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Ricci bi-conformal vector fields on non-reductive four-dimensional homogeneous spaces

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Abstract: The goal of this paper is to find the Ricci bi-conformal left invariant vector fields on the non-reductive four-dimensional homogeneous spaces. At first, we introduce some necessary definations, then we calculate the Lie derivative of the metric and the Lie derivative of the Ricci tensor. We classify the Ricci bi-conformal vector fields on non-reductive four-dimensional homogeneous spaces. Finally, we show which of them are Killing vector fields.

Keywords: Bi-conformal vector field, Conformal vector field, Killing field, Non-reductive homogeneous spaces, Pseudo-Riemannian metrics, Ricci solitons.

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1 Introduction

Let G be a Lie group and H be a closed subgroup of G. The connected pseudo-Riemannian manifold (M,g) is named to be homogeneous, when the group of isometries of (M,g) acts transitively on M. In this case, assume G is a Lie transformation group of $\frac{G}{H}$ and consider the left coset space $\frac{G}{H}$ as a smooth manifold. So the subgroup H is the isotropy subgroup of Lie group G and g is an invariant pseudo-Riemannian metric.

If the homogeneous pseudo-Riemannian manifold (M,g) can be realized as a coset space $M=\frac{G}{H}$, it is reductive. Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebra of G and H and $\mathfrak{m}$ is $Ad(H)$ (that $Ad(H)$ refers to the adjoint representation of the Lie group H on its Lie algebra g) invariant the subspace of $\mathfrak{g}$. When there is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ and $Ad(h)(\mathfrak{m})\subset m$ for all $h\in H$, it is reductive. Except for some homogeneous pseudo-Riemannian manifolds that do not accept any kind of reductive decomposition, all homogeneous Riemannian manifolds are reductive. In [26], Fels and Renner classified these spaces, up to isometry classes, and showed that one of the eight classes, $A1,\ldots,A5,B1,B2$, and $B3$, that contain both neutral signature examples and Lorentzian is isometric to the four-dimensional non-reductive homogeneous pseudo-Riemannian manifold.

The Ricci soliton studied in Lorentzian manifolds was introduced by Hamilton [28]. One of the most important and attractive topics in physics and geometry is study of the Ricci solitons which are natural generalization of Einstein metrics. On a pseudo-Riemannian manifold (M,g), it is defined by

$$\mathscr{L}_X g+S=\lambda g.$$

In this equation we consider X as a smooth vector field on M, the Lie derivative of g in the direction of X is expressed by $\mathscr{L}_X g$, the Ricci tensor is shown by S and λ is a real number [11].

Wilhelm Kal Joseph Killing, a German mathematician, made important research on the theories of Lie algebras and non-Euclidean geometry [21]. So, a Killing vector field, named after Wilhelm Killing, is a vector field on a pseudo-Riemannian manifold that preserves the metric. Killing vector fields were considered in [19].

A vector field X on a Riemannian manifold (M,g) is said to be a Killing field if the Lie derivative of the metric g with respect to X vanishes, that is

$$\mathscr{L}_X g=0,$$

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or equivalently

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0,$$

for all vector fields X, Y and in local coordinates:

$$\nabla_i X_j + \nabla_j X_i = 0,$$

where ∇ is the Levi-civita connection.

A vector field X on a Riemannian manifold (M,g) is called conformal vector field if there is a smooth function f on M that named a potential function, such that $\mathscr{L}_X g = 2fg$. If the potential function $f = 0$, X is a Killing vector field. X is a gradient conformal vector field if $X = \nabla\rho$ for some smooth functions ρ on a Riemannian manifold (M,g). If $X = \nabla\rho$ then equations $\mathscr{L}_X g = 2fg$ reduces to $\nabla^2\rho = fg$ and this implies that $\triangle\rho = nf$. Thus there is a relation between Poisson equation on the Riemannian manifold and gradient conformal vector fields on it. Conformal vector fields have been intensely studied over the past 150 years, specially in Riemannian and pseudo-Riemaninan geometry of dimension $n \geq 3$. As well, this is a special case from a mathematical point of view because a conformal vector field, given at every point, always has a gradient within the light cone. The importance of conformal vector fields or insignificant conformal transformations in differential geometry has been classified by the work of Lie, Schouten,Yano, and others. Essential conformal vector fields in Riemannian spacec were studied by Obata, Lelony-Freund, and Alekseeskii. A conformal vector field was studied in [23] completely. According to [23], the geometry of conformal vector field is separated into two classes, the first is the geometry of gradient conformal vector fields, the second is the geometry of conformal vector fields that are not closed. So, there is a relation between conformal vector fields and gradient or the geometry of closed conformal vector fields. Riemannian manifolds having gradient or closed conformal vector fields, have been researched in [24] and [25]. In [15], Fino and Calvaruso have studied Killing vector fields on non-reductive four-dimensional homogeneous spaces.

Benroummane studied about semi-symmetric curvature algebraic tensors on the vector space with metric of signature $(2,n)$ where $n \geq 2$ and a categorization of four-dimensional simply-connected semi-symmetric homogeneous neutral manifolds that are not locally symmetric, [10].

Also, Batat and Onda, [9], studied algebraic on non-symmetric simply-connected four-dimensional pseudo-Riemannian. It turns out that those of Cerny-Kowalski's types A,C and D are algebraic Ricci solitons, where as those of type B are not, and they give new examples of algebraic Ricci solitons.

