u_1,u_2\in C_x^{m-1}([a,b]\times[c,d])$, then:

$$\begin{aligned}\|Au_1(x,y)-Au_2(x,y)\| &= \|u_0(x,y)+\int_a^x g(\zeta,y)d\zeta+\int_a^x\int_a^\zeta(\xi-a)^{\beta-1}\\ &\int_c^y(\eta-c)^{\gamma-1}k(\zeta,y,\xi,\eta,u_1(\xi,\eta))\,d\eta\,d\xi\,d\zeta-u_0(x,y)-\int_a^x g(\zeta,y)d\zeta-\\ &\int_a^x\int_a^\zeta(\xi-a)^{\beta-1}\int_c^y(\eta-c)^{\gamma-1}k(\zeta,y,\xi,\eta,u_2(\xi,\eta))\,d\eta\,d\xi\,d\zeta\|\\ &= \left\|\int_a^x\int_a^\zeta\int_c^y(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}\left[k(\zeta,y,\xi,\eta,u_1(\xi,\eta))-k(\zeta,y,\xi,\eta,u_2(\xi,\eta))\right]d\eta\,d\xi\,d\zeta\right\|\\ &\le L\|u_1(x,y)-u_2(x,y)\|\int_a^x\int_a^\zeta\int_c^y(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}\,d\eta\,d\xi\,d\zeta\end{aligned} \tag{11}$$

Then after carrying some calculations over inequality (3.2), we get:

$$\|Au_1-Au_2\| \le \frac{L(x-a)^{\beta+1}(y-c)^\gamma}{\beta(\beta+1)\gamma}\|u_1-u_2\| \tag{12}$$

and when taking the supremum values of x and y on the brackets, implies to:

$$\|Au_1(x,y)-Au_2(x,y)\| \le \frac{L(b-a)^{\beta+1}(d-c)^\gamma}{\beta(\beta+1)\gamma}\|u_1-u_2\| \tag{13}$$

Since $\frac{L(b-a)^{\beta+1}(d-c)^\gamma}{\beta(\beta+1)\gamma}<1$, because $L<\frac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^\gamma}$.
Thus, the contractivity of the operator A is assured with a fixed point in which it is unique according to the "Banach fixed point theorem", which is exactly the same as the CFPIDE (2) posses a solution, which is unique.

Example 2. Consider the nonlinear integro-differential equation of fractional order:

$$\frac{\partial}{\partial x}u(x,y) = g(x,y) + I_x^{\beta} I_y^{\gamma}\left[(xy)e^{u(x,y)}\right]$$

where $g(x,y) = 3xy^2 - \frac{x^{\beta+1}}{\beta+}\frac{y^{\gamma+1}}{\gamma+1} + \frac{x^{\beta+3}}{\beta+3}\frac{y^{\gamma+3}}{\gamma+3}$ and the exact solution is given by $u(x,y) = x^2y^2$.

In order to apply the VIM and simplify the computations, we use the first degree approximation of e^u. So starting with initial gauss approximate solution $u_0(x,y) = g(x,y)$, then the first three-approximate solutions are given by:

$$u_1(x,y) = u_0(x,y) - \int_a^x \left\{\frac{\partial u_0}{\partial w}(w,y) - g(w,y) - I_w^{\beta} I_y^{\gamma}\left[\left[(xy)e^{u_0(w,y)}\right]\right]\right\} dw$$

$$u_1(x,y) = -9.09495e^{-9}x^{5.5}y^{5.7} + 1.52411e^{-7}x^{2.5}y^{1.75} - 0.0169312x^{4.5}y^{3.75} -$$
$$0.00243235x^{6.5}y^{5.75} - 0.00046176x^6y^{5.5} + 0.0609524x^{3.5}y^{3.75} -$$
$$0.00907027x^4y^{3.5} - 0.0000376436x^8y^{7.5} + 1.0x^2y^2$$

