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Bayesian and E-Bayesian Estimation for The Inverted Topp-Leone Distribution Based on Progressive Type-I Censoring Data

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Abstract: In this paper, Bayesian, E-Bayesian, and non-Bayesian estimations of the shape parameter of the inverted Topp-Leone distribution are studied based on the progressive Type I censored (PT-IC) data. The maximum likelihood estimator (MLE), Bayes, and E-Bayes estimators of the unknown parameter under the squared error loss (SEL) function, degroot loss (DL) function, and quadratic loss (QL) function are obtained. For E-Bayes estimates, we assumed three distributions for the hyper-parameters a and b. Three types of confidence intervals are discussed for the unknown parameter. The effectiveness of the suggested approaches is compared using a simulated study, and for illustration, two numerical cases have been examined.

Keywords: Inverted Topp-Leone Distribution; Bayesian Estimation; E-Bayesian Estimation; Highest Posterior Density Credible Interval; Bootstrap Confidence Interval; Progressive Type-I Censoring.

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1 Introduction

Applications to inverted distributions are numerous and include issues in the biological sciences, engineering sciences, econometrics, survey sampling, life testing, and medical research. Furthermore, it finds application in environmental studies, survival and reliability theory, and financial literature. The inverted distributions are able to model many phenomena; therefore, they are widely used by researchers. see, for example, Calabria and Pulcini [14], AL-Dayian [4], Abd EL-Kader et al. [2], AL-Dayian [5], Dey [17], Prakash [41], Aljuaid [6] and Abd Al-Fattah et al. [1]. Topp and Leone [43] presented The Topp-Leone distribution as a substitute for the beta distribution. The Topp-Leone distribution exhibits a j-shaped density function and a hazard function resembling a bathtub shape. It is advantageous for modeling phenomena with limited lifetime, and Nadarajah and Kotz [40] have investigated various properties of this distribution.

Several authors have studied the Topp-Leone distribution; see, for example, Ghitany et al. [22], Van Dorp and Kotz [44], Zhou et al. [47], Kotz and Seier [33], Vicaria et al. [45], Nadarajah [39], Al-Zahrani [9], and Gen [21].

The Inverted Topp-Leone (ITL) distribution was introduced in different forms. Muhammed [38] introduced the ITL distribution with support $x > 1$, and the pdf and the cdf of the ITL distribution can be written respectively as:

$$g(x;\beta) = 2\beta x(x-1)x^{-2\beta-1}(2x-1)^{\beta-1}; 1 < x < \infty, \beta > 0$$

and

$$G(x;\beta) = 1 - x^{-2\beta}(2x-1)^{\beta-1}; 1 < x < \infty, \beta > 0$$

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and Hassan et al. [30] also see (Muhammed [36]) introduced the ITL distribution, with support $x > 0$. the pdf and the cdf of the ITL distribution can be written respectively as

$$f(x;\beta)=2\beta x(1+2x)^{\beta-1}(1+x)^{-2\beta-1};x,\beta>0 \tag{1}$$

and

$$F(x;\beta)=1-(1+2x)^{\beta}(1+x)^{-2\beta};x\geq 0,\beta>0 \tag{2}$$

where β is the shape parameter. The behavior of the ITL distribution at various values of β was depicted in Figure (1).

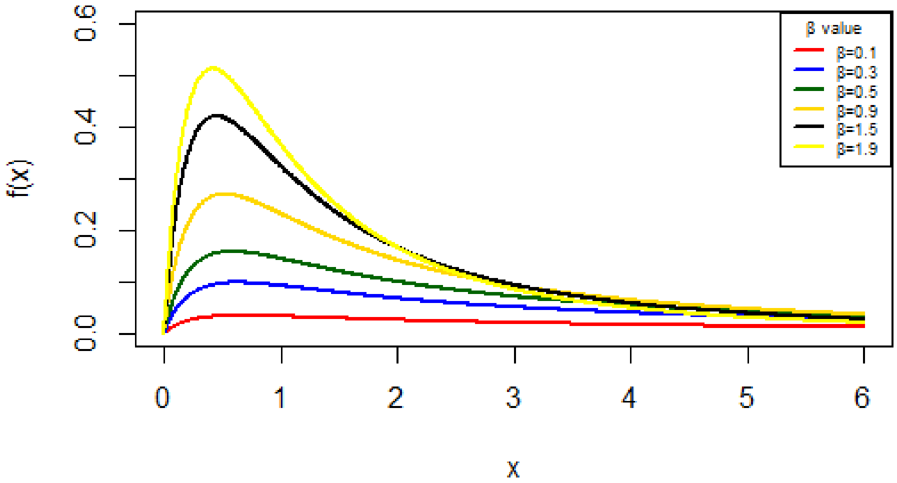


Fig. 1: ITL distribution's pdfs with different values of β

Survival function $\overline{F}(t;\beta)$ and the associated hazard function $h(t;\beta)$ are provided, respectively, by

$$\overline{F}(x;\beta)=(1+2x)^{\beta}(1+x)^{-2\beta} \tag{3}$$

and

$$h(x;\beta)=2\beta x[(1+2x)(1+x)]^{-1} \tag{4}$$

The ITL distribution is capable of modeling many lifetime and reliability data as well as natural phenomena data; see, for example, Hassan and Assar [29], Gupta and Huang [23], Al-Saiary and Bakoban [8], Almetwally et al. [7] and Hassan et al. [28]. Aijaz et al. 2020 introduced the Bayes estimate of the ITL distribution's parameter. Muhammed and Muhammed [37] discussed the Bayes and non-Bayes estimate of the ITL distribution' parameter based on randomly censored data. The life test data and the social science data can be modeled by the ITL distribution. Recently the ITL distribution has been discussed by numerous researchers; As an illustration, Aijaz et al. [3] introduced the Bayes estimated of the ITL distribution's parameter within various loss functions. Muhammed and Muhammed [37] discussed the Bayes estimates and MLEs of the ITL distribution's parameter based on random censoring samples. Yousef et al. [46] introduced the Bayes estimates for the ITL distribution parameters using MCMC method of system reliability under ranked set sampling. Hassan et al. [31] studied the Bayesian estimation of ITL distribution under progressive Type II censored (PT-IIC) data based on CRs model.

The observed data from many of applied research such as agriculture, modern industry, economics, medicine, biological sciences and actuarial science, are often not completely observed. The experiments in reliability and life testing studies takes a lot of money and time to be completely observed, so to reduce the duration and expense of the experiment, the experimenter is forced to finish the trial prior to every unit failing and the observed data named censored data. The most common and widely used censorship schemes are Type I and Type II censoring schemes. In Type I censored experiments, the experimenter finishes the experiment at a predetermined period, and in Type II censored data, the experimenter finishes after witnessing a predetermined number of units (Mann et al. [35]). Sometimes limitations (financial, material, or time) force the researcher to take out units on purpose, or units are lost prior to failure (e.g.,

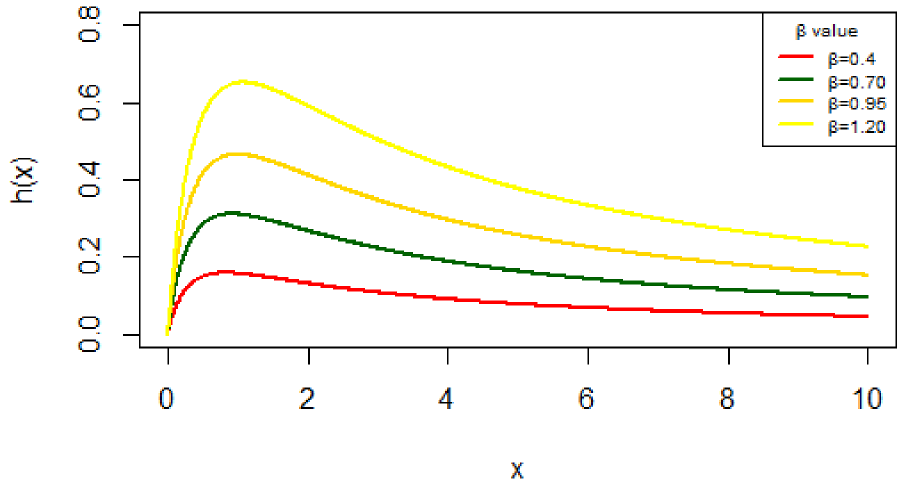


Fig. 2: ITL distribution's Hazard functions with different values of β

unintentional shattering of a unit). The deliberate removal of units from a running life experiment permits the units to be released for more trials. Progressive censoring can be used to simulate these situations.

cohen [15] expanded upon studying Type I censored data by introducing the model of PT-IC data. In the PT-IC samples technique, we remove a number of the predetermined units from the experiment at a pre-specified time. For further information on PT-IC, we refer to the monograph by Balakrishnan and Cramer [11] and to the review paper by Balakrishnan [10].

On the other hand, Herd [32] introduced a model to describe the PT-IIC data; he called it multiple censoring', and then cohen [15] named the model 'type II progressive censoring.' In the PT-IIC data, the researchers previously specified the number of failure units to finish the test, and with each failure unit, a prefixed number of surviving units are withdrawn from the test, and they finish the test after all the specified units have failed.

In this article, our objective is to use statistical inference methods to analyze the PT-IC data. Considering that the lifetimes of the experiment items independently have the ITL distribution. In Section 2, we obtain the explicit expression for the MLE. We discussed the Bayesian approach under three different loss functions in Section 3. In Section 4, we computed the E-Bayesian estimation under three different loss functions and three prior distributions for the shape parameters. In Section 5, we introduce three types of confidence intervals. In Section 6, we provide a simulation study. Two numerical examples have been presented in Section 7. Finally, some conclusions are drawn in Section 8.

2 Model description and Maximum Likelihood

To explain a life-testing process implemented with PT-IC data, we must first determine m censoring time points $T_1 < T_2 < \ldots < T_m$. Now, for $h = 1,\ 2, \ldots, m$, let v_h indicate how many items failed within the given time frame $]T_{h-1},\ T_h]$, $x_{h,l}$ indicate the l^{th} sorted failing time between these v_h units for $l = 1,\ 2,\ \ldots,\ v_h$, and R_h denote the number of items withdrawn at random i.e., censored at time T_h. Additionally, let N_h indicate the number of items functioning and staying in experiment at the beginning of the h^{th} time interval, i.e., $N_h = n - \sum_{i=1}^{h-1} v_i - \sum_{i=1}^{h-1} R_i$.

With this configuration, the following is how a PT-IC life-testing experiment works: all of the life-testing items $N_1 \equiv n$ are put under the test at time $T_0 \equiv 0$ to time T_1, at which point, a number of specified surviving items R_1 are withdrawn from the test. During this initial period, random v_1 failure times are additionally gathered. The test is then continued on $N_2 = n - v_1 - R_1$ remaining units until time T_2, at which point R_2 live units are randomly withdrawn from the test and random v_2 failure times are also collected, and so on. Finally, the number of failure times at time T_m is $r = \sum_{h=1}^{m} v_h$. Note that if $n > \sum_{h=1}^{m} v_h + R_{h-1}$ at time T_m then all the surviving units R_m are removed where $R_m = n - \sum_{h=1}^{m} v_h - \sum_{h=1}^{m} R_{h-1} = N_m - v_m$. The situation with no intermediate censoring i.e., $r_1 = r_2 = \ldots = r_{m-1} = 0$ corresponds to a life-test under the conventional Type I right censoring as a special case. See Balakrishnan et al. [12]. The likelihood function, in this case, is given by (cohen [15])

$$\mathscr{L}(x|\theta) \propto \prod_{i=1}^{r} f(x_i;\theta) \prod_{h=1}^{m} (1 - F(T_h;\theta))^{R_h} \tag{5}$$

We used henceforth x_i instead of x_{hl} to simplify the notation. Under the assumption that the lifetime of a test unit follows ITL distribution with pdf and cdf of the failure time of a test unit given in (1) and (2), respectively. Then the likelihood function under PT-IC data is given by

$$\mathscr{L}(x|\beta) \propto \left\{ \prod_{i=1}^{r} 2\beta x_i \frac{(1+2x_i)^{\beta-1}}{(1+x_i)^{2\beta+1}} \right\} \left\{ \prod_{h=1}^{m} \left[\frac{(1+2T_h)^{\beta}}{(1+T_h)^{-2\beta}} \right]^{R_h} \right\}$$

The likelihood function can be written in proportional as follows:

$$\mathscr{L}(x|\beta) \propto \beta^r \exp\left\{ -\beta \left[\sum_{i=1}^{r} \ln\left(\frac{(1+x_i)^2}{(1+2x_i)} \right) + \sum_{h=1}^{m} \ln\left(\frac{(1+T_h)^2}{(1+2T_h)} \right)^{R_h} \right] \right\} \tag{6}$$

Taking the logarithm of $\mathscr{L}(\beta)$ to obtain log-likelihood $l(x|\beta) = ln(\mathscr{L})$ as

$$l(x|\beta) \propto rln\beta - \beta \left[\sum_{i=1}^{r} \ln\left(\frac{(1+x_i)^2}{(1+2x_i)} \right) + \sum_{h=1}^{m} \ln\left(\frac{(1+T_h)^2}{(1+2T_h)} \right)^{R_h} \right] \tag{7}$$

After differentiating the $l(x|\beta$ and equating it to zero, the MLE for β can be expressed in closed form as follows.

$$\widehat{\beta} = \frac{r}{\sum_{i=1}^{r} \ln\left(\frac{(1+x_i)^2}{(1+2x_i)} \right) + \sum_{h=1}^{m} \ln\left(\frac{(1+T_h)^2}{(1+2T_h)} \right)^{R_h}} \tag{8}$$

3 Bayesian Estimation

In this section, we introduce the Bayesian inference of PT-IC data when the lifetimes follow the ITL distribution. Prior knowledge and some assumptions about the parameter can be taken into account using the Bayesian inference. Thus, the parameter β is considered to be random variables with a specified distribution.

The Bayesian estimate process for the ITL distribution's parameter under the SEL function, DL function, and QL function in closed form will be covered in this section. We assume that the ITL distribution's parameter β has a gamma conjugate prior distribution and may be expressed proportionally as follows;

$$\pi(\beta|a,b) \propto \beta^{a-1} e^{-\beta b}, \qquad \beta > 0,\ a,b > 0 \tag{9}$$

Using the likelihood function (6) and the gamma function (9), the posterior distribution of β is

$$\pi(\beta|\underline{x}) = \frac{\pi(\beta)L(\underline{x}|\beta)}{\int_0^{\infty} \pi(\beta)L(\underline{x}|\beta)d\beta}$$

.