Garcia-Parrado and Senovilla [27] introduced bi-conformal vector fields, then De et al. [22] defined Ricci bi-conformal vector fields. A Ricci bi-conformal vector field is a vector field X on a Riemannian manifold (M,g) if the following equations hold for some non-zero smooth functions α and β and any vector fields Y,Z:

$$(\mathscr{L}_X g)(Y,Z) = \alpha g(Y,Z) + \beta S(Y,Z), \tag{1}$$

and

$$(\mathscr{L}_X S)(Y,Z) = \alpha S(Y,Z) + \beta g(Y,Z), \tag{2}$$

where S is the Ricci tensor of M and $\mathscr{L}_X$ is the Lie derivation in the direction of X.

In [4], [5] and [6] have been studied Ricci bi-conformal vector fields on Lorentzian five-dimensional two-step nilpotent Lie groups, Siklos spacetimes, and homogeneous Gödel-type spacetimes, respectively. Motivated by [22], we study the Ricci bi-conformal vector fields on non-symmetric simply-connected four dimensional pseudo-Riemannian generalized symmetric spaces up to isometry.

Now in Section 2, the classification of non reductive four-dimensional homogeneous pseudo-Riemannian manifolds recalled, with the Ricci tensor, expressly explain the relation pseudo-Riemannian metrics and the Lie derivative of the metrics with reference to a vector field. In Section 3, we calculate the Lie derivative $\mathscr{L}_X g$ of the metric g with respect to the vector field $X = X_i u_i \in \mathfrak{m}$, and the Lie derivative $\mathscr{L}_X S$ of the Ricci tensor in direction X, then we show which of vectir fields is a Ricci bi-conformal vector field, and we discuss about Killing vector fields on them.

2 Non-reductive four-dimensional homogeneous spaces

The non-reductive four-dimensional homogeneous manifolds were categorized [26] corresponding to the similar non-reductive Lie algebras. We recall this categorization and, furthermore, we explicitly explain the corresponding pseudo-Riemannian metrics based on references [15], and [17] the according pseudo-Riemannian metrics, the Ricci tensor, and the Lie derivative $\mathscr{L}_X g$ of the metric tensor along a vector field $X = X_i u_i \in \mathfrak{m}$.

2.1 Lorentzian case

(A1) We know $\mathfrak{s}(2)$ can solve the 2-dimensional algebra, $\mathfrak{g}=\mathfrak{a}_1$ is the separated 5-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{s}(2)$. A basis of $\mathfrak{a}_1$ is $\{e_1,\cdots,e_5\}$, so that the non-vanishing brackets are:

$$[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_2,\ [e_2,e_3]=e_1,\ [e_4,e_5]=e_4,$$

and $\mathfrak{h}=\text{span}\{h_1=e_3+e_4\}$ is the isotropy subalgebra. Therefore, we suppose that

$$\mathfrak{m}=\text{span}\{u_1=e_1,u_2=e_2,u_3=e_5,u_4=e_3-e_4\}$$

and the isotropy description for h_1

$$H_1=\begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&0&0\\1&0&-\frac{1}{2}&0\end{pmatrix}.$$

The metric g with respect to the basis $\{u_i\}$ are obtained as follows

$$g=\begin{pmatrix}a&0&-\frac{a}{2}&0\\0&b&c&a\\-\frac{a}{2}&c&d&0\\0&a&0&0\end{pmatrix},\tag{3}$$

and they are non-degenerate whenever $a(a-4d)\neq 0$. As well, the Ricci tensor S with respect to the basis $\{u_i\}$ is described by

$$S=\begin{pmatrix}-2&0&1&0\\0&\frac{2b(a+12d)}{a(a-4d)}&-\frac{2c}{a}&-2\\1&-\frac{2c}{a}&-\frac{1}{2}&0\\0&-2&0&0\end{pmatrix}.\tag{4}$$

(A2) The one-parameter family of 5-dimensional Lie algebras $A_{5,30}$ is $\mathfrak{g}=\mathfrak{a}_2$. A basis of $\mathfrak{a}_2$ is $\{e_1,\cdots,e_5\}$ thus for any $e\in\mathbb{R}$ the non-vanishing brackets are as follows

$$\begin{aligned}&[e_1,e_5]=(e+1)e_1,\ [e_2,e_4]=e_1,\ [e_2,e_5]=ee_2,\\&[e_3,e_4]=e_2,\ [e_3,e_5]=(e-1)e_3,\ [e_4,e_5]=e_4,\end{aligned}$$

and the isotropy subalgebra is $\mathfrak{h}=\text{span}\{h_1=e_4\}$. Hence, we consider

$$\mathfrak{m}=\text{span}\{u_1=e_1,u_2=e_2,u_3=e_3,u_4=e_5\}$$

and the isotropy description for h_1 is as follows

$$H_1=\begin{pmatrix}0&-1&0&0\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Therefore, the invariant metrics are obtained as follows

$$g=\begin{pmatrix}0&0&-a&0\\0&a&0&0\\-a&0&b&c\\0&0&c&d\end{pmatrix}\tag{5}$$

as well as are non-degenerate whenever $ad\neq 0$. Furthermore, the Ricci tensor S is represented by

$$S=\begin{pmatrix}0&0&\frac{3e^2a}{d}&0\\0&-\frac{3e^2a}{d}&0&0\\\frac{3e^2a}{d}&0&-\frac{b(3e^2-3e+2)}{d}&-\frac{3e^2c}{d}\\0&0&-\frac{3e^2c}{d}&-3e^2\end{pmatrix}.\tag{6}$$