$$u_2(x,y) = -4.0798e^{-9}x^{10.5}y^{9.25} - 7.276e^{-12}x^{5.5}y^{5.75} - 9.9907e^{-7}x^{8.5}y^{7.25} -$$
$$0.000048326x^{6.5}y^{5.25} + 1.5241e^{-7}x^{2.5}y^{1.75} - 1.1111e^{-7}x^{4.5}y^{3.75} -$$
$$0.0024324x^{6.5}y^{5.75} + 0.00036941x^6y^{5.5} - 0.000073294x^7y^{5.5} + 9.2857e^{-7}x^{3.5}y^{3.7} -$$
$$2.5e^{-8}x^4y^{3.5} + 2.1772e^{-9}x^5y^{3.5} + 1.25e^{-10}x^8y^{7.5} - 0.0000045044x^9y^{7.5} + 1.0x^2y^2$$

$$u_3(x,y) = -4.0328e^{-9}x^{11.5}y^{9.25} + 7.276e^{-12}x^{10.5}y^{5.75} - 7.276e^{-12}x^{5.5}y^{5.75} +$$
$$7.9928e^{-7}x^{8.5}y^{7.25} - 1.252e^{-7}x^{9.5}y^{7.25} + 1.5241e^{-7}x^{2.5}y^{1.75} -$$
$$5.2969e^{-9}x^{4.5}y^{3.75} - 0.0024323x^{6.5}y^{5.75} + 8.5068e^{-12}x^{7.5}y^{5.25} +$$
$$6.6667e^{-9}x^6y^{5.5} - 1.4286e^{-9}x^7y^{5.5} + 5.8208e^{-11}x^{3.5}y^{3.75} - 3.638e^{-12}x^4y^{3.5} +$$
$$2.1773e^{-9}x^5y^{3.5} - 0.0000045045e^{-9}x^9y^{7.5} + 1.0x^2y^2 - 9.5884e^{-8}x^9y^7 -$$
$$1.0092e^{-9}x^{11}y^9 - 2.3775e^{-11}x^{13}y^{11}$$