The posterior distribution of β can be expressed as

$$\pi(\beta|\underline{x}) = \frac{b+\mathscr{H}^{r+a}}{\Gamma r+a} \beta^{r+a-1} e^{-\beta b+\mathscr{H}} \tag{10}$$

where,

$$\mathscr{H} = \sum_{i=1}^{r} \ln\left[\frac{(1+x_i)^2}{(1+2x_i)} \right] + \sum_{h=1}^{m} \ln\left[\frac{(1+T_h)^2}{(1+2T_h)} \right]^{R_h}$$

That mean, the posterior distribution of β follow *gamma* $(r+a, b+\mathscr{H})$.

3.1 Bayesian Estimation under SEL Function

The SEL function defined as follows:

$$L_{SEL}\left(\widetilde{\beta},\beta\right) = (\widetilde{\beta}-\beta)^2$$

where $\widetilde{\beta}$ is the Bayes estimate of β. Based on the SELF, the Bayes estimates of β is provided by

$$\widetilde{\beta}_{BSEL} = E\left(\beta|\underline{x}\right) = \int_0^\infty \beta\pi\left(\beta|\underline{x}\right)d\beta = \frac{(\mathscr{H}+b)^{(r+a)}}{\Gamma(r+a)}\int_0^\infty \beta^{(r+a+1)-1}e^{-\beta(\mathscr{H}+b)}d\beta = \frac{r+a}{\mathscr{H}+b}. \tag{11}$$

where, $\underline{x} = (x_{(1)}, x_{(2)}, \ldots,\ x_{(m)})$ is a vector of the observed lifetimes of the n units under the test and $\widetilde{\beta}_{jBSEL}$ is Bayes estimates of β based on the SEL function and E is the posterior distribution expectation.

3.2 Bayesian Estimation under DL Function

The DL function is introduced by Degroot [16] as

$$L_{DL}\left(\widetilde{\beta},\beta\right) = \left(\frac{\beta-\widetilde{\beta}}{\widetilde{\beta}}\right).$$

where $\widetilde{\beta}$ is the an estimator of β. The Bayes estimate of β under DL function is given by

$$\widetilde{\beta}_{BDL} = \frac{E\left(\beta^2|\underline{x}\right)}{E\left(\beta|\underline{x}\right)} = \frac{(r+a+1)(r+a)}{(\mathscr{H}+b)^2}\Big/\frac{(r+a)}{(\mathscr{H}+b)} = \frac{(r+a+1)}{(\mathscr{H}+b)}. \tag{12}$$

where $\widetilde{\beta}_{BDL}$ Bayes estimator of β based on DL function and E is the posterior distribution expectation.

3.3 Bayesian Estimation under QL Function

The QL function is defined by Dey [18] as follows

$$L_{QL}\left(\widetilde{\beta},\beta\right) = \left(\frac{\beta-\widetilde{\beta}}{\beta}\right)^2.$$

where $\widetilde{\beta}$ is the an estimator of β. Based on the QL function, the Bayes estimates of β_j is provided by

$$\widetilde{\beta}_{BQL} = \frac{E\left(\beta^{-1}|\underline{x}\right)}{E\left(\beta^{-2}|\underline{x}\right)} \tag{13}$$

We can get $\widetilde{\beta}_{BQL}$ by using (10) in (13) to be

$$\widetilde{\beta}_{BQL} = \frac{r+a-2}{\mathscr{H}+b} \tag{14}$$

where $\widetilde{\beta}_{jBQL}$ Bayes estimate of β under QL function and E is the posterior distribution expectation.

3.4 Hyper-Parameters Calculation

The hyper-parameters involved in priors (9) can be quickly evaluated as follow:

1. To calculate the hyper-parameter values, we need to know prior information about the ITL distribution's parameter.
2. We generate j of the random samples from the ITL distribution, where $j = 1,2,\ldots,k$ using the true value of (β).
3. We compute the maximum likelihood estimates of $(\widehat{\beta}^j)$.
4. From mean and variance of the prior distribution (gamma distribution) that is given in (see Dey [19])

$$\frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j = \frac{a}{b} \qquad \& \qquad \frac{1}{k-1}\sum_{j=1}^{k}\left(\widehat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2 = \frac{a}{b^2}\ .$$

5. We can compute the hyper-parameters values as follows

$$a = \frac{\left(\frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\widehat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2} \quad \& \quad b = \frac{\frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\widehat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\widehat{\beta}^j\right)^2} \tag{15}$$

4 E-Bayesian Estimation

Han [25] presented a new estimate technique called E-Bayesian estimation. The E-Bayesian approach relies on calculating the Bayes estimate's expectation in relation to the hyper-parameter distributions. As stated by Han [26], the prior parameters a_j and b_j should be chosen to guarantee that $\pi(\beta|a,b)$ given in (9) is decreasing function of β. The derivative of $\pi(\beta|a,b)$ w.r.t. β is

$$\frac{d\pi(\beta|a,b)}{d\beta} = \frac{b^a}{\Gamma(a)}\beta^{a-2}e^{-\beta b}[a-1-b\beta].$$

Note that $a>0,\ b>0$ and $\beta>0$ leads to $0<a<1,\ b>0$ due to $\frac{d\pi(\beta|a,b)}{d\beta}<0$, and therefore $\pi(\beta|a,b)$ is decreasing function of β_j. Suppose that a_j and b_j are independent with bivariate density function

$$\pi(a,b) = \pi_1(a)\pi_2(b). \tag{16}$$

Then the E-Bayes estimate of β can be expressed as

$$\widetilde{\beta}_{EB} = E(\beta|\underline{x}) = \iint_{\mathscr{D}} \widetilde{\beta}_B(a,b)\pi(a,b)\,da db, \tag{17}$$

where $\widetilde{\beta}_B(a,b)$ is the Bayes estimate of β given by (11), (12) and (14). For addition information see Han (2005, 2009).

4.1 E-Bayesian Estimation under SEL Function

In this subsection, the E-Bayes estimates of β is examined using three distinct distributions of the hyper parameters, a and b, in order to examine the effects of various prior distributions on the E-Bayes estimates of β. We make use of the a and b distributions listed below: (Han [27]):

$$\pi_1(a,b) = \frac{2b}{c^2}, \qquad 0<a<1,\ 0<b<c \tag{18}$$

$$\pi_2(a,b) = \frac{1}{c}, \qquad 0<a<1,\ 0<b<c \tag{19}$$

$$\pi_3(a,b) = \frac{2(c-b)}{c^2}, \quad 0<a<1,\ 0<b<c \tag{20}$$

We can compute the E-Bayes estimates of β under SEL function depending $\pi_1(a,b)$ by using (11) and (18) in (17) as follow

$$\widetilde{\beta}_{EBSEL1} = \int_0^1 \int_0^c \left[\frac{r+a}{\mathscr{H}+b}\right]\left(\frac{2b}{c^2}\right) dbda = \frac{2}{c^2}\int_0^1 r+a \int_0^c \frac{b}{\mathscr{H}+b} dbda$$

$$= \frac{2}{c^2}\left[c-\mathscr{H}\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right]\int_0^1 r+ada = \left(\frac{2r+1}{c^2}\right)\left[c-\mathscr{H}\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right] \tag{21}$$

where $\widetilde{\beta}_{EBSEL1}$ is the E-Bayes estimate under SEL function based on $\pi_1(a,b)$.

Similarly, we can compute the E-Bayes estimates of β under SEL function depending on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (11), (19) in (17) and (11), (20) in (17) respectively, as follow

$$\widetilde{\beta}_{EBSEL2} = \int_0^1 \int_0^c \left[\frac{r+a}{\mathscr{H}+b}\right]\left(\frac{1}{c}\right) dbda = \left(\frac{2r+1}{2c}\right)\left[\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right] \tag{22}$$

where $\widetilde{\beta}_{EBSEL2}$ is the E-Bayes estimate under SEL function depending on $\pi_2(a,b)$.

and

$$\widetilde{\beta}_{EBSEL3} = \int_0^1 \int_0^c \left[\frac{r+a}{\mathscr{H}+b}\right]\left(\frac{2(c-b)}{c^2}\right) dbda = \left(\frac{2r+1}{c^2}\right)\left[(c+\mathscr{H})\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)-c\right] \tag{23}$$

where $\widetilde{\beta}_{EBSEL3}$ is the E-Bayes estimate under SEL function depending on $\pi_3(a,b)$.

4.2 E-Bayesian Estimation under DL Function

We can obtain the E-Bayes estimate of β under DL function based on $\pi_1(a,b)$ by using (12) and (18) in (17) as follow

$$\widetilde{\beta}_{EBDL1} = \int_0^1 \int_0^c \left[\frac{r+a+1}{\mathscr{H}+b}\right]\left(\frac{2b}{c^2}\right) dbda = \left(\frac{2r+3}{c^2}\right)\left[c-\mathscr{H}\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right] \tag{24}$$

where $\widetilde{\beta}_{EBDL1}$ is the E-Bayes estimate under DL function depending on $\pi_1(a,b)$.

By the same way, we can obtain the E-Bayes estimates of β under DL function depending on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (12), (19) in (17) and (12), (20) in (17) respectively, as follow

$$\widetilde{\beta}_{EBDL2} = \int_0^1 \int_0^c \left[\frac{r+a+1}{\mathscr{H}+b}\right]\left(\frac{1}{c}\right) dbda = \left(\frac{2r+3}{2c}\right)\left[\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right] \tag{25}$$

where $\widetilde{\beta}_{EBDL2}$ is the E-Bayes estimator under DL function depending on $\pi_2(a,b)$.

and

$$\widetilde{\beta}_{EBDL3} = \int_0^1 \int_0^c \left[\frac{r+a+1}{\mathscr{H}+b}\right]\left(\frac{2(c-b)}{c^2}\right) dbda = \left(\frac{2r+3}{c^2}\right)\left[(c+\mathscr{H})\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)-c\right] \tag{26}$$

where $\widetilde{\beta}_{EBDL3}$ is the E-Bayes estimator under DL function depending on $\pi_3(a,b)$.

4.3 E-Bayesian Estimation under QL Function

We can get the E-Bayes estimate of β under QL function depending on $\pi_1(a,b)$ by using (14) and (18) in (17) as follow

$$\widetilde{\beta}_{EBQL1} = \int_0^1 \int_0^c \left[\frac{r+a-2}{\mathscr{H}+b}\right]\left(\frac{2b}{c^2}\right) dbda = \left(\frac{2r-3}{c^2}\right)\left[c - \mathscr{H}\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right] \tag{27}$$

where $\widetilde{\beta}_{EBQL1}$ is the E-Bayes estimate under QL function depending on $\pi_1(a,b)$. Also, we can get the E-Bayes estimates of β under QL function depending on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (14), (19) in (17) and (14), (20) in (17) respectively, as follow

$$\widetilde{\beta}_{EBQL2} = \int_0^1 \int_0^c \left[\frac{r+a-2}{\mathscr{H}+b}\right]\left(\frac{1}{c}\right) dbda = \left(\frac{2r-3}{2c}\right)\left[\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right)\right] \tag{28}$$

where $\widetilde{\beta}_{EBQL2}$ is the E-Bayes estimate under QL function depending on $\pi_2(a,b)$.
and

$$\widetilde{\beta}_{EBQL3} = \int_0^1 \int_0^c \left[\frac{r+a-2}{\mathscr{H}+b}\right]\left(\frac{2(c-b)}{c^2}\right) dbda = \left(\frac{2r-3}{c^2}\right)\left[(c+\mathscr{H})\ln\left(\frac{\mathscr{H}+c}{\mathscr{H}}\right) - c\right] \tag{29}$$

where $\widetilde{\beta}_{EBQL3}$ is the E-Bayes estimate under QL function depending on $\pi_3(a,b)$.

4.4 E-Bayesian Estimation's Properties

This subsection talk about the relationship among the E-Bayes estimates $\widetilde{\beta}_{EBSELi}$ $(i=1,2,3)$, $\widetilde{\beta}_{EBDLi}$ $(i=1,2,3)$ and $\widetilde{\beta}_{EBQLi}$ $(i=1,2,3)$.

4.4.1 Relations among $\widetilde{\beta}_{\mathbf{EBSELi}}$ $(\mathbf{i}=\mathbf{1},\mathbf{2},\mathbf{3})$

Lemma 1. *Assume* $0<c<\mathscr{H}$ *and* $\widetilde{\beta}_{EBSELi}$ $(i=1,2,3)$ *be given by Eqs. (21), (22) and (23) then*

i. $\widetilde{\beta}_{EBSEL1} > \widetilde{\beta}_{EBSEL2} > \widetilde{\beta}_{EBSEL3}$
ii. $\lim_{\mathscr{H}\to\infty} \widetilde{\beta}_{EBSEL1} = \lim_{\mathscr{H}\to\infty} \widetilde{\beta}_{EBSEL2} = \lim_{\mathscr{H}\to\infty} \widetilde{\beta}_{EBSEL3}$

Proof. i.From (21), (22) and (23), we have

$$\widetilde{\beta}_{EBSEL2} - \widetilde{\beta}_{EBSEL1} = \widetilde{\beta}_{EBSEL3} - \widetilde{\beta}_{EBSEL2} = \left(\frac{2r+1}{2c}\right)\left[\left(1+\frac{2\mathscr{H}}{c}\right)\ln\left(1+\frac{c}{\mathscr{H}}\right) - 2\right] \tag{30}$$

For $-1<x<1$, we have:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots = \sum_{\mathbb{R}=1}^{\infty} -1^{\mathbb{R}-1}\frac{x^{\mathbb{R}}}{\mathbb{R}}.$$