(A3) $\mathfrak{g} = \mathfrak{a}_3$ is the one of 5-dimensional Lie algebras $A_{5,36}$ or $A_{5,37}$ in [29]. A basis of $\mathfrak{a}_3$ is so the non-vanishing brackets are

$$[e_1,e_4] = 2e_1, \quad [e_2,e_3] = e_1, \ [e_2,e_4] = e_2,$$
$$[e_2,e_5] = -\varepsilon e_3, \ [e_3,e_4] = e_3, \ [e_3,e_5] = e_2,$$

we have $\varepsilon = 1$ for $A_{5,37}$ and we have $\varepsilon = -1$ for $A_{5,36}$ and the isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_3\}$. Thus, we consider

$$\mathfrak{m} = \text{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\},$$

and the isotropy description for h_1

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the invariant metrics are non-degenerate whenever $ab \neq 0$, calculated as follows

$$g = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix}, \tag{7}$$

The Ricci tensor S is computed by

$$S = \begin{pmatrix} 0 & 0 & 0 & -\frac{3a}{b} \\ 0 & -\frac{3a}{b} & 0 & 0 \\ 0 & 0 & -3 & -\frac{3c}{b} \\ -\frac{3a}{b} & 0 & -\frac{3c}{b} & \frac{\varepsilon b - 2d}{b} \end{pmatrix}. \tag{8}$$

(A4) The 6-dimensional Schrodinger Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$ is $\mathfrak{g} = \mathfrak{a}_4$, where $\mathfrak{n}(3)$ is the 3-dimensional Heisenberg algebra. A basis of $\mathfrak{a}_4$ is $\{e_1, \cdots, e_6\}$ so the non-vanishing brackets are

$$[e_1,e_2] = 2e_2, \ [e_1,e_3] = -2e_3, \ [e_2,e_3] = e_1, \ [e_1,e_4] = e_4,$$
$$[e_1,e_5] = -e_5, \quad [e_2,e_5] = e_4, \quad [e_3,e_4] = e_5, \ [e_4,e_5] = e_6,$$

and the isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_3 + e_6, h_2 = e_5\}$. Next, taking

$$\mathfrak{m} = \text{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 - e_6, u_4 = e_4\},$$

and the following isotropy description for h_1, h_2 are

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are obtained of the following form

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{2} \end{pmatrix}, \tag{9}$$

they are non-degenerate whenever $a \neq 0$. Hence, the Ricci tensor is calculated as follows

$$S = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -\frac{8b}{a} & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}. \tag{10}$$

(A5) The 7-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes A^1_{4,9}$ is $\mathfrak{g} = \mathfrak{a}_5$, with $A^1_{4,9}$ of [29]. A basis of $\mathfrak{a}_5$ is $\{e_1,\cdots,e_7\}$ so the non-vanishing brackets are

$[e_1,e_2] = 2e_2,\ [e_1,e_3] = -2e_3,\ [e_1,e_5] = -e_5,\ [e_1,e_6] = e_6,$
$[e_2,e_3] = e_1,\quad [e_2,e_5] = e_6,\quad [e_3,e_6] = e_5,\quad [e_4,e_7] = 2e_4,$
$[e_5,e_6] = e_4,\quad [e_5,e_7] = e_5,\quad [e_6,e_7] = e_6.$

The isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_1 + e_7, h_2 = e_3 - e_4, h_3 = e_5\}$. Hence, we take

$$\mathfrak{m} = \text{span}\{u_1 = e_1 - e_7,\ u_2 = e_2, u_3 = e_3 + e_4, u_4 = e_6\}.$$

For the isotropy description of h_1, h_2, and h_3, we get

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ H_2 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are figured as follows

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{4} & 0 \\ 0 & \frac{a}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{8} \end{pmatrix}, \tag{11}$$

and they are non-degenerate whenever $a \neq 0$. The Ricci tensor S with respect to $\{u_i\}$ is computed by

$$S = \begin{pmatrix} -12 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}. \tag{12}$$

2.2 Signature $(2,2)$ *case*

Besides cases $A1, A2, A3$, when we admit invariant metrics of neutral signature $(2,2)$, the remaining are the following.

(B1) The 5-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ is $\mathfrak{g} = \mathfrak{b}_1$. A basis of $\mathfrak{b}_1$ is so the non-vanishing brackets are

$[e_1,e_2] = 2e_2,\ [e_1,e_3] = -2e_3,\ [e_2,e_3] = e_1,\ [e_1,e_4] = e_4,$
$[e_1,e_5] = -e_5,\quad [e_2,e_5] = e_4,\quad [e_3,e_4] = e_5,$

and the isotropy subalgebra is $\mathfrak{h} = \text{span}\{h_1 = e_3\}$. Therefore, we assume

$$\mathfrak{m} = \text{span}\{u_1 = e_1,\ u_2 = e_2, u_3 = e_4, u_4 = e_5\},$$

and the following isotropy description for h_1

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The invariant metrics are computed as

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & c & a \\ a & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \tag{13}$$

so whenever $a \neq 0$ they are non-degenerate. The Ricci tensor is explained by

$$S = \begin{pmatrix} 0 & 0 & \frac{3d}{2a} & 0 \\ 0 & \frac{3(6bd-5c^2)}{2a^2} & \frac{3cd}{2a^2} & \frac{3d}{2a} \\ \frac{3d}{2a} & \frac{3cd}{2a^2} & \frac{3d^2}{2a^2} & 0 \\ 0 & \frac{3d}{2a} & 0 & 0 \end{pmatrix}. \tag{14}$$

