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A STUDY OF INFLUENCE OF NON-UNIFORM PARAMETER ON PERISTALTIC TRANSPORT OF CARREAU FLUID THROUGH A FINITE LENGTH CHANNEL: APPLICATION TO BILE FLOW

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Abstract: The present manuscript is detailed here to analyze the impact of non uniform parameter on different types of tapered duct which has a great similarity to bile flow through narrow or wider duct. For present analysis fluid is taken as non-Newtonian, also Carreau fluid model is taken into interest to describe the flow characteristic of non-Newtonian bile. The wall geometry of duct is described by the sinusoidal wave propagating along the axial direction. The governing equations of motion and continuity are simplified with the analytical approach by considering long wavelength and low Reynolds number approximation. Analytical solutions have been calculated for velocity, pressure gradient, shear stress and pressure rise also the impact of effecting parameters such as power index, Weissenberg number, amplitude ratio and non-uniform parameter are discussed for different types of ducts (i.e., converging duct, diverging duct and non-tapered duct) by plotting the graphs in MATLAB R2018b software. It is found that Bile velocity is captured maximum in case of converging duct. Also, for Newtonian bile i.e., $n = 1$ or $We = 0$ bile reaches to its maximum velocity also When bile is considered as Newtonian fluid ($n = 1$ or $We = 0$) less amount of wall shear stress S_{rz} is noticed.

Keywords: Bile flow; Carreau fluid; Peristaltic motion; Tapered duct; Shear stress.

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1 Introduction

In recent times many researchers have been attracted to the study of bile flow in the human biliary system by experimental and clinical approaches. This manuscript is designed to evaluate the bile flow through a tapered duct using an analytical approach. This manuscript deals with an application to bile flow through tapered duct because bile flow through narrow/wider duct may increase the interest of flow from cholesterol/plaque deposited or stenosed/ dilated duct or calculus duct. Nowadays big number of populations are suffering from biliary diseases, Cholelithiasis is one of them. Cholelithiasis is a disease that is initiated by the formation of gallstones in gallbladder. Gallstones size may vary from the size of grain of sand to the size of golf ball. Cholelithiasis has become worldwide health problem. There could be a situation arise when gallstone come out from the gallbladder and fall into the common bile duct via cystic duct. When such crystal / stone / calculus falls into the common bile duct, this may increase blockage level of path of bile ducts. These situations make duct wider or narrower.
Bile is one of the biofluids produced constantly via the liver that is accountable in the emulsification of fats (lipids) in the duodenum (i.e., the small intestine). The daily secreted quantity of bile in the human biliary system in a day is about one liter approximately.The Biliary system includes liver gallbladder, biliary ducts (common bile duct, hepatic duct and cystic duct) and the structure of sphincters. Biliary ducts are the medium of transportation of bile in biliary system. Gallbladder is pear formed organ positioned immediately below the liver and which regulates the bile transport. Bile