Let $x=\frac{c}{\mathscr{H}}$, when $0<c<\mathscr{H}$, $0<\frac{c}{\mathscr{H}}<1$, we get:

$$\left(1+\frac{2\mathscr{H}}{c}\right)\left[\ln\left(1+\frac{c}{\mathscr{H}}\right)\right] - 2$$

$$= \left(1+\frac{2\mathscr{H}}{c}\right)\left[\left(\frac{c}{\mathscr{H}}\right) - \frac{1}{2}\left(\frac{c}{\mathscr{H}}\right)^2 + \frac{1}{3}\left(\frac{c}{\mathscr{H}}\right)^3 - \frac{1}{4}\left(\frac{c}{\mathscr{H}}\right)^4 + \frac{1}{5}\left(\frac{c}{\mathscr{H}}\right)^5 - \ldots\right] - 2$$

$$= \left[\begin{array}{l} \left(\frac{c}{\mathscr{H}}\right) - \frac{1}{2}\left(\frac{c}{\mathscr{H}}\right)^2 + \frac{1}{3}\left(\frac{c}{\mathscr{H}}\right)^3 - \frac{1}{4}\left(\frac{c}{\mathscr{H}}\right)^4 + \frac{1}{5}\left(\frac{c}{\mathscr{H}}\right)^5 - \ldots \\ +2 - \left(\frac{c}{\mathscr{H}}\right) + \frac{2}{3}\left(\frac{c}{\mathscr{H}}\right)^2 - \frac{2}{4}\left(\frac{c}{\mathscr{H}}\right)^3 + \frac{2}{5}\left(\frac{c}{\mathscr{H}}\right)^4 - \frac{2}{6}\left(\frac{c}{\mathscr{H}}\right)^5 \ldots \end{array}\right] - 2$$

$$= -\frac{1}{2}\left(\frac{c}{\mathscr{H}}\right)^2 + \frac{1}{3}\left(\frac{c}{\mathscr{H}}\right)^3 - \frac{1}{4}\left(\frac{c}{\mathscr{H}}\right)^4 + \frac{1}{5}\left(\frac{c}{\mathscr{H}}\right)^5 - \ldots + \frac{2}{3}\left(\frac{c}{\mathscr{H}}\right)^2 - \frac{2}{4}\left(\frac{c}{\mathscr{H}}\right)^3 + \frac{2}{5}\left(\frac{c}{\mathscr{H}}\right)^4 - \frac{2}{6}\left(\frac{c}{\mathscr{H}}\right)^5 + \ldots$$

$$= \left(\frac{2}{3} - \frac{1}{2}\right)\left(\frac{c}{\mathscr{H}}\right)^2 - \left(\frac{1}{2} - \frac{1}{3}\right)\left(\frac{c}{\mathscr{H}}\right)^3 + \left(\frac{2}{5} - \frac{1}{4}\right)\left(\frac{c}{\mathscr{H}}\right)^4 - \left(\frac{2}{6} - \frac{1}{5}\right)\left(\frac{c}{\mathscr{H}}\right)^5 + \ldots$$

$$= \left(\frac{c^2}{6\mathscr{H}^2} - \frac{c^3}{6\mathscr{H}^3}\right) + \left(\frac{3c^4}{20\mathscr{H}^4} - \frac{2c^5}{15\mathscr{H}^5}\right) + \ldots$$

$$= \frac{c^2}{6\mathscr{H}^2}\left(1 - \frac{c}{\mathscr{H}}\right) + \frac{c^4}{60\mathscr{H}^4}\left(9 - \frac{8c}{\mathscr{H}}\right) + \ldots \tag{31}$$

According to (30) and (31), we get

$$\widetilde{\beta}_{EBSEL2} - \widetilde{\beta}_{EBSEL1} = \widetilde{\beta}_{EBSEL3} - \widetilde{\beta}_{EBSEL2} > 0,$$

that is

$$\widetilde{\beta}_{EBSEL1} < \widetilde{\beta}_{EBSEL2} < \widetilde{\beta}_{EBSEL3}.$$

ii.From (30) and (31), we get

$$\lim_{\mathscr{H}\to\infty}\left(\widetilde{\beta}_{EBSEL2} - \widetilde{\beta}_{EBSEL1}\right) = \lim_{\mathscr{H}\to\infty}\left(\widetilde{\beta}_{EBSEL3} - \widetilde{\beta}_{EBSEL2}\right)$$

$$= \left(\frac{2r+1}{2c}\right)\lim_{\mathscr{H}\to\infty}\left\{\frac{c^2}{6\mathscr{H}^2}\left(1 - \frac{c}{\mathscr{H}}\right) + \frac{c^4}{60\mathscr{H}^4}\left(9 - \frac{8c}{\mathscr{H}}\right) + \ldots\right\} = 0$$

That is,

$$\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBSEL1} = \lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBSEL2} = \lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBSEL3}.$$

4.4.2 Relations among $\widetilde{\beta}_{\mathbf{EBDLi}}$ $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3})$

Lemma 2. *Assume* $0 < c < \mathscr{H}$ *and* $\widetilde{\beta}_{EBDLi}$ $(i = 1, 2, 3)$ *be given by Eqs. (24), (25) and (26) then*

i. $\widetilde{\beta}_{EBDL1} > \widetilde{\beta}_{EBDL2} > \widetilde{\beta}_{EBDL3}$
ii. $\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBDL1} = \lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBDL2} = \lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBDL3}$

Proof. i.From (24), (25) and (26), we obtain

$$\widetilde{\beta}_{EBDL2} - \widetilde{\beta}_{EBDL1} = \widetilde{\beta}_{EBDL3} - \widetilde{\beta}_{EBDL2} = \left(\frac{2r+3}{2c}\right)\left[\left(1 + \frac{2\mathscr{H}}{c}\right)\ln\left(1 + \frac{c}{\mathscr{H}}\right) - 2\right] \tag{32}$$

Substituting (31) and (32), we obtain

$$\left(1 + \frac{2\mathscr{H}}{c}\right)\ln\left(1 + \frac{c}{\mathscr{H}}\right) - 2 = \frac{c^2}{6\mathscr{H}^2}\left(1 - \frac{c}{\mathscr{H}}\right) + \frac{c^4}{60\mathscr{H}^4}\left(9 - \frac{8c}{\mathscr{H}}\right) + \ldots \tag{33}$$

According to (32) and (33), we get

$$\widetilde{\beta}_{EBDL2} - \widetilde{\beta}_{EBDL1} = \widetilde{\beta}_{EBDL3} - \widetilde{\beta}_{EBDL2} > 0,$$

that is

$$\widetilde{\beta}_{EBDL1} < \widetilde{\beta}_{EBDL2} < \widetilde{\beta}_{EBDL3}.$$

ii.From (31) and (32), we get

$$\lim_{\mathscr{H}\to\infty}\left(\widetilde{\beta}_{EBDL2}-\widetilde{\beta}_{EBDL1}\right)=\lim_{\mathscr{H}\to\infty}\left(\widetilde{\beta}_{EBDL3}-\widetilde{\beta}_{EBDL2}\right)$$
$$=\left(\frac{2r+3}{2c}\right)\lim_{\mathscr{H}\to\infty}\left\{\frac{c^2}{6\mathscr{H}^2}\left(1-\frac{c}{\mathscr{H}}\right)+\frac{c^4}{60\mathscr{H}^4}\left(9-\frac{8c}{\mathscr{H}}\right)+\dots\right\}=0$$

That is,

$$\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBDL1}=\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBDL2}=\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBDL3}.$$

4.4.3 Relations among $\widetilde{\beta}_{\mathbf{EBQLi}}$ $(\mathbf{i}=\mathbf{1},\mathbf{2},\mathbf{3})$

Lemma 3.*Lemma 3.Assume* $0<c<\mathscr{H}$ *and* $\widetilde{\beta}_{EBQLi}$ $(i=1,2,3)$ *be given by Eqs. (27), (28) and (29) then*

i.$\widetilde{\beta}_{EBQL1} > \widetilde{\beta}_{EBQL2} > \widetilde{\beta}_{EBQL3}$
ii.$\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBQL1}=\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBQL2}=\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBQL3}$

Proof. i.From (27), (28) and (29), we obtain

$$\widetilde{\beta}_{EBQL2}-\widetilde{\beta}_{EBQL1}=\widetilde{\beta}_{EBQL3}-\widetilde{\beta}_{EBQL2}=\left(\frac{2r-3}{2c}\right)\left[\left(1+\frac{2\mathscr{H}}{c}\right)\ln\left(1+\frac{c}{\mathscr{H}}\right)-2\right] \tag{34}$$

Substituting (31) and (34), we obtain

$$\left(1+\frac{2\mathscr{H}}{c}\right)\ln\left(1+\frac{c}{\mathscr{H}}\right)-2=\frac{c^2}{6\mathscr{H}^2}\left(1-\frac{c}{\mathscr{H}}\right)+\frac{c^4}{60\mathscr{H}^4}\left(9-\frac{8c}{\mathscr{H}}\right)+\dots \tag{35}$$

According to (34) and (35), we get

$$\widetilde{\beta}_{EBQL2}-\widetilde{\beta}_{EBQL1}=\widetilde{\beta}_{EBQL3}-\widetilde{\beta}_{EBQL2}>0,$$

that is

$$\widetilde{\beta}_{EBQL1} < \widetilde{\beta}_{EBQL2} < \widetilde{\beta}_{EBQL3}.$$

ii.From (34) and (35), we get

$$\lim_{\mathscr{H}\to\infty}\left(\widetilde{\beta}_{EBQL2}-\widetilde{\beta}_{EBQL1}\right)=\lim_{\mathscr{H}\to\infty}\left(\widetilde{\beta}_{EBQL3}-\widetilde{\beta}_{EBQL2}\right)$$
$$=\left(\frac{2r-3}{2c}\right)\lim_{\mathscr{H}\to\infty}\left\{\frac{c^2}{6\mathscr{H}^2}\left(1-\frac{c}{\mathscr{H}}\right)+\frac{c^4}{60\mathscr{H}^4}\left(9-\frac{8c}{\mathscr{H}}\right)+\dots\right\}=0$$

That is,

$$\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBQL1}=\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBQL2}=\lim_{\mathscr{H}\to\infty}\widetilde{\beta}_{EBQL3}.$$

5 Confidence Intervals (CIs)

We introduce three types of CIs. The first one is depending on the asymptotic distribution of $\widehat{\beta}$, while the second one are the bootstrap CIs, and finally, the highest posterior density credible (HPDC) intervals.

5.1 The Approximate CIs

We drive the approximate CIs of the unknown parameter β according to the asymptotic distributions of the MLE of the parameter because we cannot derive the distribution of this estimator.

The second derivative for $\mathscr{L}(x|\beta)$ is obtained by

$$\frac{d^2\mathscr{L}(x|\beta)}{d\beta^2} = -\frac{r}{\beta^2}.$$

The Fisher information matrix is

$$I\left(\widehat{\beta}\right) = -\frac{d^2\mathscr{L}(x|\beta)}{d\beta^2}\bigg|_{\beta=\widehat{\beta}} = \frac{r}{\widehat{\beta}^2}.$$

The variance of $\widehat{\beta}$ is

$$V\left(\widehat{\beta}\right) = \frac{1}{I(\widehat{\beta})} = \frac{\widehat{\beta}^2}{r}.$$

Assume the regularity conditions are satisfied, the sampling distribution of $\frac{\widehat{\beta}-\beta}{\sqrt{V\left(\widehat{\beta}\right)}}$ a standard normal distribution employed in approximate. A large $(1-\alpha)100\%$ CIs for β is provided by

$$\widehat{\beta}_L, \widehat{\beta}_U = \widehat{\beta} \mp Z_{\frac{\alpha}{2}}\sqrt{V\left(\widehat{\beta}\right)}.$$

where $(1-\alpha)$ is the confidence coefficient and $Z_{\frac{\alpha}{2}}$ is the standard normal random variable.

5.2 Bootstrap Confidence Intervals

Although the bootstrap CIs are approximations, they are generally more accurate than standard intervals. Compared to a point estimate, a parametric bootstrap interval and other confidence intervals reveal far more information regarding the populace value of the amount of attention. There are two main forms of parametric bootstrap techniques:

(i)The percentile bootstrap technique (Boot-p) was introduced by Efron [20].
(ii)The bootstrap-t technique (Boot-t) was introduced by Hall [24].

– **Boot-P Intervals**

The boot-p technique is relatively simple and builds CIs directly from the estimated parameters' percentiles of the bootstrap distribution. The following steps give it:

I.A sample of size n is generated from ITL distribution by using the true value (β_0).
II.A PT-IC sample is generated from the generated sample $X = x_1, x_2 \ldots x_n$ and then computes the MLE $\widehat{\beta}$ of the parameter β as will be shown in Section 6.
III.Again, an independent bootstrap sample $X^* = x_1{}^*, x_2{}^* \ldots x_n{}^*$ is generated using $\widehat{\beta}$.
IV.A PT-IC sample is generated from the previous bootstrap sample, and then the bootstrap MLE $\widehat{\beta}^*$ of parameter β is computed.
V.Repeat steps III-IV, B times representing B bootstrap MLEs $\widehat{\beta}^*$ based on B different bootstrap samples, $i = 1, 2, \ldots, B$.
VI.Arrange all $\widehat{\beta}^*$ in ascending order to obtain the bootstrap sample.i.e., $\widehat{\beta}_1^* \leq \widehat{\beta}_{(2)}^* \leq \ldots \leq \widehat{\beta}_B^*$.
VII.An approximate $100(1-\omega)\%$ boot-p confidence interval for β is obtained by

$$\left(\widehat{\beta}^*_{\left[\left(\frac{\omega}{2}\right)\times B\right]}, \widehat{\beta}^*_{\left[\left(1-\frac{\omega}{2}\right)\times B\right]}\right).$$

Where $\frac{\omega}{2}$ is the quantity that helps to determine the bootstrap points.

– **Boot-t Intervals**

The following steps give the Boot-t confidence interval:

I. Steps I to IV of the Boot-p and Boot-t techniques are the same.
II. Determine the Boot-t statistic for β as follows

$$T^* = \frac{\widehat{\beta}_b^* - \widehat{\beta}}{\sqrt{v\widehat{\beta}_b^*}}$$

for $\widehat{\beta}_b^*$ where b = 1,2,... B.

III. To get a set of Boot-t statistics, $T^*{}_i; i = 1,2,\ldots,\ B$ repeat steps 2 and 3, B times.
IV. Assume $T^*{}_{(1)} \le T^*{}_{(2)} \le \ldots \le T^*{}_{(B)}$ be the values in order of $T^*{}_i; i = 1,2,\ldots,B$.
V. Then, the approximate $100(1-\omega)\%$ Boot-t CIs for parameter β is given by

$$\left(\widehat{\beta} - \widehat{T}^*_{[(1-\frac{\omega}{2})\times B]}\sqrt{v\left(\widehat{\beta}\right)}, \widehat{\beta} - \widehat{T}^*_{[(\frac{\omega}{2})\times B]}\sqrt{v\left(\widehat{\beta}\right)}\right)$$

5.3 HPDC Intervals.

We generate MCMC samples from the posterior distribution (10) to calculate the HPDC intervals for parameter β under Bayesian estimation by using the following algorithm (see Chen and Shao [34]):

1. Generate β from (10) $[gamma\ (r+a, b+\mathscr{H})]$.
2. Repeat step 1 M times and obtain $\beta_1,\ldots,\ \beta_M$.
3. Sort $\left\{\beta_{(j)},\ j = 1,2,\ldots M\right\}$ to get the values in order as $\widetilde{\beta}_{(1)} < \widetilde{\beta}_{(2)} < \ldots < \widetilde{\beta}_{(M)}$.
4. For $0 < \gamma < 1$, compute the $100 \times (1-\gamma)\%$ credible intervals $\left(\widetilde{\beta}_{(j)}, \widetilde{\beta}_{((j)+[(1-\gamma)M])}\right)$; for $j = 1,2,\ldots,M-[(1-\gamma)M]$.

   Here, $[x]$ denotes the integer part of x.

5. The $100 \times (1-\gamma)\%$ HPDC interval is the one with the smallest interval width between all credible intervals obtained in Step 3.