(B2) The 6-dimensional Schrodinger Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$ is $\mathfrak{g} = \mathfrak{b}_2$ is with the isotropy subalgebra $\mathfrak{h} = \text{span}\{h_1 = e_3 - e_6, h_2 = e_5\}$. Therefore, we compute that

$$\mathfrak{m} = \text{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 + e_6, u_4 = e_4\},$$

and the following isotropy description for h_1, h_2 are

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are represented by

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{2} \end{pmatrix}, \tag{15}$$

so they are non-degenerate whenever $a \neq 0$. The Ricci tensor is described by

$$S = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -\frac{8b}{a} & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}. \tag{16}$$

(B3) The 6-dimensional Lie algebra $(\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}$ is $\mathfrak{g} = \mathfrak{b}_3$. A basis of $\mathfrak{b}_3$ is $\{e_1, \cdots, e_6\}$ so the non-vanishing brackets are

$$\begin{aligned} &[e_5, e_2] = e_1, \quad [e_5, e_3] = -e_4, \quad [e_6, e_2] = -2e_6, \\ &[e_6, e_3] = -e_2, \ [e_6, e_4] = e_1, \quad [e_1, e_2] = -e_1, \\ &[e_1, e_3] = e_4, \quad [e_2, e_3] = -2e_3, \ [e_2, e_4] = -e_4, \end{aligned}$$

and the isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_5, h_2 = e_6\}$. Thus, we consider

$$\mathfrak{m} = \text{span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_4\},$$

and the following isotropy description for h_1, h_2 are

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are described as follows

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \tag{17}$$

so they are non-degenerate whenever $a \neq 0$. The Ricci tensor is calculated by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

3 The main results and their proofs

In this section, we calculate the Lie derivative of the metric g and the Lie derivative of the Ricci tensor. Also, we give the classification of homogeneous generalized Ricci solitons of these spaces.

Case (A1)

We consider Lie algebra of type (**A1**). For any vector field $X = X_i u_i \in \mathfrak{m}$, where $X_i \in \mathbb{R}$ by $(\mathscr{L}_X g)(e_i,e_j) = g(\nabla_{e_i}X, e_j) + g(e_i, \nabla_{e_j}X)$ the Lie derivative of the metric g with respect to the vector field X, (see [30]), is given by

$$\mathscr{L}_X g = \begin{pmatrix} 0 & 2bX_2 & 2cX_2 & aX_2 \\ 2bX_2 & -4bX_1 & -2cX_1 - aX_4 & -aX_1 + \frac{1}{2}aX_3 \\ 2cX_2 & -2cX_1 - aX_4 & 0 & \frac{a}{2}X_2 \\ aX_2 & -aX_1 + \frac{a}{2}X_3 & \frac{a}{2}X_2 & 0 \end{pmatrix}. \tag{19}$$

Further, using the formula $(\mathscr{L}_X S)(e_i,e_j) = X(S(e_i,e_j)) - S(\mathscr{L}_X e_i, e_j) - S(e_i, \mathscr{L}_X e_j)$ the Lie derivative of the Ricci tensor in direction X, (see [30]), is determined by

$$\mathscr{L}_X S = \begin{pmatrix} 0 & \frac{4b(a+12d)}{a(a-4d)}X_1 - 2X_4 & 0 & \frac{4(a-c)-2a}{a}X_2 - 4(\frac{a-c}{a})X_4 \\ \frac{4b(a+12d)}{a(a-4d)}X_1 - 2X_4 & \frac{-8b(a+12d)}{a(a-4d)}X_1 & -2X_4 & S_{24} \\ 0 & -2X_4 & 0 & 0 \\ \frac{4(a-c)-2a}{a}X_2 - 4(\frac{a-c}{a})X_4 & S_{24} & 0 & 2X_1 + X_2 \end{pmatrix}, \tag{20}$$

where $S_{24} = (\frac{-4b(a+12d)+4(c+a)(a-4d)}{a(a-4d)})X_1 - 2X_3 + X_4$. Applying (3), (4), and (19) into (1) and (2), we obtain

$$\begin{cases} \frac{a}{2}X_2 = 0, \\ -\frac{a}{2}\alpha + \beta = 2cX_2, \\ a\alpha - 2\beta = 0, \\ b\alpha + \frac{2b(a+12d)}{a(a-4d)}\beta = -4bX_1, \\ c\alpha - \frac{2c}{a}\beta = -2cX_1 - aX_4, \\ a\alpha - 2\beta = -aX_1 + \frac{a}{2}X_3, \\ d\alpha - \frac{1}{2}\beta = 0. \end{cases} \tag{21}$$

Also, substituting (3), (4), and (20) into (1) and (2), we get

$$\begin{cases} \frac{2b(a+12d)}{a(a-4d)}\alpha + b\beta = \frac{-8b(a+12d)}{a(a-4d)}X_1, \\ -\frac{1}{2}\alpha + d\beta = 0, \\ -8(\frac{a-c}{a})X_1 + 2X_2 = 0, \\ -2\alpha + a\beta = 0, \\ -2\alpha + a\beta = S_{24}, \\ \alpha - \frac{a}{2}\beta = 0, \\ -\frac{2c}{a}\alpha + c\beta = -2X_4, \\ -2X_4 + \frac{4b(a+12d)}{a(a-4d)}X_2 = 0, \\ 2(\frac{2(a-c)-a}{a})X_2 - 4(\frac{a-c}{a})X_4 = 0. \end{cases} \tag{22}$$