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plays a critical role in absorbing protein, vitamins D, E, K and A which can be soluble in lipids. There are number of papers dedicated on the rheological properties of bile flow modelling in different circumstances of biliary system using analytical, computational, and experimental approaches. Gottschalk and Lochner [1] examined 33 samples and reported that post-operative T-tube bile is Maxwell fluid. Atabi et al. [2] investigated flow of bile in patient specific cystic duct using experimental approach and revealed that presence of gallstone may lead to increase resistance Atabi et al. [3] compared clinical and CFD (computational Fluid Dynamics) results of bile in human cystic duct and showed CFD is a relevant mechanism to investigate the function of biliary system. Luo et al. [4] worked to understand the biomechanical behaviour of human biliary system. Ooi et al. [5] studied bile flow in human cystic duct and found there is a great influence of valve of heister on the resistance to bile flow. Kuchumov et al. [6] presented a model for pathological bile flow in the major duodenal papilla duct with stone and it was shown that pathological bile behaves like non-Newtonian fluid. Reinkart et al. [7] stated that bile from the common bile duct has generally a low viscosity, but that it can significantly increase by way of mucus secretion and turn out to be a non-Newtonian fluid with unfavorable effects on bile glide. Tomizawa et al. [8] presented a paper and concluded acute cholangitis suffers without common bile duct dilation had been barely more youthful and exhibited substantially higher ranges of C-creative protein than sufferers with common bile duct dilatation. Those statistics suggest that prognosis of acute cholangitis in patients without common bile duct dilatation must to be based totally on a combination of medical symptoms and laboratory statistics. The aim of this manuscript is to analyze the impact of Carreau fluid on peristaltic motion because lithogenic bile behaves as Carreau fluid [9]. A peristaltic motion is generated by the wavy propagation along the surface the channel/duct that has built serious attention for the researcher working in field of physiological fluid dynamics. Peristaltic motion is one of the most important keystones to developing the science and engineering in the current research. Peristalsis mechanism also plays a significant role in transporting the biological fluid inside the living organism such as urine transportation from the kidney to bladder, chyme transportation in gastrointestinal track, blood transportation in blood vessels (artery, vein and capillary), bile flow in ducts etc. In the present time many artificial equipment like heart pumping machine, dialysis machine etc. are designed on the basis of working of peristaltic pumping. Few investigations on peristaltic motion of fluid have been done by considering fluid as Newtonian and non-Newtonian fluid. Latham [10] investigated the fluid mechanism of peristaltic transport. Lew et al. [11] analyzed a mathematical model for peristaltic motion of duodenum (small intestine). Misra and Pandey [12] studied properties of peristaltic flow through tapered duct. Also, peristaltic transport of bio-fluid has been investigated by Eytan [13], Mishra and Pandey [14], Rao and Mishra [15] by long wavelength and low Reynolds number approximation. Maiti and Mishra [16] studied peristaltic motion of bile in common bile duct with the presence of gallstone and it is obtained presence of gallstone leads to decrease the bile velocity. The non-Newtonian behaviour of peristaltic transport using Carreau fluid, power law fluid, Herschel Bulkley fluid, Bingham fluid etc. has been examined by some researchers by theoretical, numerical and experimental approaches. Vajrevelu et al. [17] studied peristaltic transport of Casson fluid through an elastic channel. Haricharan et al. [18] considered peristaltic flow of Bingham fluid through non uniform channel. Lithogenic bile (Diseased bile) behaves like Carreau fluid found by Kuchumov [9]. There are few articles have been solved for peristaltic transport of Carreau fluid with different configuration of channel. Noreen [19] investigated the effect of heat and mass transfer on Carreau fluid for blood through tapered stenosed artery also Noreen et al. [20] explored the analytical solution for the heat transform on peristaltic transport of Carreau fluid in a wavy microchannel. Ali and Hayat [21] worked on peristaltic flow of Carreau fluid in an asymmetric channel. Hayat et al. [22] considered non-Newtonian fluid as Carreau fluid in a tube with various wave form. Ellahi et al. [23] discussed peristaltic flow of non-Newtonian fluid (Jeffery fluid) through porous medium with partial slip condition as an application in industrial filters. Tripathi et al. [24] studied peristaltic flow of Bergers fluid and the study is relatable to movement of chyme in duodenum (small intestine). Kumari et al. [25] presented a paper that deals with the peristaltic motion of fluid with variable viscosity as an application to bile flow in duct. Rawat et al. [26] studied a mathematical model for peristaltic transport of non-Newtonian bile through calculus duct with heat transfer and wall slip conditions and concluded that velocity and pressure gradient are affected by heat and slip parameters.