   For computing the above algorithm, "HDInterval" package of R can be implemented.

6 Simulation Study

The several estimation techniques for points and intervals for the ITL distribution's parameter that were discussed in the previous sections are compared in this section. In order to verify the conduct of the recommended techniques and evaluate the statistical results of the estimators of the ITL distribution based on the PT-IC data, we utilized the Monte Carlo simulation. All computations will be carried out with the use of the statistical programming language R. The following procedures are accustomed to get the outcomes of the simulation for the MLEs, Bayesian, and E-Bayesian estimates: The following steps obtain the simulations results for MLE, Bayesian and E-Bayesian estimation:

i. The ITL distribution's parameter β was estimated for a number of replications $k = 5000$ for sample sizes $n = 25,\ 50,\ 100$.
ii. Specify true values for parameter β as 0.3, 0.5, 1, 1.5, 3 and 5.
iii. Generate n standard uniform variates i.e. $U \sim Uniform\ 0,\ 1$.
iv. Generate a sample of size n from the ITL β distribution through the use of the equation

$$x = 1 - u^{-\frac{1}{\beta}}\sqrt{1-1-u^{\frac{1}{\beta}}}\left[1+\sqrt{1-1-u^{\frac{1}{\beta}}}\right].$$

v. Determine the PT-IC data as follow:

1.We have three stages of PT-IC are $m = 3,\ 4,\ 5$.
2.Determine the censoring times $T_1,\ T_2,\ \ldots, T_m$ based on the following quantile function:
3.When $m = 3$ thus $h = 10\%,\ 40\%,\ 60\%$,
4.When $m = 4$ thus $h = 10\%,\ 30\%,\ 50\%,\ 80\%$,
5.When $m = 5$ thus $h = 10\%,\ 25\%,\ 40\%, 60\%,\ 80\%$.

where $j = 1,\ 2,\ \ldots,\ m$.

6.The withdrawn units R_h are assumed with different sample sizes in Table (1).

vi.Obtain the MLE $\widehat{\beta}$ from (8).
vii.Compute the Bayes and E-Bayes estimates as follow

1. For Bayesian estimation we compute the hyper parameters a and b by using the equation in (15).
2. For E-Bayesian estimation, let $c = 2$ upper limit of the hyper parameter b.
3. Under the SEL function, we get the estimates $\widetilde{\beta}_{BSEL}$, $\widetilde{\beta}_{EBSEL1}$, $\widetilde{\beta}_{EBSEL2}$ and $\widetilde{\beta}_{EBSEL3}$ from (11), (21), (22), (23).
4. Under the DL function, we get the estimates $\widetilde{\beta}_{BDL}$, $\widetilde{\beta}_{EBDL1}$, $\widetilde{\beta}_{EBDL2}$ and $\widetilde{\beta}_{EBDL3}$ from (12), (24), (25), (26).
5. Under the SEL function, we get the estimates $\widetilde{\beta}_{BQL}$, $\widetilde{\beta}_{EBQL1}$, $\widetilde{\beta}_{EBQL2}$ and $\widetilde{\beta}_{EBQL3}$ from (14), (27), (28), (29).

viii.Obtain the mean squared error MSEs, bias for MLE, Bayesian and E-Bayesian estimation
ix.Average interval lengths (AILs) and coverage probability (CP) are computed for approximate CIs, HPDC intervals and Bootstrap CIs.
x.Repeat steps i-ix 5000 times.

From tabulated values in Tables (2)-(9), it can be noticed that:

i.Based on the results that shown in Tables (2)-(9), the Bayesian and E-Bayesian estimation have smaller MSEs as compare with MLE in all stages and all true value 0.5, 1, 1.5.
ii.Depending on MSEs, higher values of n lead to better estimates.
iii.Through the true value 1.5 for the unknown parameter under Type I censoring data the first scheme I in Table (1) we note that, the E-Bayes estimates under SEL function, DL function and QL function have smaller MSEs compare with the corresponding Bayes estimates in all stages $m = 3, 4, 5$ but the opposite in scheme II PT-IC data.
iv.Through the true value 0.5 and 1 for the unknown parameter we note that, the Bayes estimates under SEL function, DL function and QL function have smaller MSEs and bias compare with the corresponding E-Bayes estimates in all stages $m = 3, 4, 5$ except some cases the E-Bayes estimates under QL function have small MSEs and bias as compare with the corresponding Bayes estimate.
v.In comparing the different E-Bayes estimates, we note that the E-Bayes estimates under SEL function, DL function and QL function can be ordered due to having smaller MSEs and bias to be $\widetilde{\beta}_{EBQL} < \widetilde{\beta}_{EBSEL} < \widetilde{\beta}_{EBDL}$.
vi.In comparing the E-Bayes estimates under three different distributions for the shape parameters $a,\ b$, we note that the E-Bayes estimates under SEL function, DL function and QL function can be ordered due to having smaller MSEs and bias to be $\widetilde{\beta}_{EBSEL1} < \widetilde{\beta}_{EBSEL2} < \widetilde{\beta}_{EBSEL3}$, $\widetilde{\beta}_{EBDL1} < \widetilde{\beta}_{EBDL2} < \widetilde{\beta}_{EBDL3}$ and $\widetilde{\beta}_{EBQL1} < \widetilde{\beta}_{EBQL2} < \widetilde{\beta}_{EBQL3}$.
vii.It can also be noticed that the CPs and AILs under the HPDC intervals are better than those of approximate CIs for MLE and bootstrap p, t, respectively.

6.1 Tables

m	Schemes	Removed items		
		Removed items R_h at different sample size		
		N=25	N=50	N=100
3	I	0,0,R_m	0,0,R_m	0,0,R_m
	II	2,2,R_m	6,6,R_m	10,10,R_m
4	III	0,0,0,R_m	0,0,0,R_m	0,0,0,R_m
	IV	2,2,2,R_m	6,6,6,R_m	10,10,10,R_m
5	V	0,0,0,0,R_m	0,0,0,0,R_m	0,0,0,0,R_m
	VI	2,2,2,2,R_m	6,6,6,6,R_m	8,8,8,8,R_m

Table 1: The removed units at different stages during the test with different sample size.

n	m	SC	S	MLE	BSEL	E-BSEL1	E-BSEL2	E-BSEL3	BDL	E-BDL1	E-BDL2	E-BSDL3	BQL	E-BQL1	E-BQL2	E-BQL3
25	**3**	**I**	**MSE**	**0.0030**	**0.0015**	**0.0032**	**0.0033**	**0.0033**	**0.0019**	**0.0042**	**0.0043**	**0.0044**	**0.0009**	**0.0017**	**0.0018**	**0.0018**
			Bias	**0.0363**	**0.0334**	**0.0390**	**0.0395**	**0.0399**	**0.0381**	**0.0480**	**0.0485**	**0.0490**	**0.0242**	**0.0211**	**0.0215**	**0.0219**
		II	**MSE**	**0.0015**	**0.0005**	**0.0016**	**0.0017**	**0.0017**	**0.0007**	**0.0023**	**0.0023**	**0.0023**	**0.0003**	**0.0009**	**0.0009**	**0.0010**
			Bias	**0.0167**	**0.0144**	**0.0197**	**0.0201**	**0.0205**	**0.0191**	**0.0288**	**0.0292**	**0.0296**	**0.0050**	**0.0015**	**0.0018**	**0.0021**
	4	**III**	**MSE**	**0.0027**	**0.0013**	**0.0028**	**0.0029**	**0.0029**	**0.0015**	**0.0035**	**0.0036**	**0.0036**	**0.0009**	**0.0018**	**0.0018**	**0.0018**
			Bias	**0.0331**	**0.0304**	**0.0351**	**0.0355**	**0.0358**	**0.0338**	**0.0417**	**0.0421**	**0.0424**	**0.0236**	**0.0220**	**0.0223**	**0.0226**
		IV	**MSE**	**0.0008**	**0.0002**	**0.0008**	**0.0008**	**0.0008**	**0.0002**	**0.0010**	**0.0010**	**0.0010**	**0.0002**	**0.0007**	**0.0007**	**0.0007**
			Bias	**0.0032**	**0.0017**	**0.0055**	**0.0058**	**0.0060**	**0.0051**	**0.0121**	**0.0124**	**0.0126**	**0.0050**	**0.0076**	**0.0074**	**0.0072**
	5	**V**	**MSE**	**0.0012**	**0.0005**	**0.0013**	**0.0013**	**0.0013**	**0.0006**	**0.0016**	**0.0016**	**0.0017**	**0.0003**	**0.0007**	**0.0007**	**0.0008**
			Bias	**0.0194**	**0.0179**	**0.0214**	**0.0217**	**0.0219**	**0.0209**	**0.0273**	**0.0276**	**0.0279**	**0.0118**	**0.0096**	**0.0098**	**0.0100**
		VI	**MSE**	**0.0008**	**0.0002**	**0.0008**	**0.0008**	**0.0008**	**0.0002**	**0.0009**	**0.0009**	**0.0009**	**0.0004**	**0.0009**	**0.0009**	**0.0009**
			Bias	**0.0058**	**0.0075**	**0.0034**	**0.0032**	**0.0029**	**0.0041**	**0.0032**	**0.0035**	**0.0037**	**0.0143**	**0.0167**	**0.0165**	**0.0163**
50	**3**	**I**	**MSE**	**0.0028**	**0.0020**	**0.0029**	**0.0029**	**0.0029**	**0.0022**	**0.0034**	**0.0034**	**0.0035**	**0.0016**	**0.0020**	**0.0020**	**0.0021**
			Bias	**0.0434**	**0.0419**	**0.0448**	**0.0451**	**0.0453**	**0.0443**	**0.0496**	**0.0498**	**0.0501**	**0.0371**	**0.0353**	**0.0355**	**0.0358**
		II	**MSE**	**0.0005**	**0.0001**	**0.0005**	**0.0005**	**0.0005**	**0.0001**	**0.0006**	**0.0006**	**0.0006**	**0.0001**	**0.0004**	**0.0004**	**0.0004**
			Bias	**0.0033**	**0.0024**	**0.0049**	**0.0051**	**0.0053**	**0.0046**	**0.0094**	**0.0096**	**0.0097**	**0.0022**	**0.0040**	**0.0039**	**0.0037**
	4	**III**	**MSE**	**0.0016**	**0.0011**	**0.0017**	**0.0017**	**0.0017**	**0.0012**	**0.0019**	**0.0019**	**0.0019**	**0.0009**	**0.0012**	**0.0012**	**0.0012**
			Bias	**0.0316**	**0.0306**	**0.0327**	**0.0328**	**0.0330**	**0.0322**	**0.0359**	**0.0361**	**0.0362**	**0.0272**	**0.0261**	**0.0262**	**0.0264**
		IV	**MSE**	**0.0004**	**0.0002**	**0.0004**	**0.0004**	**0.0004**	**0.0002**	**0.0003**	**0.0003**	**0.0003**	**0.0003**	**0.0005**	**0.0005**	**0.0005**
			Bias	**0.0133**	**0.0139**	**0.0121**	**0.0120**	**0.0119**	**0.0122**	**0.0088**	**0.0087**	**0.0086**	**0.0172**	**0.0186**	**0.0185**	**0.0184**
	5	**V**	**MSE**	**0.0009**	**0.0005**	**0.0009**	**0.0010**	**0.0010**	**0.0006**	**0.0011**	**0.0011**	**0.0011**	**0.0004**	**0.0007**	**0.0007**	**0.0007**
			Bias	**0.0218**	**0.0209**	**0.0228**	**0.0229**	**0.0230**	**0.0225**	**0.0258**	**0.0260**	**0.0261**	**0.0179**	**0.0167**	**0.0169**	**0.0170**
		VI	**MSE**	**0.0005**	**0.0002**	**0.0005**	**0.0005**	**0.0005**	**0.0002**	**0.0006**	**0.0006**	**0.0007**	**0.0001**	**0.0004**	**0.0004**	**0.0004**
			Bias	**0.0108**	**0.0100**	**0.0119**	**0.0120**	**0.0122**	**0.0116**	**0.0151**	**0.0153**	**0.0154**	**0.0067**	**0.0054**	**0.0056**	**0.0057**
100	**3**	**I**	**MSE**	**0.0020**	**0.0017**	**0.0021**	**0.0021**	**0.0021**	**0.0018**	**0.0023**	**0.0023**	**0.0023**	**0.0015**	**0.0017**	**0.0017**	**0.0017**
			Bias	**0.0403**	**0.0396**	**0.0410**	**0.0411**	**0.0412**	**0.0408**	**0.0433**	**0.0434**	**0.0436**	**0.0373**	**0.0363**	**0.0364**	**0.0366**
		II	**MSE**	**0.0004**	**0.0002**	**0.0004**	**0.0004**	**0.0004**	**0.0002**	**0.0005**	**0.0005**	**0.0005**	**0.0001**	**0.0003**	**0.0003**	**0.0003**
			Bias	**0.0100**	**0.0094**	**0.0108**	**0.0109**	**0.0110**	**0.0105**	**0.0130**	**0.0131**	**0.0132**	**0.0071**	**0.0063**	**0.0063**	**0.0064**
	4	**III**	**MSE**	**0.0011**	**0.0009**	**0.0012**	**0.0012**	**0.0012**	**0.0010**	**0.0013**	**0.0013**	**0.0013**	**0.0008**	**0.0010**	**0.0010**	**0.0010**
			Bias	**0.0301**	**0.0297**	**0.0306**	**0.0307**	**0.0308**	**0.0305**	**0.0322**	**0.0323**	**0.0324**	**0.0281**	**0.0274**	**0.0274**	**0.0275**
		IV	**MSE**	**0.0002**	**0.0064**	**0.0002**	**0.0064**	**0.0002**	**0.0064**	**0.0002**	**0.0064**	**0.0002**	**0.0064**	**0.0002**	**0.0064**	**0.0002**
			Bias	**0.0001**	**0.0067**	**0.0001**	**0.0067**	**0.0001**	**0.0067**	**0.0001**	**0.0067**	**0.0001**	**0.0067**	**0.0001**	**0.0067**	**0.0001**
	5	**V**	**MSE**	**0.0007**	**0.0005**	**0.0007**	**0.0007**	**0.0007**	**0.0006**	**0.0008**	**0.0008**	**0.0008**	**0.0005**	**0.0006**	**0.0006**	**0.0006**
			Bias	**0.0222**	**0.0218**	**0.0227**	**0.0228**	**0.0229**	**0.0226**	**0.0243**	**0.0243**	**0.0244**	**0.0203**	**0.0197**	**0.0197**	**0.0198**
		VI	**MSE**	**0.0004**	**0.0003**	**0.0004**	**0.0004**	**0.0005**	**0.0003**	**0.0005**	**0.0005**	**0.0005**	**0.0002**	**0.0003**	**0.0004**	**0.0004**
			Bias	**0.0152**	**0.0147**	**0.0157**	**0.0158**	**0.0158**	**0.0155**	**0.0172**	**0.0173**	**0.0174**	**0.0132**	**0.0126**	**0.0126**	**0.0127**