By solving the above systems, we get $\alpha = \beta = 0$ and $X_1 = X_2 = X_3 = X_4 = 0$. Thus, we have the following theorem:

Theorem 1. *Suppose that (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{g}_1$. Then (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if $X_1 = X_2 = X_3 = X_4 = 0$ and $\alpha = \beta = 0$.*

Since $\alpha = \beta = 0$, according to the defination of Killing vector fields $\mathscr{L}_X g = 0$, we have the following corollary:

Corollary 1. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g} = \mathfrak{a}_1$ is a Killing vector field.*

Case (A2)

We consider Lie algebra of type (**A2**). For any vector field $X = X_i u_i \in \mathfrak{m}$, where $r = e+1$ and $s = e-1$, the Lie derivative of the metric g with respect to the vector field X, is given by

$$\mathscr{L}_X g = \begin{pmatrix} 0 & 0 & -2eaX_4 & asX_3 \\ 0 & 2eaX_4 & 0 & -eaX_2 \\ -2eaX_4 & 0 & 2bsX_4 & arX_1 - s(bX_3 - cX_4) \\ asX_3 & -eaX_2 & arX_1 - s(bX_3 - cX_4) & -2csX_3 \end{pmatrix}, \tag{23}$$

Further, the Lie derivative of the Ricci tensor is calculated as follows

$$\mathscr{L}_X S=\begin{pmatrix} 0 & 0 & \frac{6e^3a}{d}X_4 & -\frac{3re^2a}{d}X_3 \\ 0 & -\frac{6e^3a}{d}X_4 & 0 & \frac{3e^3a}{d}X_2 \\ \frac{6e^3a}{d}X_4 & 0 & -\frac{2bs(e-1)(3e^3-3e+2)}{d}X_4 & -\frac{3re^2a}{d}X_1+\frac{bs(3e^2-3e+2)}{d}X_3 \\ -\frac{3re^2a}{d}X_3 & \frac{3e^3a}{d}X_2 & -\frac{3re^2a}{d}X_1+\frac{bs(3e^2-3e+2)}{d}X_3 & 0 \end{pmatrix}, \tag{24}$$

Substituting (5), (6), and (23) into (1) and (2), we get

$$\begin{cases} a\alpha-\frac{3e^2a}{d}\beta=2eaX_4, \\ a(e-1)X_3=0, \\ -eaX_2=0, \\ b\alpha-\frac{b(3e^2-3e+2)}{d}\beta=2b(e-1)X_4, \\ c\alpha-\frac{3e^2c}{d}\beta=a(e+1)X_1-(e-1)(bX_3-cX_4), \\ d\alpha-3e^2\beta=-2(e-1)cX_3, \end{cases} \tag{25}$$

Now, by applying (5), (6), and (24) into (1) and (2), we have

$$\begin{cases} -\frac{3e^2a}{d}\alpha+a\beta=-6\frac{e^3a}{d}X_4, \\ -\frac{b(3e^2-3e+2)}{d}\alpha+b\beta=-\frac{2b(e-1)(3e^2-3e+2)}{d}X_4, \\ \frac{-3(e-1)e^2a}{d}X_3=0, \\ \frac{3e^3a}{d}X_2=0, \\ -\frac{3e^2c}{d}\alpha+c\beta=-\frac{3(e+1)e^2a}{d}X_1+\frac{b(e-1)(3e^2-3e+2)}{d}X_3, \\ -3e^2\alpha+d\beta=0. \end{cases} \tag{26}$$

By solving the above systems, we have the following theorem:

Theorem 2. *Assume that (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{g}_2$. Then (M,g) has a Ricci bi-conformal vector fields $X=X_i\partial_i$ if and only if the following cases are true:*

(1) $e=0, \alpha=\beta=0, ad\neq 0, X_1=\frac{c}{a}X_4, X_3=0$ and for any X_2,
(2) $e=1, \alpha=\frac{3}{d}\beta, X_1=X_2=X_4=0$ and for any X_3, such that $(d^2-9)\beta=0$ and $b\beta=0$,
(3) $e=-1, X_2=X_3=X_4=0$ and for any X_1, such that $(-9+d^2)\beta=0$, $bd=0$, and $\alpha=\frac{3}{d}\beta$,
(4) $e\neq 0,1,-1, X_1=X_2=X_3=X_4=0$, such that $(-9e^4+d^2)\beta=0$, $b(-3e+1)\beta=0$, and $\alpha=\frac{3e^2}{d}\beta$.

Since $\alpha=\beta=0$, and α is the coefficient of β, according to the defination of Killing vector fields $\mathscr{L}_X g=0$, we have the following corollary:

Corollary 2. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g}=\mathfrak{a}_2$ is a Killing vector field for $\beta=0$.*

Case (A3)
We consider Lie algebra of type (**A3**). For any arbitrary vector field $X=X_iu_i\in\mathfrak{m}$, we compute

$$\mathscr{L}_X g=\begin{pmatrix} 0 & 0 & 0 & 2aX_3 \\ 0 & 2aX_3 & -aX_2 & 0 \\ 0 & -aX_2 & 0 & -2aX_1 \\ 2aX_3 & 0 & -2aX_1 & 0 \end{pmatrix}. \tag{27}$$