As per our gained knowledge from literature, no paper is available that deals with the comparison of the peristaltic flow of fluid flow through different types of ducts. Therefore, motivated by above discussion, it is significant to study the peristaltic transport of Carreau fluid as lithogenic bile through a tapered duct which has an important physiological application such as bile flow through the narrow/wider or plaque deposited duct or calculus duct. The problem is solved under the assumption of long wavelength and low Reynolds number. The regular perturbation technique is utilized to determine the solution of governing equations of continuity and motion. The impact of various parameters effecting the problem are studied analytically and analyzed by plotting the solution in MATLAB R2018b software.

2 Formation of the problem

Here we consider the peristaltic flow of lithogenic bile (Carreau fluid) with density ρ through a tapered duct of finite length. Sinusoidal waves of constant speed c are propagating along the surface of the duct wall. The wavelength is

$(\bar{R},\bar{Z})$	Cylindrical co-ordinate in fixed wave frame
$(\bar{U},\bar{W})$	Velocity components along $\bar{R}$ and $\bar{Z}$ directions
$(r^{'},z^{'})$	Cylindrical co-ordinate in wave frame of reference
$(u^{'},w^{'})$	Velocity components along $r^{'}$ and $z^{'}$ directions
c	wave speed
p	fluid pressure
g	acceleration due to gravity
m	non-uniform parameter
n	power index
δ	wave number
λ	wavelength
ρ	density of fluid
Θ	average flux
ϕ	amplitude ratio
Re	Reynolds number
We	Weissenberg number

Table 1: Nomenclature

comparable with the channel (duct) length thus, the wave number is small and Reynolds number is negligible. We have considered cylindrical co-ordinate system (r,θ,z) , z axis is considered along the axis of the duct. Here we take a stationary frame of reference $(\bar{R},\bar{Z})$ and let $(\bar{U},\bar{W})$ be the velocity components in fixed frame of reference $(\bar{R},\bar{Z})$. The wall surface geometry is taken as .

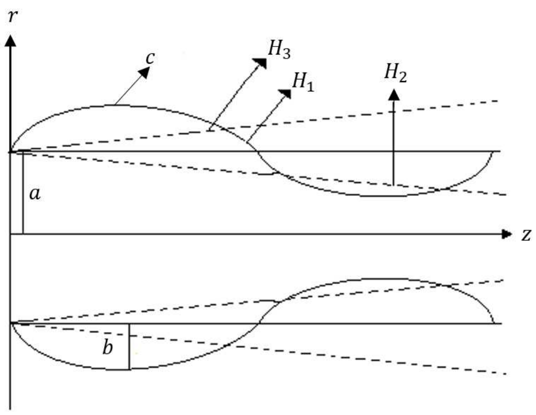

Fig. 1: Physical sketch of the Problem.

$$H_1(\bar{Z},\bar{t}) = a + b\sin\frac{2\pi}{\lambda}(\bar{Z}-c\bar{t})(\text{non-tapered duct [27]}) \quad (1)$$

$$H_2(\bar{Z},\bar{t}) = a - m^{'}\bar{Z} + b\sin\frac{2\pi}{\lambda}(\bar{Z}-c\bar{t})\text{Converging duct [9]} \quad (2)$$

$$H_3(\bar{Z},\bar{t}) = a + m^{'}\bar{Z} + b\sin\frac{2\pi}{\lambda}(\bar{Z}-c\bar{t})\text{Diverging duct [28]} \quad (3)$$

In which a be the inlet radius, b be the wave amplitude, $m^{'}$ be the non-uniform parameter of tapered duct, λ be the wavelength and c be the wave velocity.
The equations governing the motion may be written as discussed assumptions [9]

$$\frac{1}{\bar{R}}\frac{\partial(\bar{R}\bar{U})}{\partial\bar{R}} + \frac{\partial\bar{W}}{\partial\bar{Z}} = 0 \quad (4)$$

$$\rho\left(\frac{\partial\bar{W}}{\partial\bar{t}}+\bar{U}\frac{\partial\bar{W}}{\partial\bar{R}}+\bar{W}\frac{\partial\bar{W}}{\partial\bar{Z}}\right)=-\frac{\partial\bar{p}}{\partial\bar{Z}}+\frac{1}{\bar{R}}\frac{\partial(\bar{R}\bar{S}_{\bar{R}\bar{Z}})}{\partial\bar{R}}+\frac{\partial(\bar{S}_{\bar{Z}\bar{Z}})}{\partial\bar{Z}} \tag{5}$$