Note: SC- Schemes; S- Statistics

Table 2: MSEs and bias for MLE, Bayes and E-Bayes Estimate under SEL, DL and QL function of β under PT-IC data at $\beta_0 = 0.1$.

n	m	SC	S	MLE	BSEL	E-BSEL1	E-BSEL2	E-BSEL3	BDL	E-BDL1	E-BDL2	E-BSDL3	BQL	E-BQL1	E-BQL2	E-BQL3
25	3	I	MSE	0.091	0.048	0.072	0.080	0.087	0.059	0.096	0.105	0.114	0.030	0.038	0.043	0.047
			Bias	0.207	0.192	0.185	0.196	0.207	0.216	0.229	0.241	0.253	0.144	0.096	0.106	0.116
		II	MSE	0.041	0.013	0.033	0.036	0.039	0.018	0.044	0.049	0.054	0.007	0.021	0.022	0.023
			Bias	0.084	0.072	0.070	0.079	0.088	0.096	0.113	0.123	0.132	0.025	0.016	0.009	0.001
	4	III	MSE	0.055	0.031	0.047	0.051	0.054	0.037	0.060	0.065	0.069	0.021	0.027	0.030	0.032
			Bias	0.166	0.157	0.152	0.160	0.167	0.173	0.184	0.192	0.200	0.123	0.089	0.095	0.102
		IV	MSE	0.019	0.005	0.016	0.017	0.018	0.006	0.020	0.022	0.023	0.004	0.015	0.015	0.016
			Bias	0.020	0.012	0.013	0.019	0.025	0.030	0.045	0.051	0.058	0.022	0.052	0.046	0.041
	5	V	MSE	0.040	0.019	0.034	0.037	0.039	0.023	0.044	0.047	0.050	0.012	0.020	0.022	0.023
			Bias	0.124	0.115	0.113	0.119	0.126	0.131	0.142	0.149	0.156	0.083	0.053	0.059	0.065
		VI	MSE	0.016	0.004	0.014	0.014	0.015	0.004	0.015	0.016	0.017	0.007	0.019	0.018	0.018
			Bias	0.021	0.028	0.027	0.021	0.016	0.011	0.006	0.011	0.017	0.063	0.092	0.087	0.082
50	3	I	MSE	0.060	0.040	0.054	0.056	0.059	0.045	0.064	0.067	0.070	0.031	0.037	0.039	0.041
			Bias	0.194	0.186	0.183	0.189	0.194	0.198	0.206	0.211	0.217	0.163	0.138	0.144	0.149
		II	MSE	0.013	0.003	0.012	0.012	0.012	0.004	0.013	0.014	0.015	0.003	0.011	0.011	0.011
			Bias	0.018	0.013	0.013	0.017	0.021	0.024	0.035	0.039	0.043	0.010	0.030	0.027	0.023
	4	III	MSE	0.037	0.026	0.035	0.036	0.037	0.028	0.040	0.042	0.043	0.021	0.025	0.026	0.027
			Bias	0.155	0.151	0.149	0.153	0.156	0.159	0.165	0.169	0.172	0.134	0.117	0.120	0.124
		IV	MSE	0.010	0.006	0.010	0.010	0.010	0.005	0.009	0.009	0.008	0.009	0.015	0.015	0.014
			Bias	0.067	0.070	0.068	0.066	0.063	0.061	0.052	0.049	0.047	0.086	0.100	0.098	0.096
	5	V	MSE	0.024	0.015	0.023	0.024	0.025	0.017	0.027	0.028	0.029	0.012	0.016	0.017	0.017
			Bias	0.115	0.111	0.110	0.113	0.117	0.119	0.125	0.129	0.132	0.096	0.080	0.083	0.086
		VI	MSE	0.018	0.015	0.018	0.017	0.017	0.013	0.015	0.014	0.014	0.019	0.026	0.026	0.025
			Bias	0.113	0.116	0.114	0.112	0.109	0.107	0.097	0.094	0.092	0.134	0.148	0.146	0.144
100	3	I	MSE	0.047	0.038	0.045	0.046	0.047	0.040	0.050	0.051	0.052	0.033	0.036	0.037	0.038
			Bias	0.191	0.187	0.186	0.189	0.191	0.193	0.197	0.200	0.203	0.176	0.163	0.166	0.168
		II	MSE	0.008	0.003	0.008	0.008	0.008	0.004	0.009	0.009	0.010	0.002	0.006	0.006	0.006
			Bias	0.043	0.040	0.040	0.042	0.044	0.046	0.051	0.053	0.056	0.029	0.018	0.020	0.022
	4	III	MSE	0.030	0.024	0.029	0.029	0.030	0.025	0.031	0.032	0.033	0.022	0.024	0.024	0.025
			Bias	0.153	0.150	0.150	0.151	0.153	0.154	0.158	0.160	0.161	0.142	0.134	0.135	0.137
		IV	MSE	0.004	0.002	0.004	0.004	0.004	0.002	0.004	0.004	0.004	0.003	0.006	0.005	0.005
			Bias	0.033	0.035	0.034	0.033	0.032	0.031	0.026	0.025	0.023	0.043	0.050	0.049	0.048
	5	V	MSE	0.017	0.013	0.016	0.017	0.017	0.013	0.018	0.019	0.019	0.011	0.013	0.014	0.014
			Bias	0.108	0.106	0.106	0.107	0.109	0.110	0.113	0.115	0.116	0.098	0.091	0.092	0.094
		VI	MSE	0.045	0.043	0.044	0.043	0.042	0.039	0.037	0.037	0.036	0.051	0.059	0.058	0.057
			Bias	0.203	0.205	0.202	0.200	0.198	0.196	0.184	0.182	0.180	0.224	0.237	0.236	0.234

Note: SC- Schemes; S- Statistics

Table 3: MSEs and bias for MLE, Bayes and E-Bayes Estimate under SEL, DL and QL function of β under PT-IC data at $\beta_0 = 0.5$.

n	m	SC	S	MLE	BSEL	E-BSEL1	E-BSEL2	E-BSEL3	BDL	E-BDL1	E-BDL2	E-BSDL3	BQL	E-BQL1	E-BQL2	E-BQL3
25	3	I	MSE	0.350	0.188	0.196	0.237	0.283	0.230	0.265	0.318	0.375	0.118	0.101	0.123	0.149
			Bias	0.411	0.382	0.286	0.327	0.368	0.430	0.369	0.413	0.456	0.286	0.120	0.156	0.191
		II	MSE	0.161	0.053	0.092	0.110	0.130	0.071	0.123	0.149	0.178	0.029	0.071	0.076	0.082
			Bias	0.171	0.148	0.077	0.111	0.144	0.195	0.159	0.195	0.230	0.054	0.086	0.057	0.029
	4	III	MSE	0.214	0.121	0.138	0.160	0.184	0.144	0.180	0.207	0.236	0.081	0.079	0.092	0.107
			Bias	0.327	0.308	0.245	0.273	0.301	0.341	0.306	0.335	0.364	0.240	0.124	0.149	0.174
		IV	MSE	0.071	0.017	0.052	0.057	0.062	0.021	0.059	0.067	0.076	0.017	0.062	0.060	0.060
			Bias	0.033	0.018	0.024	0.002	0.020	0.052	0.037	0.060	0.084	0.050	0.147	0.127	0.108
	5	V	MSE	0.156	0.074	0.101	0.117	0.134	0.091	0.132	0.152	0.173	0.047	0.061	0.070	0.079
			Bias	0.245	0.228	0.175	0.199	0.224	0.259	0.232	0.258	0.284	0.164	0.060	0.082	0.105
		VI	MSE	0.062	0.018	0.054	0.055	0.057	0.016	0.052	0.056	0.061	0.029	0.083	0.079	0.076
			Bias	0.046	0.060	0.097	0.077	0.056	0.026	0.036	0.014	0.008	0.129	0.221	0.204	0.186
50	3	I	MSE	0.244	0.165	0.181	0.200	0.220	0.184	0.217	0.239	0.261	0.130	0.122	0.136	0.151
			Bias	0.394	0.378	0.332	0.352	0.373	0.402	0.375	0.397	0.418	0.331	0.244	0.264	0.283
		II	MSE	0.049	0.012	0.039	0.042	0.045	0.014	0.043	0.047	0.052	0.011	0.042	0.041	0.042
			Bias	0.032	0.023	0.006	0.009	0.023	0.045	0.036	0.051	0.067	0.023	0.091	0.077	0.063
	4	III	MSE	0.155	0.108	0.123	0.133	0.143	0.119	0.143	0.155	0.167	0.088	0.087	0.095	0.103
			Bias	0.318	0.308	0.277	0.291	0.305	0.325	0.309	0.323	0.338	0.275	0.214	0.228	0.241
		IV	MSE	0.041	0.024	0.044	0.042	0.041	0.020	0.037	0.036	0.035	0.034	0.064	0.062	0.059
			Bias	0.129	0.136	0.151	0.142	0.132	0.119	0.120	0.110	0.100	0.169	0.215	0.206	0.197
	5	V	MSE	0.095	0.058	0.075	0.082	0.088	0.065	0.089	0.097	0.104	0.045	0.052	0.057	0.062
			Bias	0.225	0.216	0.190	0.203	0.215	0.231	0.220	0.232	0.245	0.185	0.132	0.143	0.155
		VI	MSE	0.072	0.060	0.078	0.075	0.072	0.052	0.065	0.062	0.059	0.077	0.113	0.109	0.105
			Bias	0.228	0.233	0.246	0.237	0.229	0.216	0.213	0.204	0.195	0.269	0.313	0.305	0.297
100	3	I	MSE	0.186	0.150	0.159	0.168	0.177	0.159	0.177	0.186	0.195	0.133	0.128	0.135	0.143
			Bias	0.381	0.374	0.350	0.361	0.371	0.385	0.373	0.383	0.394	0.351	0.306	0.316	0.326
		II	MSE	0.033	0.013	0.027	0.029	0.031	0.015	0.031	0.034	0.036	0.010	0.022	0.023	0.024
			Bias	0.088	0.083	0.067	0.075	0.083	0.094	0.089	0.097	0.105	0.060	0.023	0.031	0.039
	4	III	MSE	0.118	0.096	0.104	0.109	0.114	0.101	0.114	0.119	0.124	0.086	0.086	0.090	0.094
			Bias	0.305	0.300	0.285	0.292	0.299	0.308	0.301	0.308	0.315	0.284	0.253	0.259	0.266
		IV	MSE	0.017	0.008	0.018	0.018	0.017	0.007	0.016	0.016	0.016	0.010	0.023	0.023	0.022
			Bias	0.065	0.068	0.077	0.072	0.067	0.060	0.061	0.056	0.051	0.085	0.109	0.105	0.100
	5	V	MSE	0.070	0.051	0.061	0.064	0.067	0.055	0.068	0.071	0.074	0.045	0.049	0.051	0.054
			Bias	0.219	0.215	0.202	0.208	0.214	0.222	0.217	0.223	0.229	0.199	0.172	0.178	0.184
		VI	MSE	0.179	0.173	0.187	0.181	0.176	0.159	0.160	0.155	0.150	0.205	0.248	0.242	0.236
			Bias	0.409	0.413	0.419	0.412	0.405	0.394	0.384	0.377	0.369	0.450	0.489	0.482	0.476

Note: SC- Schemes; S- Statistics

Table 4: MSEs and bias for MLE, Bayes and E-Bayes Estimate under SEL, DL and QL function of β under PT-IC data at $\beta_0 = 1$.

n	m	SC	S	MLE	BSEL	E-BSEL1	E-BSEL2	E-BSEL3	BDL	E-BDL1	E-BDL2	E-BSDL3	BQL	E-BQL1	E-BQL2	E-BQL3
25	3	I	MSE	0.818	0.448	0.323	0.433	0.561	0.511	0.446	0.587	0.750	0.284	0.168	0.222	0.291
			Bias	0.636	0.592	0.341	0.426	0.511	0.665	0.460	0.550	0.641	0.447	0.103	0.177	0.251
		II	MSE	0.357	0.118	0.154	0.193	0.245	0.159	0.197	0.259	0.335	0.065	0.154	0.156	0.168
			Bias	0.258	0.223	0.032	0.100	0.169	0.294	0.147	0.221	0.295	0.082	0.200	0.142	0.083
	4	III	MSE	0.520	0.289	0.254	0.318	0.391	0.343	0.332	0.411	0.500	0.196	0.148	0.184	0.227
			Bias	0.507	0.476	0.306	0.365	0.425	0.527	0.394	0.456	0.519	0.374	0.130	0.183	0.237
		IV	MSE	0.160	0.039	0.106	0.114	0.128	0.048	0.111	0.129	0.153	0.037	0.147	0.138	0.133
			Bias	0.058	0.036	0.088	0.041	0.005	0.088	0.001	0.050	0.100	0.066	0.265	0.224	0.184
	5	V	MSE	0.367	0.175	0.183	0.227	0.277	0.213	0.238	0.294	0.358	0.112	0.117	0.139	0.166
			Bias	0.377	0.349	0.202	0.255	0.308	0.396	0.285	0.340	0.396	0.253	0.036	0.084	0.131
		VI	MSE	0.144	0.041	0.127	0.124	0.126	0.037	0.111	0.118	0.130	0.064	0.208	0.191	0.177
			Bias	0.061	0.082	0.194	0.150	0.107	0.030	0.105	0.058	0.011	0.186	0.373	0.336	0.298
50	3	I	MSE	0.528	0.360	0.322	0.376	0.434	0.403	0.390	0.452	0.518	0.284	0.211	0.250	0.292
			Bias	0.583	0.561	0.434	0.478	0.522	0.596	0.497	0.543	0.589	0.491	0.307	0.348	0.390
		II	MSE	0.112	0.029	0.081	0.087	0.095	0.034	0.086	0.097	0.110	0.025	0.094	0.092	0.091
			Bias	0.063	0.048	0.038	0.006	0.027	0.082	0.024	0.058	0.092	0.020	0.162	0.132	0.103
	4	III	MSE	0.328	0.229	0.222	0.251	0.282	0.253	0.262	0.295	0.330	0.186	0.155	0.177	0.201
			Bias	0.462	0.448	0.363	0.393	0.423	0.473	0.409	0.440	0.471	0.399	0.271	0.299	0.328
		IV	MSE	0.092	0.055	0.108	0.101	0.094	0.046	0.089	0.084	0.079	0.077	0.158	0.148	0.138
			Bias	0.196	0.205	0.255	0.235	0.214	0.180	0.209	0.187	0.166	0.255	0.348	0.329	0.310
	5	V	MSE	0.220	0.134	0.147	0.167	0.189	0.150	0.175	0.198	0.223	0.104	0.103	0.117	0.132
			Bias	0.343	0.330	0.256	0.283	0.310	0.353	0.300	0.327	0.355	0.284	0.169	0.195	0.221
		VI	MSE	0.164	0.135	0.193	0.181	0.170	0.117	0.160	0.149	0.140	0.173	0.275	0.260	0.246
			Bias	0.343	0.351	0.395	0.376	0.357	0.325	0.346	0.326	0.306	0.404	0.492	0.475	0.458
100	3	I	MSE	0.418	0.334	0.323	0.350	0.378	0.354	0.359	0.388	0.418	0.296	0.257	0.280	0.304
			Bias	0.569	0.558	0.493	0.516	0.539	0.575	0.526	0.549	0.572	0.523	0.427	0.449	0.471
		II	MSE	0.076	0.030	0.056	0.061	0.067	0.035	0.064	0.070	0.078	0.022	0.046	0.049	0.052
			Bias	0.134	0.126	0.079	0.097	0.115	0.143	0.111	0.130	0.148	0.093	0.015	0.032	0.049
	4	III	MSE	0.267	0.217	0.217	0.231	0.246	0.229	0.238	0.253	0.269	0.195	0.178	0.191	0.204
			Bias	0.458	0.451	0.407	0.423	0.438	0.463	0.431	0.447	0.462	0.426	0.360	0.375	0.390
		IV	MSE	0.039	0.018	0.043	0.041	0.040	0.016	0.039	0.037	0.036	0.023	0.056	0.053	0.051
			Bias	0.097	0.102	0.130	0.119	0.108	0.090	0.107	0.095	0.084	0.127	0.178	0.167	0.156
	5	V	MSE	0.154	0.113	0.122	0.132	0.141	0.120	0.136	0.146	0.157	0.098	0.097	0.105	0.113
			Bias	0.324	0.317	0.280	0.294	0.307	0.328	0.302	0.316	0.330	0.294	0.236	0.249	0.262
		VI	MSE	0.399	0.385	0.439	0.421	0.403	0.353	0.378	0.361	0.344	0.456	0.576	0.558	0.539
			Bias	0.608	0.615	0.645	0.629	0.614	0.587	0.594	0.577	0.561	0.670	0.746	0.733	0.719