As well, the Lie derivative of the Ricci tensor is obtained by

$$\mathscr{L}_X S=\begin{pmatrix} 0 & 0 & \frac{-6a}{b}X_3 & 0 \\ 0 & 0 & \frac{3\varepsilon c}{b}X_4 & 0 \\ \frac{-6a}{b}X_3 & \frac{3\varepsilon c}{b}X_4 & \frac{12a}{b}X_1 & \frac{-3\varepsilon c}{b}X_2 \\ 0 & 0 & \frac{-3\varepsilon c}{b}X_2 & 0 \end{pmatrix}. \tag{28}$$

Inserting (7), (8), and (27) into (1) and (2), we infer

$$\begin{cases} a\alpha - \frac{3a}{b}\beta = 2aX_3, \\ aX_2 = 0, \\ b\alpha - 3\beta = 0, \\ c\alpha - \frac{3c}{b}\beta = -2aX_1, \\ d\alpha - \frac{\varepsilon b - 2d}{b}\beta = 0. \end{cases} \tag{29}$$

So, by applying (7), (8), and (28) into (1) and (2), we have

$$\begin{cases} -\frac{3a}{b}\alpha + a\beta = 0, \\ \frac{3\varepsilon c}{b}X_4 = 0, \\ -3\alpha + b\beta = \frac{12a}{b}X_1, \\ -\frac{3c}{b}\alpha + c\beta = \frac{-3\varepsilon c}{b}X_2, \\ \frac{-6a}{b}X_3 = 0, \\ \frac{\varepsilon b - 2d}{b}\alpha + d\beta = 0. \end{cases} \tag{30}$$

By solving the above systems, we have the following theorem:

Theorem 3. *Consider that (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{g}_3$. So (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if $X_1 = X_2 = X_3 = 0, cX_4 = 0, (-9+b^2)\beta = 0$ and $\alpha = \frac{3}{b}\beta$.*

Since α is the coefficient of β, according to the defination of Killing vector fields $\mathscr{L}_X g = 0$, we obtain the following corollary:

Corollary 3. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g} = \mathfrak{a}_3$ is a Killing vector field for $\beta = 0$.*

Case (A4)

We consider Lie algebra of type (**A4**). For any arbitrary vector field $X = X_i u_i \in \mathfrak{m}$, the Lie derivative is computed as follows

$$\mathscr{L}_X g = \begin{pmatrix} 0 & 2bX_2 & aX_2 & \frac{a}{2}X_4 \\ 2bX_2 & -4bX_1 & -aX_1 & 0 \\ aX_2 & -aX_1 & 0 & 0 \\ \frac{a}{2}X_4 & 0 & 0 & -aX_1 \end{pmatrix}. \tag{31}$$

Thus, the Lie derivative of the Ricci tensor is computed as

$$\mathscr{L}_X S = \begin{pmatrix} 0 & 9X_3 & -6X_2 & \frac{-3}{2}X_4 \\ 9X_3 & 32\frac{b}{a}X_1 & 12X_1 & 0 \\ -6X_2 & 12X_1 & 6X_2 & 0 \\ \frac{-3}{2}X_4 & 0 & 0 & -3X_1 \end{pmatrix}. \tag{32}$$

Substituting (9), (10), and (31) into (1) and (2), we get

$$\begin{cases} a\alpha - 3\beta = 0, \\ 2bX_2 = 0, \\ aX_2 = 0, \\ \frac{a}{2}X_4 = 0, \\ b\alpha - \frac{8b}{a}\beta = -4bX_1, \\ a\alpha - 3\beta = -aX_1, \\ \frac{a}{2}\alpha - \frac{3}{2}\beta = -aX_1. \end{cases} \tag{33}$$

Also, by applying (9), (10), and (32) into (1) and (2), we deduce

$$\begin{cases} -3\alpha + a\beta = 0, \\ 9X_3 = 0, \\ -6X_2 = 0, \\ -\frac{3}{2}X_4 = 0, \\ -\frac{8b}{a}\alpha + b\beta = \frac{32b}{a}X_1, \\ -3\alpha + a\beta = 12X_1, \\ 6X_2 = 0, \\ -\frac{3}{2}\alpha + \frac{a}{2}\beta = -3X_1. \end{cases} \tag{34}$$

By solving the above systems we have the following theorem:

Theorem 4. *Suppose that (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{g}_4$. So (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if the following cases hold:*

(1) $\alpha = \beta = 0, X_1 = X_2 = X_3 = X_4 = 0,$
(2) $\alpha = \pm\beta, \beta \neq 0, a = \pm 3, b = 0, X_1 = X_2 = X_3 = X_4 = 0.$

Since $X = 0$, according to the defination of Killing vector fields $\mathscr{L}_X g = 0$, we have the following corollary:

Corollary 4. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g} = \mathfrak{a}_4$ is a Killing vector field.*

Case (A5)
We consider Lie algebra of type $(\mathbf{A5})$. We have the Lie derivative for any arbitrary vector field $X = X_i u_i \in \mathfrak{m}$ as follows

$$\mathscr{L}_X g = \begin{pmatrix} 0 & \frac{a}{2}X_3 & 0 & \frac{a}{4}X_4 \\ \frac{a}{2}X_3 & 0 & -\frac{a}{2}X_1 & 0 \\ 0 & -\frac{a}{2}X_1 & 0 & 0 \\ \frac{a}{4}X_4 & 0 & 0 & -\frac{a}{2}X_1 \end{pmatrix}. \tag{35}$$

So, the Lie derivative of the Ricci tensor is described by

$$\mathscr{L}_X S = \begin{pmatrix} 0 & -6X_3 & -18X_2+3X_3 & 0 \\ -6X_3 & -24X_3 & 12X_1 & 0 \\ -18X_2+3X_3 & 12X_1 & -6X_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