$$\rho\left(\frac{\partial\bar{U}}{\partial\bar{t}}+\bar{U}\frac{\partial\bar{W}}{\partial\bar{Z}}+\bar{U}\frac{\partial\bar{U}}{\partial\bar{R}}\right)=-\frac{\partial\bar{p}}{\partial\bar{R}}+\frac{1}{\bar{R}}\frac{\partial(\bar{R}\bar{S}_{\bar{R}\bar{R}})}{\partial\bar{R}}+\frac{\partial(\bar{S}_{\bar{R}\bar{Z}})}{\partial\bar{Z}} \tag{6}$$

Where ρ is the density, $\bar{U}$ and $\bar{R}$ are the velocity components, $\bar{p}$is the pressure of the fluid. The flow is unsteady in the laboratory frame ($\bar{R}$,$\bar{Z}$).
The transformation from fixed frame to wave frame is given by

$$z^{'}=\bar{Z}-c\bar{t},r^{'}=\bar{R},w'=\bar{W}-\bar{t},u^{'}=\bar{U},p^{'}=\bar{p},t=\bar{t} \tag{7}$$

Where $\bar{U}$,$\bar{W}$ and $u^{'}$,$w^{'}$ are the velocity components in the fixed frame and wave frame of reference respectively.
Carreau fluid is one of the kinds of generalized Newtonian fluid where viscosity depends on the shear rate. Although this type of fluid is known as non- Newtonian in nature. Carreau fluid is useful in describing flow behaviour of fluids in the high shear rate region [30].
The constitutive equation of Carreau fluid is described as

$$\bar{S}_{ij}=-p\delta_{ij}+\bar{\tau}_{ij} \tag{8}$$

$$\bar{\tau}_{ij}=\left[\eta_\infty+(\eta_0-\eta_\infty)(1+(\Gamma\dot{\gamma})^2)^{\frac{n-1}{2}}\right]\bar{\dot{\gamma}}_{ij} \tag{9}$$

where p is the pressure, δ_{ij} is kronecker delta, δ_{ij} is the extra stress tensor, η_0 and η_∞ are the zero-shear rate viscosity and the infinite shear rate viscosity respectively, is the time constant, n is the power index.
is defind as follows:

$$\bar{\dot{\gamma}}=\sqrt{\frac{1}{2}\sum\sum\bar{\dot{\gamma}}_{ij}\bar{\dot{\gamma}}_{ji}}=\sqrt{\frac{1}{2}\Pi} \tag{10}$$

Where Π is the second invariant strain rate tensor. In case of η_∞ and applying Taylors expansion, equation (9) converted into:

$$\bar{\tau}_{ij}=\eta_0\left[1+\frac{n-1}{2}(\Gamma\dot{\gamma})^2)\right]\bar{\dot{\gamma}}_{ij} \tag{11}$$

Here let us introduce some non-dimensional variables

$$z=\frac{z^{'}}{\lambda},r=\frac{r^{'}}{a},w=\frac{w^{'}}{c},u=\frac{\lambda u^{'}}{ac},m=\frac{m^{'}\lambda}{a},\delta=\frac{a}{\lambda},\phi=\tfrac{b}{a},h_1=\frac{H_1}{a},h_2=\frac{H_2}{a},$$

$$h_3=\frac{H_3}{a},p=\frac{a^2p^{'}}{\lambda c},We=\frac{c\Gamma}{a},S_{rr}=\frac{\lambda\bar{S}_{r^{'}r^{'}}}{c\eta_0},S_{rz}=\frac{\lambda\bar{S}_{r^{'}z^{'}}}{c\eta_0} \tag{12}$$