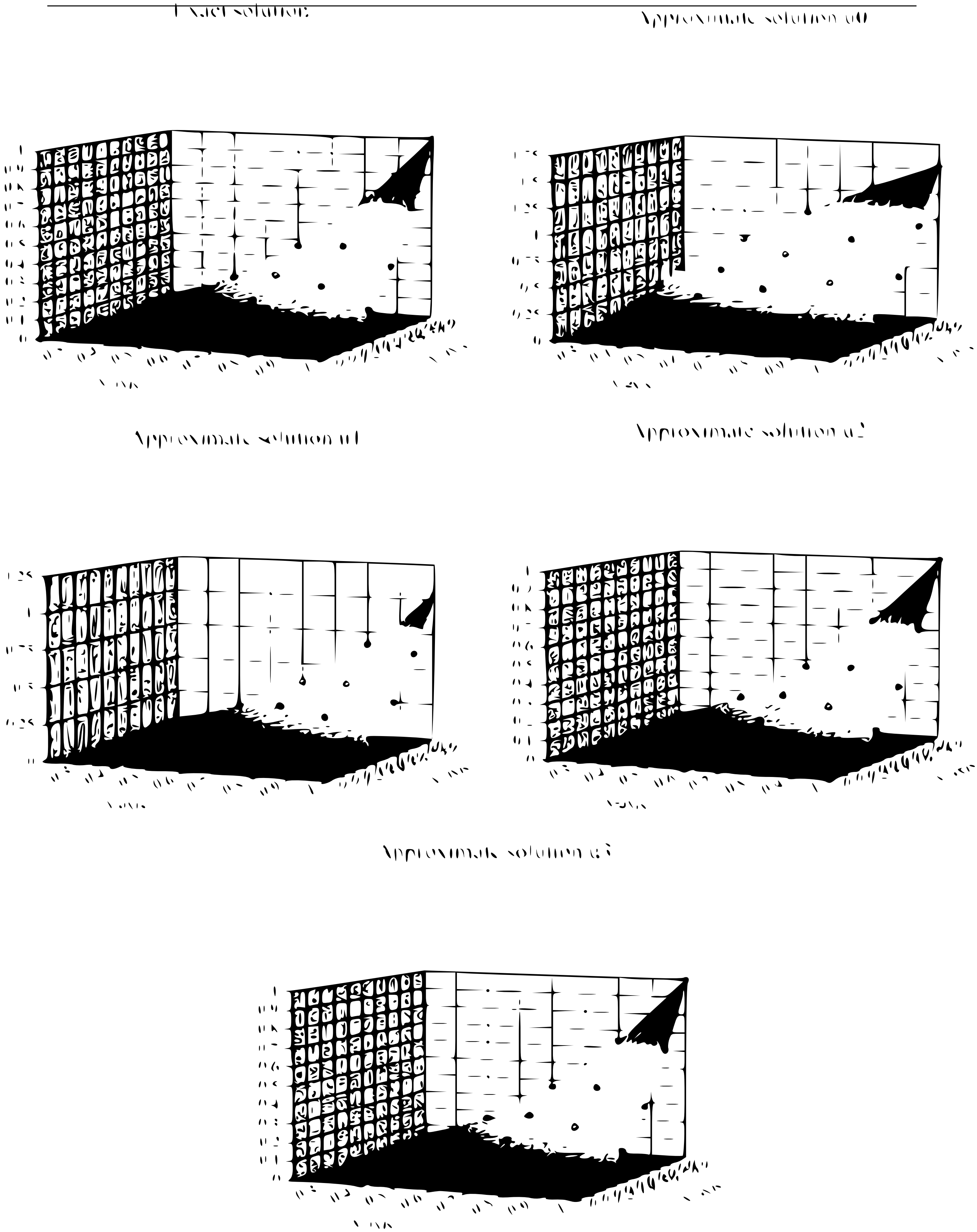

Fig. 2: Exact and approximate solutions of Example 2.

6 Existence and Uniqueness Theorem of Problem 3

It is important to note that, and as the main result of this paper, equation (3) is generalization equation of these equations (1) and (2) corresponding to problems 1 and 2, respectively and hence the existence and uniqueness of a unique solution for the most generalized case of CFIPDE (3) given by:

$$D_x^{\alpha}u(x,y)=g(x,y)+I_{\xi}^{\beta}I_{\eta}^{\gamma}k(x,y,\xi,\eta,u(\xi,\eta)),\ \alpha,\beta,\gamma\in\mathbb{R}^+,x\in[a,b],y\in[c,d]$$

where k is the "kernel function", g is a given function and u is an unknown real valued function that must be determined as the the unique solution of equation (3), $\beta,\gamma<\alpha$, $m-1<\alpha\leq m$, $m\in\mathbb{N}$. D_x^{α} denote the conformable fractional order derivative and I_{ξ}^{β}, I_{η}^{γ} denotes the conformable fractional order integral operators.

Theorem 4.*Consider the multidimensional integro-differential equation of fractional order given by equation (3) and supposing that the kernel k is Lipschitzian with respect to u and having the Lipschitz constant L, so that* $L<\dfrac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^{\gamma}}$, *then equation (3) has a unique solution.*

*Proof.*Applying the conformable integration I_x^{α}, $\alpha>0$ to the both sides of the CFPIDE (3), getting:

$$I_x^{\alpha}D_x^{\alpha}u(x,y)=I_x^{\alpha}g(x,y)+I_x^{\alpha}I_{\xi}^{\beta}I_{\eta}^{\gamma}k(x,y,\xi,\eta,u(\xi,\eta))$$

and so:

$$u(x,y)=u_0(x,y)+I_x^{\alpha}g(x,y)+I_x^{\alpha}I_{\xi}^{\beta}I_{\eta}^{\gamma}k(x,y,\xi,\eta,u(\xi,\eta))$$

$$u(x,y)=u_0(x,y)+I_x^{\alpha}g(x,y)+\int_a^x\int_a^{\zeta}\int_c^y(\zeta-a)^{\alpha-1}(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}k(\zeta,y,\xi,\eta,u(\xi,\eta))\,d\eta d\xi d\zeta \tag{14}$$