Substituting (11), (12), and (35), into (1) and (2), we get

$$\begin{cases} a\alpha - 12\beta = 0, \\ \frac{a}{2}X_3 = 0, \\ \frac{a}{4}X_4 = 0, \\ \frac{a}{4}\alpha - 3\beta = -\frac{a}{2}X_1, \\ \frac{a}{8}\alpha - \frac{3}{2}\beta = -\frac{a}{2}X_1, \end{cases} \tag{37}$$

So,by inserting (11), (12), and (36) into (1) and (2), we infer

$$\begin{cases} -12\alpha + a\beta = 0, \\ -6X_3 = 0, \\ -18X_2 + 3X_3 = 0, \\ -24X_3 = 0, \\ -3\alpha + \frac{a}{4}\beta = 12X_1, \\ -6X_1 = 0, \qquad -\frac{3}{2}\alpha + \frac{a}{8}\beta = 0. \end{cases} \tag{38}$$

By solving the above systems, we obtain:

Theorem 5. *Let (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{g}_5$. So (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if the following cases are true:*

(1) $a \neq \pm 12$, $\alpha = \beta = 0, a \neq 0, X_1 = X_2 = X_3 = X_4 = 0,$
(2) $\alpha = \pm\beta, \beta \neq 0, a = \pm 12, X_1 = X_2 = X_3 = X_4 = 0.$

Since $X = 0$, according to the defination of Killing vector fields $\mathscr{L}_X g = 0$, we have the following corollary:

Corollary 5. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g} = \mathfrak{a}_5$ is a Killing vector field.*

Case (B1)
We consider Lie algebra of type $(\mathbf{B1})$. For any vector field $X = X_i u_i \in \mathfrak{m}$, we get

$$\mathscr{L}_X g = \begin{pmatrix} 2aX_3 & 2bX_2 + cX_3 & -aX_1 + 2cX_2 + dX_3 & aX_2 \\ 2bX_2 + cX_3 & -4bX_1 + 2cX_4 & -3cX_1 + dX_4 & -aX_1 - cX_2 \\ -aX_1 + 2cX_2 + dX_3 & -3cX_1 + dX_4 & -2dX_1 & -dX_2 \\ aX_2 & -aX_1 - cX_2 & -dX_2 & 0 \end{pmatrix}. \tag{39}$$

Thus, the Lie derivative of the Ricci tensor is represented by

$$\mathscr{L}_X S = \begin{pmatrix} 0 & \frac{3(6bd-5c^2)}{a^2}X_2 + \frac{3d}{2a}X_3 & \frac{3d}{a}X_2 & 0 \\ \frac{3(6bd-5c^2)}{a^2}X_2 + \frac{3d}{2a}X_3 & -\frac{6(6bd-5c^2)}{a^2}X_1 + \frac{3d}{2a}X_4 & \frac{-9d}{2a}X_1 & \frac{-3d}{2a}X_2 \\ \frac{3d}{a}X_2 & \frac{-9d}{2a}X_1 & 0 & 0 \\ 0 & \frac{-3d}{2a}X_2 & 0 & 0 \end{pmatrix}. \tag{40}$$

Substituting (13), (14), and (39), into (1) and (2), we get

$$\begin{cases} 2aX_3 = 0, \\ 2bX_2 + cX_3 = 0, \\ a\alpha + \frac{3d}{2a}\beta = -aX_1 + 2cX_2 + dX_3, \\ aX_2 = 0, \\ b\alpha + \frac{3(6bd-5c^2)}{2a^2}\beta = -4bX_1 + 2cX_4, \\ c\alpha + \frac{3cd}{2a^2}\beta = -3cX_1 + dX_4, \\ a\alpha + \frac{3d}{2a}\beta = -aX_1 - cX_2, \\ d\alpha + \frac{3d^2}{2a^2}\beta = -2dX_1, \\ -dX_2 = 0. \end{cases} \tag{41}$$

Also, by applying (13), (14), and (40) into (1) and (2), we have

$$\begin{cases} \frac{3(6bd-5c^2)}{a^2}X_2 + \frac{3d}{2a}X_3 = 0, \\ \frac{3(6bd-5c^2)}{2a^2}\alpha + b\beta = \frac{-6(6bd-5c^2)}{a^2}X_1 + \frac{3d}{2a}X_4, \\ \frac{3d}{2a}\alpha + a\beta = \frac{3d}{a}X_2, \\ \frac{3cd}{2a^2}\alpha + c\beta = \frac{-9d}{2a}X_1, \\ \frac{3d}{2a}\alpha + a\beta = \frac{-3d}{2a}X_2, \\ \frac{3d^2}{2a^2}\alpha + d\beta = 0. \end{cases} \tag{42}$$

By solving the above systems, we derive the following theorem:

Theorem 6. *Suppose that (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{b}_1$. Then (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if the following cases hold:*

(1) $\alpha = \beta = 0, d \neq 0, X_1 = X_2 = X_3 = X_4 = 0,$
(2) $\alpha = \beta = 0, d = 0, X_1 = X_2 = X_3 = 0$ *and for all* $X_4, a, b, c,$
(3) $\alpha \neq 0, b = c = d = 0, X_1 = -\alpha, X_2 = X_3 = 0,$ *and for any* X_4.