Note: SC- Schemes; S- Statistics

Table 5: MSEs and bias for MLE, Bayes and E-Bayes Estimate under SEL, DL and QL function of β under PT-IC data at $\beta_0 = 1.5$.

n	m	SC	S	MLE	BSEL	E-BSEL1	E-BSEL2	E-BSEL3	BDL	E-BDL1	E-BDL2	E-BSDL3	BQL	E-BQL1	E-BQL2	E-BQL3
25	**3**	**I**	**MSE**	**2.317**	**1.365**	**0.360**	**0.603**	**0.992**	**1.693**	**0.486**	**0.865**	**1.410**	**0.828**	**0.355**	**0.371**	**0.497**
			Bias	**1.103**	**1.037**	**0.100**	**0.350**	**0.601**	**1.175**	**0.300**	**0.566**	**0.833**	**0.759**	**0.300**	**0.082**	**0.136**
		II	**MSE**	**1.314**	**0.453**	**0.432**	**0.454**	**0.593**	**0.620**	**0.382**	**0.513**	**0.779**	**0.239**	**0.783**	**0.633**	**0.567**
			Bias	**0.520**	**0.457**	**0.336**	**0.118**	**0.100**	**0.597**	**0.135**	**0.099**	**0.334**	**0.176**	**0.738**	**0.552**	**0.367**
	4	**III**	**MSE**	**1.960**	**1.127**	**0.426**	**0.665**	**0.997**	**1.343**	**0.561**	**0.885**	**1.311**	**0.760**	**0.311**	**0.401**	**0.566**
			Bias	**1.006**	**0.951**	**0.231**	**0.429**	**0.628**	**1.053**	**0.388**	**0.597**	**0.805**	**0.748**	**0.084**	**0.095**	**0.274**
		IV	**MSE**	**0.523**	**0.135**	**0.472**	**0.402**	**0.385**	**0.160**	**0.367**	**0.349**	**0.390**	**0.146**	**0.831**	**0.679**	**0.568**
			Bias	**0.071**	**0.033**	**0.496**	**0.344**	**0.193**	**0.133**	**0.340**	**0.179**	**0.018**	**0.167**	**0.808**	**0.675**	**0.543**
	5	**V**	**MSE**	**1.673**	**0.797**	**0.418**	**0.602**	**0.869**	**0.963**	**0.511**	**0.770**	**1.120**	**0.520**	**0.374**	**0.427**	**0.547**
			Bias	**0.809**	**0.748**	**0.099**	**0.282**	**0.465**	**0.845**	**0.250**	**0.442**	**0.634**	**0.554**	**0.203**	**0.038**	**0.127**
		VI	**MSE**	**0.702**	**0.160**	**0.685**	**0.590**	**0.551**	**0.149**	**0.531**	**0.486**	**0.505**	**0.249**	**1.154**	**0.980**	**0.848**
			Bias	**0.106**	**0.150**	**0.658**	**0.510**	**0.362**	**0.046**	**0.497**	**0.339**	**0.181**	**0.359**	**0.978**	**0.851**	**0.723**
50	**3**	**I**	**MSE**	**2.240**	**1.536**	**0.752**	**1.046**	**1.398**	**1.712**	**0.933**	**1.279**	**1.687**	**1.215**	**0.477**	**0.676**	**0.925**
			Bias	**1.205**	**1.161**	**0.594**	**0.753**	**0.912**	**1.232**	**0.712**	**0.876**	**1.040**	**1.019**	**0.358**	**0.507**	**0.655**
		II	**MSE**	**0.396**	**0.103**	**0.310**	**0.289**	**0.297**	**0.123**	**0.268**	**0.275**	**0.314**	**0.092**	**0.475**	**0.404**	**0.358**
			Bias	**0.108**	**0.081**	**0.313**	**0.198**	**0.084**	**0.149**	**0.198**	**0.079**	**0.040**	**0.055**	**0.541**	**0.437**	**0.332**
	4	**III**	**MSE**	**1.276**	**0.842**	**0.528**	**0.686**	**0.871**	**0.931**	**0.632**	**0.815**	**1.025**	**0.679**	**0.365**	**0.477**	**0.613**
			Bias	**0.884**	**0.854**	**0.477**	**0.585**	**0.694**	**0.903**	**0.562**	**0.674**	**0.785**	**0.756**	**0.305**	**0.408**	**0.511**
		IV	**MSE**	**0.370**	**0.238**	**0.582**	**0.505**	**0.440**	**0.199**	**0.481**	**0.415**	**0.361**	**0.330**	**0.830**	**0.734**	**0.648**
			Bias	**0.419**	**0.435**	**0.675**	**0.602**	**0.530**	**0.386**	**0.589**	**0.513**	**0.438**	**0.533**	**0.848**	**0.780**	**0.713**
	5	**V**	**MSE**	**0.875**	**0.486**	**0.371**	**0.474**	**0.598**	**0.548**	**0.439**	**0.561**	**0.706**	**0.376**	**0.277**	**0.342**	**0.427**
			Bias	**0.645**	**0.616**	**0.286**	**0.382**	**0.479**	**0.662**	**0.367**	**0.466**	**0.565**	**0.524**	**0.123**	**0.215**	**0.307**
		VI	**MSE**	**0.402**	**0.135**	**0.221**	**0.243**	**0.284**	**0.162**	**0.228**	**0.267**	**0.326**	**0.096**	**0.253**	**0.243**	**0.249**
			Bias	**0.251**	**0.226**	**0.078**	**0.012**	**0.101**	**0.274**	**0.006**	**0.099**	**0.191**	**0.130**	**0.248**	**0.163**	**0.078**
100	**3**	**I**	**MSE**	**1.739**	**1.428**	**0.977**	**1.157**	**1.352**	**1.512**	**1.098**	**1.292**	**1.504**	**1.268**	**0.760**	**0.911**	**1.077**
			Bias	**1.178**	**1.158**	**0.853**	**0.940**	**1.027**	**1.193**	**0.917**	**1.005**	**1.094**	**1.088**	**0.726**	**0.810**	**0.894**
		II	**MSE**	**0.241**	**0.084**	**0.150**	**0.162**	**0.184**	**0.098**	**0.155**	**0.176**	**0.206**	**0.062**	**0.161**	**0.158**	**0.163**
			Bias	**0.196**	**0.181**	**0.033**	**0.030**	**0.094**	**0.215**	**0.027**	**0.092**	**0.157**	**0.115**	**0.155**	**0.094**	**0.033**
	4	**III**	**MSE**	**1.053**	**0.865**	**0.654**	**0.751**	**0.856**	**0.911**	**0.724**	**0.829**	**0.940**	**0.778**	**0.526**	**0.610**	**0.701**
			Bias	**0.916**	**0.903**	**0.698**	**0.757**	**0.816**	**0.927**	**0.744**	**0.804**	**0.863**	**0.853**	**0.606**	**0.664**	**0.721**
		IV	**MSE**	**0.172**	**0.066**	**0.215**	**0.197**	**0.183**	**0.058**	**0.192**	**0.177**	**0.167**	**0.084**	**0.276**	**0.250**	**0.228**
			Bias	**0.163**	**0.174**	**0.317**	**0.273**	**0.230**	**0.149**	**0.270**	**0.226**	**0.182**	**0.224**	**0.410**	**0.368**	**0.326**
	5	**V**	**MSE**	**0.635**	**0.485**	**0.376**	**0.439**	**0.507**	**0.516**	**0.424**	**0.492**	**0.565**	**0.425**	**0.293**	**0.345**	**0.402**
			Bias	**0.675**	**0.663**	**0.484**	**0.536**	**0.588**	**0.686**	**0.527**	**0.580**	**0.633**	**0.617**	**0.397**	**0.448**	**0.499**
		VI	**MSE**	**0.362**	**0.231**	**0.201**	**0.238**	**0.280**	**0.253**	**0.230**	**0.272**	**0.319**	**0.192**	**0.155**	**0.182**	**0.215**
			Bias	**0.447**	**0.436**	**0.266**	**0.316**	**0.366**	**0.459**	**0.310**	**0.361**	**0.411**	**0.389**	**0.179**	**0.227**	**0.276**

Note: SC- Schemes; S- Statistics

Table 6: MSEs and bias for MLE, Bayes and E-Bayes Estimate under SEL, DL and QL function of β under PT-IC data at $\beta_0 = 3$.

n	m	SC	S	MLE	BSEL	E-BSEL1	E-BSEL2	E-BSEL3	BDL	E-BDL1	E-BDL2	E-BSDL3	BQL	E-BQL1	E-BQL2	E-BQL3
25	3	I	MSE	9.756	4.475	1.114	1.374	2.428	5.484	0.982	1.664	3.244	2.810	1.890	1.455	1.622
			Bias	1.978	1.803	0.559	0.003	0.566	2.043	0.273	0.326	0.925	1.324	1.132	0.642	0.152
		II	MSE	3.335	1.219	1.908	1.264	1.137	1.688	1.358	0.998	1.235	0.622	3.519	2.451	1.755
			Bias	0.869	0.772	1.206	0.733	0.260	1.008	0.918	0.409	0.100	0.301	1.782	1.380	0.979
	4	III	MSE	5.236	3.014	0.705	0.951	1.655	3.595	0.694	1.193	2.196	2.025	1.055	0.861	1.040
			Bias	1.637	1.545	0.276	0.172	0.619	1.713	0.045	0.424	0.894	1.209	0.737	0.333	0.071
		IV	MSE	1.559	0.406	2.061	1.442	1.112	0.507	1.559	1.098	0.963	0.383	3.404	2.539	1.895
			Bias	0.203	0.136	1.289	0.932	0.575	0.308	1.055	0.675	0.295	0.207	1.758	1.446	1.133
	5	V	MSE	4.262	2.131	0.889	0.983	1.475	2.578	0.807	1.123	1.877	1.391	1.357	1.069	1.104
			Bias	1.309	1.210	0.449	0.034	0.380	1.370	0.227	0.208	0.643	0.890	0.893	0.519	0.144
		VI	MSE	1.591	0.484	3.083	2.300	1.755	0.405	2.419	1.758	1.371	0.820	4.745	3.784	3.001
			Bias	0.321	0.387	1.646	1.325	1.004	0.216	1.414	1.071	0.728	0.727	2.110	1.833	1.557
50	3	I	MSE	6.849	4.675	1.079	1.884	3.030	5.194	1.342	2.332	3.686	3.726	0.752	1.217	1.981
			Bias	2.100	2.018	0.485	0.873	1.261	2.138	0.665	1.066	1.466	1.778	0.125	0.488	0.850
		II	MSE	1.259	0.288	1.479	1.154	0.990	0.316	1.199	0.964	0.904	0.309	2.221	1.742	1.398
			Bias	0.067	0.017	1.005	0.738	0.472	0.129	0.833	0.555	0.277	0.208	1.347	1.104	0.860
	4	III	MSE	4.256	2.912	0.967	1.495	2.201	3.196	1.158	1.785	2.597	2.388	0.697	1.042	1.547
			Bias	1.646	1.589	0.506	0.788	1.071	1.673	0.642	0.931	1.221	1.422	0.234	0.502	0.771
		IV	MSE	1.031	0.612	2.158	1.752	1.420	0.512	1.828	1.461	1.176	0.854	2.933	2.457	2.045
			Bias	0.642	0.674	1.362	1.176	0.991	0.591	1.226	1.034	0.841	0.841	1.633	1.461	1.290
	5	V	MSE	2.334	1.415	0.557	0.797	1.167	1.592	0.635	0.949	1.399	1.096	0.498	0.601	0.821
			Bias	1.112	1.068	0.140	0.384	0.627	1.145	0.267	0.517	0.766	0.914	0.114	0.118	0.349
		VI	MSE	1.154	0.408	0.645	0.597	0.661	0.489	0.573	0.589	0.723	0.286	0.897	0.731	0.665
			Bias	0.460	0.420	0.435	0.209	0.017	0.501	0.303	0.071	0.162	0.259	0.699	0.486	0.273
100	3	I	MSE	4.767	3.806	1.705	2.314	3.027	4.033	1.945	2.611	3.385	3.374	1.287	1.785	2.381
			Bias	1.922	1.885	1.048	1.269	1.489	1.943	1.148	1.372	1.596	1.769	0.848	1.061	1.274
		II	MSE	0.735	0.312	0.371	0.373	0.436	0.365	0.350	0.388	0.488	0.224	0.470	0.407	0.399
			Bias	0.445	0.424	0.212	0.043	0.126	0.480	0.115	0.058	0.230	0.311	0.408	0.245	0.083
	4	III	MSE	3.014	2.389	1.314	1.674	2.083	2.515	1.466	1.853	2.293	2.148	1.045	1.349	1.701
			Bias	1.514	1.486	0.911	1.066	1.220	1.527	0.985	1.141	1.298	1.405	0.764	0.915	1.066
		IV	MSE	0.426	0.189	0.769	0.640	0.537	0.165	0.672	0.558	0.471	0.248	0.997	0.839	0.705
			Bias	0.313	0.330	0.740	0.628	0.515	0.288	0.666	0.551	0.437	0.412	0.889	0.780	0.672
	5	V	MSE	1.771	1.255	0.723	0.934	1.185	1.338	0.814	1.048	1.322	1.097	0.569	0.738	0.943
			Bias	1.076	1.051	0.550	0.686	0.822	1.089	0.619	0.757	0.894	0.975	0.412	0.545	0.677
		VI	MSE	1.002	0.624	0.374	0.484	0.629	0.682	0.420	0.550	0.717	0.516	0.312	0.383	0.486
			Bias	0.731	0.712	0.230	0.360	0.490	0.750	0.300	0.432	0.564	0.634	0.090	0.216	0.343