Now, assume the integral operator A related to (14) as:

$$Au(x,y)=u_0(x,y)+I_x^{\alpha}g(x,y)+\int_a^x\int_a^{\zeta}\int_c^y(\zeta-a)^{\alpha-1}(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}k(\zeta,y,\xi,\eta,u(\xi,\eta))\,d\eta d\xi d\zeta$$

and taking $u_1,u_2\in C_x^{m-1}([a,b]\times[c,d])$, then after carrying out some simplifications:

$$\|Au_1(x,y)-Au_2(x,y)\|=\|\int_a^x\int_a^{\zeta}\int_c^y(\zeta-a)^{\alpha-1}(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}\{k(\zeta,y,\xi,\eta,u_1(\xi,\eta))-\quad k(\zeta,y,\xi,\eta,u_2(\xi,\eta))\}\,d\eta d\xi d\zeta\|$$

$$\begin{aligned}\|Au_1(x,y)-Au_2(x,y)\|&=\|\int_a^x\int_a^{\zeta}\int_c^y(\zeta-a)^{\alpha-1}(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}\{k(\zeta,y,\xi,t,u_1(\xi,\eta))-\\&\quad k(\zeta,y,\xi,\eta,u_2(\xi,\eta))\}d\eta d\xi d\zeta\|\\&\leq\int_a^x\int_a^{\zeta}\int_c^y(\zeta-a)^{\alpha-1}(\xi-a)^{\beta-1}(\eta-c)^{\gamma-1}\|\{k(\zeta,y,\xi,\eta,u_1(\xi,\eta))-\\&\quad k(\zeta,y,\xi,\eta,u_2(\xi,\eta))\}\|\,d\eta d\xi d\zeta\\&\leq\frac{L(x-a)^{\alpha+\beta}(y-c)^{\gamma}}{\beta(\alpha+\beta)\gamma}\|u_1-u_2\|\\&\leq\frac{L(b-a)^{\alpha+\beta}(d-c)^{\gamma}}{\beta(\alpha+\beta)\gamma}\|u_1-u_2\|\end{aligned}$$

Since $\dfrac{L(b-a)^{\beta+1}(d-c)^{\gamma}}{\beta(\beta+1)\gamma}<1$, because $L<\dfrac{\beta(\beta+1)\gamma}{(b-a)^{\beta+1}(d-c)^{\gamma}}$
Then A is a contraction mapping and by "Banach fixed point theorem" it has a unique fixed point, which is equivalent to that the CFPIDE (3) has a unique solution.

*Example 3.*Consider the nonlinear integro-differential equation:

$$T_x^{\alpha}u(x,y)=g(x,y)+I_x^{\beta}I_y^{\gamma}\left[(xy)e^{u(x,y)}\right]$$

where $g(x,y) = 2y^2x^{2-\alpha} - \dfrac{x^{\beta+1}}{\beta+1}\dfrac{y^{\gamma+1}}{\gamma+1} + \dfrac{x^{\beta+3}}{\beta+3}\dfrac{y^{\gamma+3}}{\gamma+3}$ and the exact solution is given for comparison purpose $u(x,y) = x^2y^2$.

For simplicity of the calculations, use the first degree Manchurian series approximation of e^u, i.e., let $e^u = 1+u$ and starting with the initial guess solution $u_0(x,y) = g(x,y)$, then the first variational iteration approximate solution using the below equation is given by:

$$u_1(x,y) = u_0(x,y) - I_x^\alpha\left[T_x^\alpha u_0(x,y) - g(x,y) - I_x^\beta I_y^\gamma\left[xy(1+u_0(x,y))\right]\right]$$

$$= 2y^2x^{2-\alpha} - \frac{x^{\beta+1}}{\beta+1}\frac{y^{\gamma+1}}{\gamma+1} + \frac{x^{\beta+3}}{\beta+3}\frac{y^{\gamma+3}}{\gamma+3} -$$

$$I_y^\alpha\left[T_x^\alpha\left[2y^2x^{2-\alpha} - \frac{x^{\beta+1}}{\beta+1}\frac{y^{\gamma+1}}{\gamma+1} + \frac{x^{\beta+3}}{\beta+3}\frac{y^{\gamma+3}}{\gamma+3}\right] -\right.$$