According to the defination of Killing vector fields, we have the following corollary:

Corollary 6. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g} = \mathfrak{b}_1$ satisfied in (1) and (2) is a Killing vector field.*

Case (B2)

We consider Lie algebra of type (**B2**). We have the Lie derivative of the metric for any vector field $X = X_i u_i \in \mathfrak{m}$ as follows

$$\mathscr{L}_X g = \begin{pmatrix} 0 & 2bX_2 & aX_2 & -\frac{a}{2}X_4 \\ 2bX_2 & -4bX_1 & -aX_1 & 0 \\ aX_2 & -aX_1 & 0 & 0 \\ -\frac{a}{2}X_4 & 0 & 0 & aX_1 \end{pmatrix}. \tag{43}$$

Hence, the Lie derivative of the Ricci tensor is represented by

$$\mathscr{L}_X S = \begin{pmatrix} 0 & \frac{-16b}{a}X_2 + 3X_3 & -9X_2 & \frac{3}{2}X_4 \\ \frac{-16b}{a}X_2 + 3X_3 & 32\frac{b}{a}X_1 - 3X_3 & 12X_1 & 0 \\ -9X_2 & 12X_1 & 0 & 0 \\ \frac{3}{2}X_4 & 0 & 0 & -3X_1 \end{pmatrix}. \tag{44}$$

Substituting (15), (16), and (43), into (1) and (2), we get

$$\begin{cases} 2bX_2 = 0, \\ aX_2 = 0, \\ a\alpha - 3\beta = 0, \\ -\frac{a}{2}X_4 = 0, \\ b\alpha - \frac{8b}{a}\beta = -4bX_1, \\ a\alpha - 3\beta = -aX_1, \\ \frac{-a}{2}\alpha + \frac{3}{2}\beta = aX_1, \end{cases} \tag{45}$$

Also, substituting (15), (16), and (44) into (1) and (2), we conclude

$$\begin{cases} \frac{-16b}{a}X_2 + 3X_3 = 0, \\ \frac{3}{2}X_4 = 0, \\ -3\alpha + a\beta = 0, \\ -\frac{8b}{a}\alpha + b\beta = \frac{32b}{a}X_1 - 3X_3, \\ -3\alpha + a\beta = 12X_1, \\ -9X_2 = 0 \\ \frac{3}{2}\alpha - \frac{a}{2}\beta = -3X_1. \end{cases} \tag{46}$$

By solving the above systems, we get the following theorem:

Theorem 7. *Assume (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{b}_2$. Then (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if the following cases are true:*

(1) $a \neq \pm 3$, $\alpha = \beta = 0, X_1 = X_2 = X_3 = X_4 = 0$ *for all* b,

(2) $\alpha = \pm\beta, \beta \neq 0, a = \pm 3, b = 0, X_1 = X_2 = X_3 = X_4 = 0.$

Since $X = 0$, according to the defination of Killing vector fields $\mathscr{L}_X g = 0$, we have the following corollary:

Corollary 7. *Any Ricci bi-conformal vector field X on space of type $\mathfrak{g} = \mathfrak{b}_2$ is a Killing vector field.*

Case (B3)

We consider Lie algebra of type (**B3**). For any vector field $X = X_i u_i \in \mathfrak{m}$, we infer

$$\mathscr{L}_X g = \begin{pmatrix} 0 & -aX_3 & aX_2 & 0 \\ -aX_3 & -2aX_4 & -2bX_3 & aX_2 \\ aX_2 & -2bX_3 & 4bX_2 & 0 \\ 0 & aX_2 & 0 & 0 \end{pmatrix}. \tag{47}$$

As well, the Lie derivative of the Ricci tensor is obtained by

$$\mathscr{L}_X S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

Putting (17), (18), (47), and (48) into (1) and (2), we infer

$$\begin{cases} aX_3 = 0, \\ \alpha = X_2, \\ aX_4 = 0, \\ bX_3 = 0, \\ b\alpha = 4bX_2. \end{cases} \tag{49}$$

By inserting (17), (18), and (48) into (1) and (2), we get

$$\beta = 0. \tag{50}$$

By solving the above systems, the following theorem is obtained:

Theorem 8. *Consider (M,g) is a non-reductive four-dimensional homogeneous space of type $\mathfrak{b}_3$. Then (M,g) has a Ricci bi-conformal vector fields $X = X_i\partial_i$ if and only if $X_3 = X_4 = 0, \alpha = X_2, \beta = 0$ for all X_1, b, c, d and α.*

For $\alpha = 0$, according to the defination of Killing vector fields $\mathscr{L}_X g = 0$, we have the following corollary:

Corollary 8. *Any Ricci bi-conformal vector fields X on space of type $\mathfrak{g} = \mathfrak{b}_3$ is a Killing vector field for $\alpha = 0$.*

4 Conclusion

The main study of the paper is to classify Ricci bi-conformal vector fields on non-reductive four-dimensional homogeneous spaces. The non-reductive four-dimensional homogeneous pseudo-Riemannian manifolds are classified with the Ricci tensor, and expressly explain the relation pseudo-Riemannian metrics and the Lie derivative of the metrics with reference to a vector field. Then we calculated the Lie derivative of the metric g with respect to the vector field X, and the Lie derivative of the Ricci tensor in direction X. Then we prove which of vector fields are Ricci bi-conformal vector fields, also we show which of them are Killing vector fields. It is also suggested that the Ricci bi-conformal vector fields can be studied on the non-reductive four-dimensional homogeneous spaces associated to the canonical connections, Kobayashi-Nomizu connections, and Schouten Van-Kampen connections.

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