Note: SC- Schemes; S- Statistics

Table 7: MSEs and bias for MLE, Bayes and E-Bayes Estimate under SEL, DL and QL function of β under PT-IC data at $\beta_0 = 5$.

n	m	SC	S	$\beta_0 = 0.1$				B = $\beta_0 = 0.5$				$\beta_0 = 1$			
				APP	**HPDC**	**Poot-P**	**Poot-T**	**APP**	**HPDC**	**Poot-P**	**Poot-T**	**APP**	**HPDC**	**Poot-P**	**Poot-T**
25	**3**	**I**	**AILs**	**0.139**	**0.096**	**0.211**	**0.123**	**0.721**	**0.406**	**0.524**	**0.534**	**1.438**	**0.779**	**1.054**	**1.065**
			CPs	**95**	**99**	**90.5**	**100**	**96.9**	**97.7**	**92.4**	**91.8**	**97.4**	**98.02**	**93.3**	**92.7**
		II	**AILs**	**0.13**	**0.09**	**0.148**	**0.129**	**0.647**	**0.341**	**0.578**	**0.583**	**1.297**	**0.684**	**1.114**	**1.127**
			CPs	**99**	**100**	**89**	**94.5**	**97.7**	**98.32**	**90.8**	**89.2**	**97.74**	**97.86**	**92**	**92**
	4	**III**	**AILs**	**0.057**	**0.04**	**0.078**	**0.048**	**0.587**	**0.313**	**0.551**	**0.554**	**1.169**	**0.623**	**1.072**	**1.084**
			CPs	**96**	**100**	**92.5**	**100**	**97.62**	**98.2**	**91**	**89**	**97.4**	**97.68**	**91**	**90.3**
		IV	**AILs**	**0.103**	**0.072**	**0.1**	**0.105**	**0.522**	**0.253**	**0.598**	**0.595**	**1.038**	**0.5**	**1.24**	**1.214**
			CPs	**99**	**100**	**90.5**	**87.5**	**98.6**	**98.4**	**93**	**92.3**	**98.66**	**98.46**	**91.7**	**91.7**
	5	**V**	**AILs**	**0.105**	**0.073**	**0.091**	**0.098**	**0.55**	**0.295**	**0.532**	**0.541**	**1.097**	**0.582**	**1.057**	**1.067**
			CPs	**99**	**100**	**91**	**100**	**97.56**	**97.44**	**93.7**	**92.3**	**97.5**	**98.32**	**91.7**	**91.7**
		VI	**AILs**	**0.099**	**0.069**	**0.045**	**0.034**	**0.503**	**0.234**	**0.628**	**0.611**	**1.003**	**0.46**	**1.199**	**1.177**
			CPs	**97.5**	**100**	**99**	**100**	**98.8**	**97.44**	**92.3**	**93**	**98.98**	**97.94**	**90.7**	**91**
50	**3**	**I**	**AILs**	**0.103**	**0.072**	**0.154**	**0.087**	**0.498**	**0.284**	**0.3669**	**0.371**	**1.001**	**0.564**	**0.72**	**0.729**
			CPs	**95.5**	**99**	**88.5**	**100**	**96.68**	**97.64**	**93.2**	**91.8**	**96.76**	**98.32**	**92**	**90.7**
		II	**AILs**	**0.085**	**0.059**	**0.083**	**0.086**	**0.424**	**0.212**	**0.3809**	**0.384**	**0.846**	**0.423**	**0.771**	**0.774**
			CPs	**98.5**	**100**	**92**	**91**	**97.94**	**97.7**	**94**	**92**	**98.06**	**97.82**	**92**	**91.7**
	4	**III**	**AILs**	**0.082**	**0.057**	**0.113**	**0.07**	**0.407**	**0.221**	**0.363**	**0.367**	**0.819**	**0.442**	**0.727**	**0.733**
			CPs	**95**	**99.5**	**92.5**	**100**	**97.14**	**98.32**	**92.7**	**92.3**	**97**	**97.82**	**91.3**	**90.7**
		IV	**AILs**	**0.066**	**0.046**	**0.051**	**0.072**	**0.333**	**0.149**	**0.433**	**0.435**	**0.669**	**0.298**	**0.89**	**0.883**
			CPs	**99**	**100**	**93**	**97.5**	**99.02**	**97.22**	**93.7**	**93.7**	**98.92**	**98.2**	**90.3**	**90.3**
	5	**V**	**AILs**	**0.076**	**0.053**	**0.095**	**0.093**	**0.382**	**0.203**	**0.363**	**0.367**	**0.761**	**0.404**	**0.722**	**0.733**
			CPs	**97**	**99.5**	**87.5**	**100**	**97.22**	**97.76**	**88.3**	**87.7**	**97.24**	**97.34**	**94.3**	**94**
		VI	**AILs**	**0.076**	**0.053**	**0.095**	**0.093**	**0.323**	**0.136**	**0.445**	**0.446**	**0.646**	**0.277**	**0.908**	**0.901**
			CPs	**97**	**99.5**	**87.5**	**100**	**99.26**	**98.14**	**89**	**89**	**98.8**	**97.28**	**91**	**89.3**
100	**3**	**I**	**AILs**	**0.071**	**0.05**	**0.105**	**0.059**	**0.35**	**0.199**	**0.2569**	**0.258**	**0.7**	**0.387**	**0.509**	**0.511**
			CPs	**99.5**	**99.5**	**92**	**100**	**96.42**	**97.66**	**84.2**	**93.4**	**96.4**	**98.58**	**90.7**	**90.7**
		II	**AILs**	**0.062**	**0.043**	**0.066**	**0.059**	**0.306**	**0.153**	**0.2618**	**0.262**	**0.614**	**0.307**	**0.516**	**0.519**
			CPs	**99**	**99.5**	**88**	**97.5**	**97.98**	**98.2**	**92.2**	**88.6**	**97.6**	**97.36**	**93.7**	**91.7**
	4	**III**	**AILs**	**0.057**	**0.04**	**0.078**	**0.048**	**0.286**	**0.155**	**0.257**	**0.258**	**0.573**	**0.309**	**0.514**	**0.516**
			CPs	**96**	**100**	**92.5**	**100**	**96.8**	**97.76**	**92.7**	**91.7**	**96.88**	**97.98**	**88.7**	**88.7**
		IV	**AILs**	**0.049**	**0.034**	**0.041**	**0.049**	**0.243**	**0.112**	**0.297**	**0.296**	**0.486**	**0.223**	**0.587**	**0.59**
			CPs	**97**	**99**	**94**	**60**	**98.74**	**97.9**	**93.7**	**93**	**98.6**	**97.84**	**92**	**91.7**
	5	**V**	**AILs**	**0.054**	**0.038**	**0.094**	**0.088**	**0.267**	**0.142**	**0.257**	**0.258**	**0.535**	**0.283**	**0.504**	**0.51**
			CPs	**99**	**99.5**	**91.5**	**100**	**97.48**	**97.86**	**89.3**	**88.7**	**97.26**	**98.18**	**92**	**91.7**
		VI	**AILs**	**0.053**	**0.037**	**0.047**	**0.041**	**0.29**	**0.108**	**0.311**	**0.313**	**0.579**	**0.214**	**0.622**	**0.622**
			CPs	**97**	**99.5**	**99**	**99.2**	**99.7**	**98.3**	**91**	**90.7**	**99.55**	**97.15**	**95**	**94.7**

Note: SC- Schemes; S- Statistics

Table 8: The AILs and the corresponding CPs of the approximate confidence intervals, the HPDC intervals and Bootstrap CIs of β under PT-IC data.

n	m	SC	S	$\beta_0 = 1.5$				B = $\beta_0 = 3$				$\beta_0 = 5$			
				APP	**HPDC**	**Poot-P**	**Poot-T**	**APP**	**HPDC**	**Poot-P**	**Poot-T**	**APP**	**HPDC**	**Poot-P**	**Poot-T**
25	**3**	**I**	**AILs**	**2.177**	**1.197**	**1.577**	**1.601**	**4.181**	**2.896**	**6.308**	**3.651**	**7.111**	**4.935**	**11.007**	**6.333**
			CPs	**96.82**	**98.58**	**92**	**92**	**98**	**99**	**89.5**	**100**	**95.5**	**99.5**	**92**	**99.5**
		II	**AILs**	**1.946**	**0.999**	**1.697**	**1.703**	**3.891**	**2.69**	**4.517**	**3.694**	**6.512**	**4.497**	**7.676**	**6.365**
			CPs	**98.04**	**97.98**	**91**	**90.7**	**98**	**100**	**89.5**	**95.5**	**98.5**	**99.5**	**92**	**98.5**
	4	**III**	**AILs**	**1.768**	**0.976**	**1.623**	**1.631**	**3.53**	**2.458**	**4.88**	**3.052**	**5.847**	**4.062**	**8.174**	**5.049**
			CPs	**97.1**	**97.84**	**93.3**	**92.7**	**98**	**100**	**93.5**	**100**	**97.5**	**99.5**	**92**	**100**
		IV	**AILs**	**1.565**	**0.747**	**1.859**	**1.826**	**3.072**	**2.129**	**2.965**	**3.033**	**5.24**	**3.631**	**5.046**	**5.174**
			CPs	**98.68**	**98.6**	**92.3**	**92.3**	**98.5**	**99.5**	**93.5**	**92**	**99**	**100**	**91**	**89**
	5	**V**	**AILs**	**1.653**	**0.89**	**1.613**	**1.627**	**3.356**	**2.338**	**3.062**	**3.338**	**5.559**	**3.866**	**6.667**	**6.976**
			CPs	**97.5**	**97.42**	**92.3**	**91.7**	**96**	**99.5**	**91**	**94**	**96.5**	**99**	**91**	**95**
		VI	**AILs**	**1.514**	**0.722**	**1.788**	**1.764**	**3.04**	**2.109**	**0.972**	**1.059**	**4.943**	**3.425**	**2.281**	**1.806**
			CPs	**98.62**	**97.52**	**91.3**	**91.7**	**99.5**	**100**	**98**	**98**	**99**	**100**	**98**	**100**
50	**3**	**I**	**AILs**	**1.496**	**0.826**	**1.079**	**1.094**	**3.02**	**2.108**	**4.556**	**2.608**	**5.1**	**3.564**	**7.632**	**4.217**
			CPs	**96.82**	**97.84**	**88.3**	**87.7**	**95**	**100**	**90**	**100**	**97.5**	**99**	**89.5**	**100**
		II	**AILs**	**1.28**	**0.626**	**1.142**	**1.148**	**2.55**	**1.777**	**2.525**	**2.59**	**4.176**	**2.911**	**4.175**	**4.262**
			CPs	**98.06**	**98.04**	**93**	**92.3**	**99**	**100**	**93**	**92.5**	**98.5**	**99.5**	**95**	**95**
	4	**III**	**AILs**	**1.219**	**0.66**	**1.081**	**1.091**	**2.414**	**1.69**	**3.303**	**2.062**	**4.13**	**2.891**	**5.737**	**3.55**
			CPs	**96.76**	**97.58**	**92**	**92**	**97.5**	**100**	**93.5**	**100**	**95**	**99**	**92**	**100**
		IV	**AILs**	**1.001**	**0.444**	**1.318**	**1.327**	**1.976**	**1.381**	**1.48**	**2.088**	**3.341**	**2.335**	**2.52**	**3.56**
			CPs	**99.04**	**97.56**	**91.7**	**91.3**	**99.5**	**100**	**91**	**93**	**98.5**	**99.5**	**91**	**48.5**
	5	**V**	**AILs**	**1.146**	**0.611**	**1.092**	**1.101**	**2.265**	**1.589**	**2.054**	**1.98**	**3.798**	**2.659**	**2.236**	**3.99**
			CPs	**97.3**	**97.94**	**91.7**	**91.3**	**97**	**100**	**95**	**96**	**97**	**99**	**91.5**	**92**
		VI	**AILs**	**0.968**	**0.417**	**1.345**	**1.346**	**2.193**	**1.534**	**0.448**	**0.655**	**3.677**	**2.572**	**1.258**	**0.866**
			CPs	**99.1**	**97.98**	**92**	**91.7**	**98.5**	**99.5**	**99.5**	**95**	**98.5**	**100**	**98**	**100**
100	**3**	**I**	**AILs**	**1.049**	**0.593**	**0.767**	**0.769**	**2.118**	**1.486**	**3.203**	**1.774**	**3.509**	**2.463**	**5.388**	**2.996**
			CPs	**96.54**	**98.32**	**93**	**92**	**97**	**99.5**	**91**	**100**	**97**	**100**	**91.5**	**100**
		II	**AILs**	**0.922**	**0.456**	**0.776**	**0.781**	**1.804**	**1.264**	**1.917**	**1.736**	**3.069**	**2.15**	**3.292**	**2.978**
			CPs	**98.02**	**97.72**	**91.3**	**91**	**98.5**	**100**	**91**	**98.5**	**99**	**100**	**95**	**98.5**
	4	**III**	**AILs**	**0.859**	**0.459**	**0.778**	**0.779**	**1.718**	**1.206**	**2.381**	**1.474**	**2.858**	**2.009**	**3.943**	**2.461**
			CPs	**97.34**	**97.56**	**89.7**	**89.3**	**98.5**	**100**	**95**	**100**	**96.5**	**99.5**	**93.5**	**100**
		IV	**AILs**	**0.729**	**0.337**	**0.883**	**0.886**	**1.473**	**1.035**	**1.253**	**1.492**	**2.441**	**1.714**	**2.047**	**2.455**
			CPs	**98.54**	**97.66**	**92.7**	**92**	**98**	**100**	**90.5**	**94.5**	**99.5**	**100**	**94.5**	**86.5**
	5	**V**	**AILs**	**0.8**	**0.429**	**0.761**	**0.765**	**1.613**	**1.134**	**1.711**	**1.536**	**2.666**	**1.873**	**3.555**	**2.234**
			CPs	**96.9**	**97.56**	**92.7**	**92.3**	**97.5**	**99**	**95**	**96**	**96**	**99**	**91**	**93**
		VI	**AILs**	**0.871**	**0.335**	**0.927**	**0.933**	**1.573**	**1.105**	**0.307**	**0.302**	**2.615**	**1.836**	**0.581**	**0.52**
			CPs	**99.45**	**98.2**	**94.3**	**93.3**	**98.5**	**99**	**98.5**	**100**	97	100	99	100

Note: SC- Schemes; S- Statistics

Table 9: The AILs and the corresponding CPs of the approximate confidence intervals, the HPDC intervals and Bootstrap CIs of β under PT-IC data.