$$\left[2y^2x^{2-\alpha} - \frac{x^{\beta+1}}{\beta+1}\frac{y^{\gamma+1}}{\gamma+1} + \frac{x^{\beta+3}}{\beta+3}\frac{y^{\gamma+3}}{\gamma+3}\right] -$$

$$\left. I_x^\beta I_y^\gamma\left[xy + 2y^3x^{3-\alpha} - \frac{x^{\beta+2}}{\beta+1}\frac{y^{\gamma+2}}{\gamma+1} + \frac{x^{\beta+4}}{\beta+3}\frac{y^{\gamma+4}}{\gamma+3}\right]\right]$$

and carrying out recursively order fractional order derivative and and integrals with orders $\alpha = 0.8$, $\beta = 0.5$ and $\gamma = 0.75$, we get the final from of $u_1(x,y)$ to be as follows:

$$u_1(x,y) = 1.81826\times10^{-8}x^{5.5}y^{5.75} - 0.000477683x^{5.8}y^{5.5} - 0.0177187x^{4.3}y^{3.75} + 1.0x^{2.0}y^2 -$$
$$0.00250957x^{6.3}y^{5.75} + 9.31323\times10^{-10}x^{1.2}y^2 + 0.0564374x^{3.5}y^{3.75} -$$
$$0.00954771x^{3.8}y^{3.5} - 0.0000386088x^{7.8}y^{7.5}$$

To find the second approximate solution $u_2(x,y)$, substitute$n = 1$, we get:

$$u_2(x,y) = 7.27596\times10^{-12}x^{5.5}y^{5.75} + 0.000353838x^{5.8}y^{5.5} - 0.00000111428x^{8.1}y^{7.25} -$$
$$0.0000562515x^{6.1}y^{5.25} - 4.65079\times10^{-8}x^{4.3}y^{3.75} - 0.00000498821x^{8.6}y^{7.5} + 1.0x^{2.0}y^2 -$$
$$0.00250957x^{6.3}y^{5.75} - 0.0000841583x^{6.6}y^{5.5} - 2.42514\times10^{-15}x^{1.2}y^2 +$$
$$5.82077\times10^{-11}x^{3.5}y^{3.75} - 4.94072\times10^{-8}x^{9.6}y^{9.25}$$

and similarly if $n = 2$, then:

$$u_3(x,y) = -4.89841\times10^{-9}x^{10.9}y^{9.25} - 7.20333\times10^{-12}x^{5.5}y^{5.75} + 2.27374\times10^{-13}$$
$$x^{5.8}y^{5.5} + 8.25388\times10^{-7}x^{8.1}y^{7.25} - 1.61021\times10^{-7}x^{8.9}y^{7.25} - 7.27596\times10^{-12}x^{4.3}$$
$$y^{3.75} - 0.00000498821x^{8.6}y^{7.5} + 1.0x^{2.0}y^2 - 0.00250957x^{6.3}y^{5.75} - 3.03032\times10^{-10}x^{6.6}y^{5.5} -$$
$$1.25876\times10^{-7}x^{8.4}y^7 - 1.7528\times10^{-21}x^{1.2}y^2 - 1.24008\times10^{-9}x^{10.4}y^9 -$$
$$3.40038\times10^{-11}x^{11.9}y^{11} + 1.77679\times10^{-14}x^{7.8}y^{7.5}$$

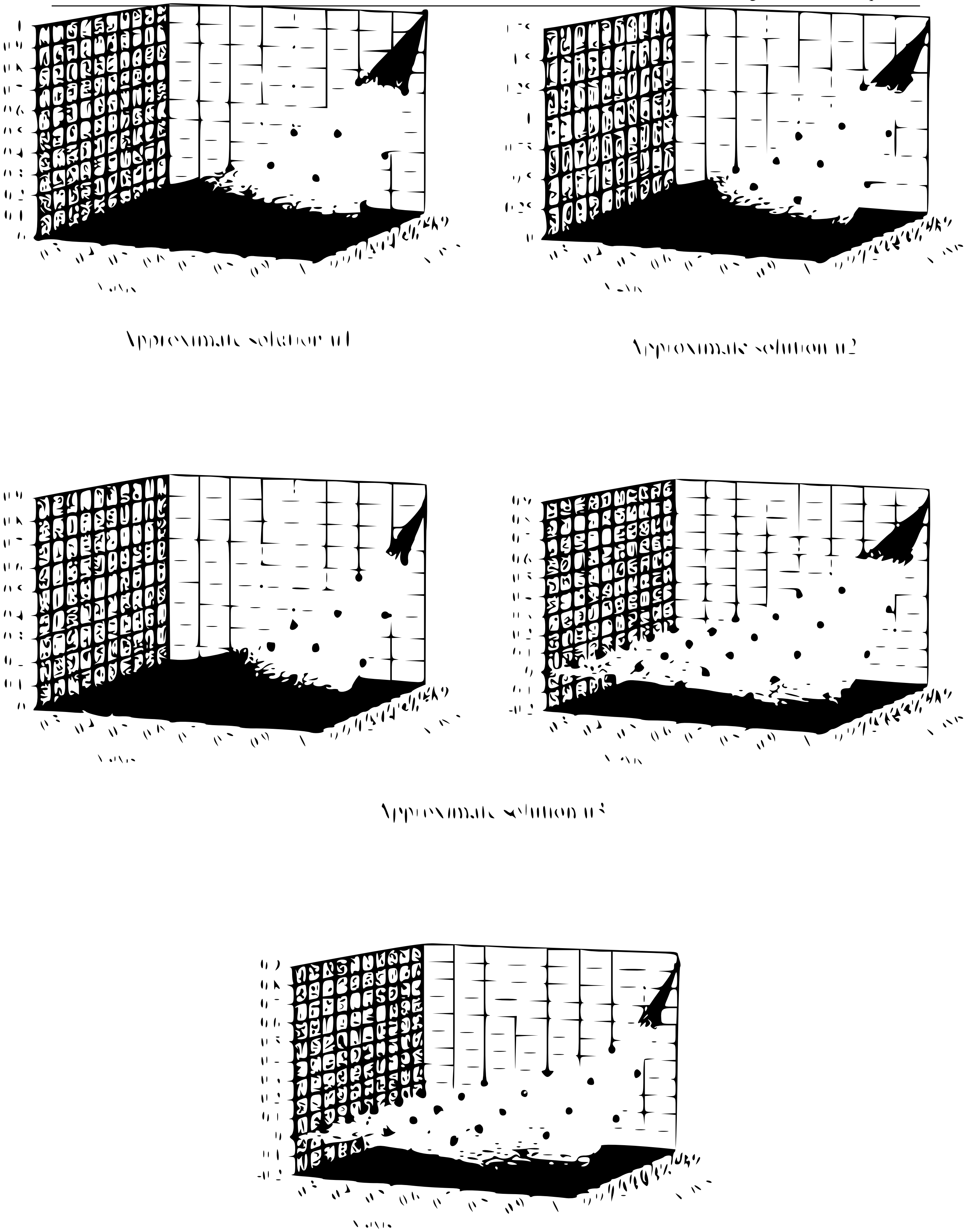

Fig. 3: Exact and approximate solutions of Example 3.

7 Concluding Remarks

Evaluation or approximating unique solutions of differential equations which are ordinary or partial with fractional order or integral or integro differential equations of fractional order are among the most intriguing areas of fractional calculus and relatively for fractional order ordinary and/or integral equations, which is related to the importance of understanding the physical and dynamical behavior of problems that arise in physics and engineering applications. With these considerations in mind, this article sheds light on the use of conformable fractional derivatives and integrals to theoretically investigate the existence and uniqueness of solutions for two-dimensional CFPIDEs by employing the "Banach fixed point theorem" in their proofs. In this article, the statement and proof of three Banach fixed point theorems are established in order to formulate and verify three types of problems CFPIDE's existence and uniqueness theorems, in which the third problem is considered as a generalization of the first and second problems by selecting appropriate values of α, β, and γ.

Declarations

Competing interests: The authors declare no competing interests.
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Availability of data and materials: The data and materials used in this study are available upon request from the corresponding author.
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