7 Applications to Real Data Sets.

For the sake of illustration, we employed two real data analysis, which we used to evaluate the results of the MLEs for the ITL distribution under PT-IC data. The ITL distribution will be compared with further inverted distributions. for example inverse exponential (IE), inverse weibull (IW), inverted Lindley (IL) and inverse Rayleigh (IR).

For the data set, the unknown parameter of the previous distributions the MLE, the Kolmogorov-Smirnov (KS) statistic, Akaike information criterion (AIC) , Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) will be computed.

1.**The First Data Set: Vinyl Chloride Data.**

We evaluate vinyl chloride data received from clean upgradient monitoring wells in *mg/L*; see Bhaumik et. al. [13]. The data set is given as: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2. The Table (10) provides an overview of the findings:

From Table (10), we note that the vinyl chloride data can be modeled by the ITL distribution where P-Value is 0.9232 and MLE of β is $\widehat{\beta} = 2.1172$

To fit the given data set graphically, we plot the graph and the fitted pdf lines corresponding to the ITL, IW, IE, IR, and IL distributions; we also plot the empirical cdf and fitted cdfs corresponding to identical distributions. Figure (3) shows the lines fitted for cdfs and pdfs for the vinyl chloride data set and the corresponding distributions. The numbers in Table (10) also indicate that the ITL distribution provides a better fit than other distributions for this data set.

We generate different PT-IC samples from the original vinyl chloride data. Withdrawn units are treated as in the simulation study in Table (1) for n = 25. In addition, different stages of censoring are proposed at $m = 3$, $m = 4$ and $m = 5$ then compute the censoring times as shown in section 6.

In Table (12), we determine the MLE's value of the parameter β, its associated CIs, bootstrap CIs Boot-p, Boot-t and HPDC intervals. We also compute Bayes and E-Bayes estimates. For Bayesian estimation we compute the hyper parameters by using the MLE estimates for the parameter β and its variance through the three stages $m = 3, 4, 5$.

1.**The Second Data Set: Annual Rainfall Data.**

These data show the total amount of rain that fell in January between 1880 and 1916, measured in inches at Los Angeles Civic Center. Center introduced by Selim [42]. These data display the annual remission times for a set of 37 observations. The collection of data is provided as: 1.33, 1.43, 1.01, 1.62, 3.15, 1.05, 7.72, 0.20, 6.03, 0.25, 7.83, 0.25, 0.88, 6.29, 0.94, 5.84, 3.23, 3.70, 1.26, 2.64, 1.17, 2.49, 1.62, 2.10, 0.14, 2.57, 3.85, 7.02, 5.04, 7.27, 1.53, 6.70, 0.07, 2.01, 10.35, 5.42, 13.3. The Table (11) provides an overview of the findings:

From Table (11), we note the annual rainfall data can be modeled by the ITL distribution where P-Value is 0.50, and the MLE of β is $\widehat{\beta} = 1.28$.

We plot the graph and the fitted pdf lines corresponding to the ITL, IW, IE, IR, and IL distributions to fit the given dataset graphically. Also, we plot the empirical cdf and fitted cdfs corresponding to the same distributions. Figure (4) shows the lines fitted for cdfs and pdfs for the annual rainfall data set and the corresponding distributions. The numbers in Table (11) also indicate that the ITL distribution provides a better fit than other distributions for this data set.

We generate different progressive censoring Type I samples from the annual rainfall data. Removed items are considered as in the simulation study in Table (1) for n = 25. In addition, different stages of censoring are proposed at $m = 3$, $m = 4$ and $m = 5$ then compute the censoring times as shown in section 6.

In Table (12), we determined the MLE' value of the parameter β, its associated CIs, bootstrap CIs Boot-p and Boot-t and HPDC intervals. Additionally, we compute Bayes and E-Bayes estimates. For Bayesian estimation we compute the hyper parameters by using the MLE estimates for the parameter β and its variance through the three stages $m = 3, 4, 5$.

7.1 Tables and Figures

Distribution		Measures						
		P-value	K-S	-2log L	AIC	BIC	AICc	HQIC
ITL	2.1172	0.9232	0.092	-111.220	-109.220	-107.694	-111.102	-108.700
IW	0.6174	0.7740	0.113	-117.253	-113.253	-110.200	-115.900	-112.212
	0.8805							
IE	0.5727	0.4553	0.147	-118.386	-116.386	-114.860	-118.268	-115.866
IR	0.3389	0.0000	0.485	-186.702	-184.702	-183.176	-186.584	-184.184
IL	0.8773	0.1681	0.191	-123.627	-121.627	-120.101	-123.510	-121.107

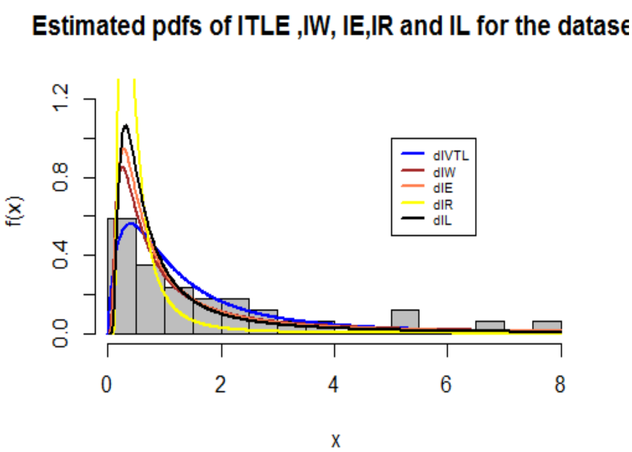
Table 10: The goodness of fit test values for Vinyl Chloride data Set to the ITL distribution.

Distribution	MLE	Measures						
		P-value	K-S	-2log L	AIC	BIC	AICc	HQIC
ITL	1.28	0.50	0.14	-176.1	354.2	355.8	352.3	354.8
IW	1.04	0.14	0.19	-186.9	377.7	380.9	375.0	378.8
	0.71							
IE	0.76	0.0	0.3	-198.6	399.1	400.7	397.2	399.7
IR	0.34	0.0	0.7	-339.3	680.6	682.2	678.7	681.1
IL	1.12	0.0	0.3	-210.4	422.8	424.4	420.9	423.3

Table 11: The Values of Goodness of Fit Test for Annual Rainfall Data to the ITL distribution.

The Results of Vinyl Chloride Data Set																	
m	**SC**	**Estimates**												**Confidence intervals**			
		MLE	**BSEL**	**E-BSEL1**	**E-BSEL2**	**E-BSEL3**	**BDL**	**E-BDL1**	**E-BDL2**	**E-BSDL3**	**BQL**	**E-BQL1**	**E-BQL2**	**E-BQL3**	**AILs of APP**	**AILs of HPDC**	**AILs of Poot-P**	**AILs of Poot-T**
3	**I**	**1.995**	**1.995**	**1.688**	**1.779**	**1.871**	**2.085**	**1.834**	**1.934**	**2.034**	**1.813**	**1.394**	**1.47**	**1.546**	**2.38**	**1.109**	**1.298**	**1.489**
	II	**1.454**	**1.454**	**1.25**	**1.318**	**1.386**	**1.545**	**1.397**	**1.473**	**1.549**	**1.272**	**0.956**	**1.008**	**1.06**	**2.044**	**1.14**	**1.744**	**2.309**
4	**III**	**1.995**	**1.995**	**1.78**	**1.845**	**1.91**	**2.054**	**1.882**	**1.95**	**2.019**	**1.878**	**1.577**	**1.634**	**1.691**	**1.909**	**0.892**	**1.145**	**1.459**
	IV	**1.185**	**1.185**	**1.086**	**1.122**	**1.159**	**1.239**	**1.18**	**1.22**	**1.26**	**1.078**	**0.897**	**0.927**	**0.957**	**1.414**	**0.903**	**1.466**	**1.794**
5	**V**	**1.753**	**1.753**	**1.59**	**1.641**	**1.692**	**1.805**	**1.681**	**1.735**	**1.789**	**1.65**	**1.408**	**1.453**	**1.499**	**1.677**	**0.864**	**1.083**	**1.3**
	VI	**1.325**	**1.325**	**1.196**	**1.241**	**1.285**	**1.385**	**1.3**	**1.349**	**1.397**	**1.204**	**0.988**	**1.025**	**1.062**	**1.58**	**0.938**	**1.012**	**1.476**
The Results of Annual Rainfall Data Set																	
3	**I**	**1.32**	**1.32**	**1.245**	**1.271**	**1.297**	**1.35**	**1.306**	**1.333**	**1.36**	**1.25**	**1.123**	**1.147**	**1.171**	**1.163**	**0.705**	**0.864**	**0.908**
	II	**1.08**	**1.08**	**1.019**	**1.041**	**1.062**	**1.11**	**1.08**	**1.104**	**1.127**	**1.01**	**0.895**	**0.914**	**0.934**	**1.061**	**0.716**	**1.247**	**1.446**
4	**III**	**1.25**	**1.25**	**1.204**	**1.222**	**1.24**	**1.28**	**1.247**	**1.266**	**1.285**	**1.2**	**1.116**	**1.133**	**1.15**	**0.95**	**0.587**	**0.828**	**0.822**
	IV	**1.03**	**1.03**	**0.991**	**1.006**	**1.3021**	**1.05**	**1.035**	**1.051**	**1.067**	**0.98**	**0.903**	**0.917**	**0.931**	**0.864**	**0.602**	**0.965**	**0.899**
5	**V**	**1.19**	**1.18**	**1.141**	**1.157**	**1.173**	**1.21**	**1.182**	**1.199**	**1.216**	**1.14**	**1.058**	**1.073**	**1.088**	**0.898**	**0.572**	**0.845**	**0.847**
	VI	**1.03**	**1.03**	**0.993**	**1.009**	**1.025**	**1.06**	**1.039**	**1.056**	**1.072**	**0.98**	**0.901**	**0.915**	**0.929**	**0.888**	**0.608**	**1.052**	**1.141**

Table 12: MLE, Bayes and E-Bayes Estimates under SEL, DL and QL function and approximate CIs, HPDC intervals and Bootstrap CIs of the ITL distribution parameter under Two Real Data Set.



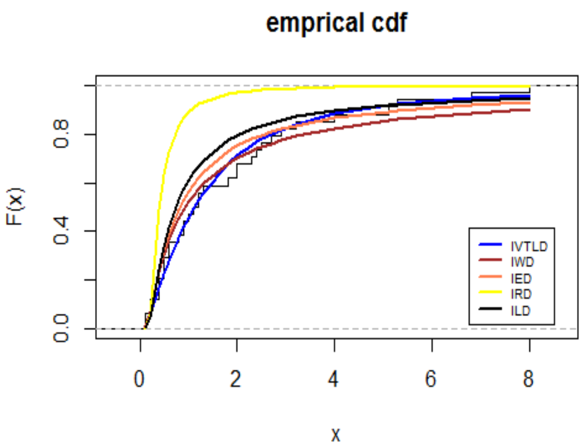


Fig. 3: Estimated pdf and cdf for the vinyl chloride data set with corresponding distributions

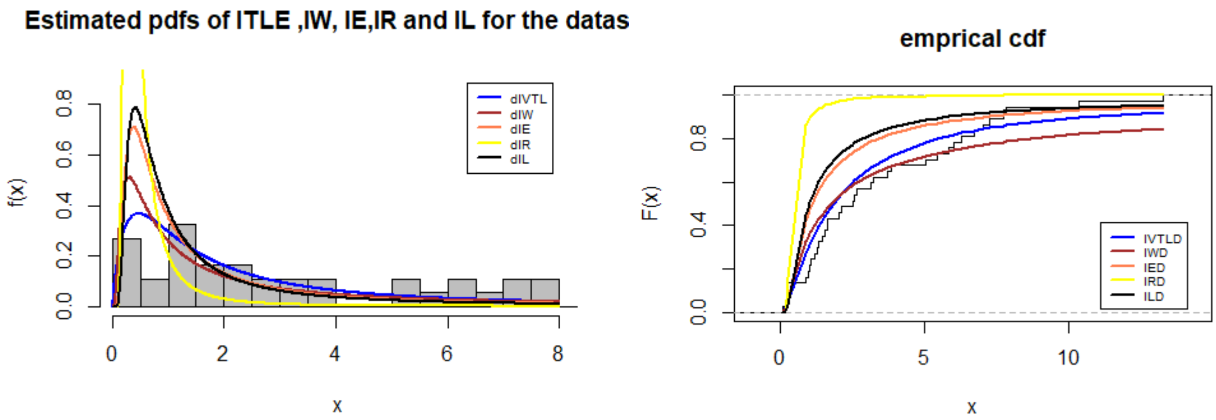


Fig. 4: Estimated pdf and cdf for the annual rainfall data set with corresponding distributions.

8 Conclusion

In this paper, we have derived the MLE, Bayesian, and E-Bayesian estimates under the SEL function, DL function, and QL function for the unknown parameter of the ITL distribution under PT-IC data. The approximate CIs for MLE and HPDC intervals for Bayes estimates are computed; moreover, bootstrapped p-t intervals are acquired. We perform some simulations to see the performances of the MLE, Bayes, and E-Bayes estimates under PT-IC samples. Two real data sets have been re-analyzed based on PT-IC data. The simulation results indicate that the performance of estimates under E-Bayesian estimation is better than Bayesian estimation in some cases and better than MLE in all cases. On the other hand, the Bayesian estimation is better than the E-Bayesian estimation in some cases and better than MLE in all cases. The values of hyper-parameters in Bayesian estimation were determined for simulation and real data based on the MLE and its variance by using the equation in (15).

Declarations

Competing interests: The researchers declare themselves to be free from any conflicts.
Authors' contributions: All authors contributed equally to all aspects of the research, including conceptualization and design, data collection and analysis, drafting the manuscript, and review and editing. All authors approved the final version for publication.
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