Making use of non-dimensional variables detailed in equation (12), equation (1)-(6) converted into

$$h_1=1+\phi\sin(2\pi z) \tag{13}$$

$$h_2=1-mz+\phi\sin(2\pi z) \tag{14}$$

$$h_3=1+mz+\phi\sin(2\pi z) \tag{15}$$

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial(w)}{\partial z}=0 \tag{16}$$

$$Re\delta\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial(rS_{rz})}{\partial r}+\delta\frac{\partial(S_{zz})}{\partial z} \tag{17}$$

$$Re\delta^3\left(w\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial r}+\delta^2\frac{\partial(rS_{rr})}{\partial r}+\delta^2\frac{\partial(S_{rz})}{\partial z} \tag{18}$$

To reduce the complexity of the problem, lubrication theory approximation is taken of infinitesimally small wall curvature ($\delta\to 0$) and small Reynolds number ($Re\to 0$), The approximations assume that inertial effects are negligible and that the dominant axial scale is much larger than the dominant radial scale [9], thus equations (17) - (18) takes the form

$$\frac{\partial p}{\partial z}=\frac{1}{r}\frac{\partial(rS_{rz})}{\partial r} \tag{19}$$

$$\frac{\partial p}{\partial r} = 0 \tag{20}$$

Where $S_{rz} = S_{zr} = \left(1 + \frac{n-1}{2} We^2 \left(\frac{\partial w}{\partial r}\right)^2\right)\left(\frac{\partial w}{\partial r}\right)$

From equation (19) and (20), it is clear

$$\frac{dp}{dz} = \frac{1}{r}\frac{\partial}{\partial z}\left[r\left(1 + \frac{n-1}{2} We^2 \left(\frac{\partial w}{\partial r}\right)^2\right)\left(\frac{\partial w}{\partial r}\right)\right] \tag{21}$$

The appropriate boundary conditions are

$$\frac{\partial w}{\partial r} = 0 at r = 0 \tag{22}$$

$$w = -1 at r = h \tag{23}$$

where $h = h_1, h_2$ and h_3

The dimensionless instantaneous flux in the fixed frame is

$$Q = 2\int_0^H \bar{W}\bar{R} d\bar{R} \tag{24}$$

where H is the function of $\bar{Z}$ and $\bar{t}$.

The dimensionless volume flow rate in the wave frame of reference is given by

$$q = 2\int_0^H w^{'} r^{'} dr^{'} \tag{25}$$

where H is the function of $z^{'}$.

substituing equation (7) into equation (24) and make use of equation (25), we find

$$Q = q + cH \tag{26}$$

The time average flux Θ over a single period T of peristaltic wave is

$$\bar{Q} = \frac{1}{T}\int_0^T Q dt \tag{27}$$

substituing equation (26) into equation (27)

$$\bar{Q} = q + c\left(a^2 + \frac{b^2}{2}\right) \tag{28}$$

dimensionless mean flow Θ in the fixed frame and F in the wave frame as

$$\Theta = \frac{\bar{Q}}{ca^2}, F = \frac{q}{ca^2} \tag{29}$$

Thus equation (28) reduces to

$$\Theta = F + 1 + \frac{\phi^2}{2} \tag{30}$$

3 Solution of the problem

Since the problems involved are nonlinear. It appears difficult to find the exact solutions. Thus, we are interested to derive the approximate analytical solutions by perturbation analysis. Perturbation analysis is better than other methods as it can be used to solve any set of complicated problems containing small parameters. The expressions developed by the perturbation method are not exact but these can lead to accurate results when the expansion parameter is small. Since Eq. (2.21) is non-linear in Weissenberg number and hence cannot be solved to obtain an exact solution. Therefore, we use the perturbation analysis and expand w, p and F by considering We^2 as perturbation parameter.

$$w = w_o + We^2 w_1 + O(We^4) \tag{31}$$

$$p = p_o + We^2 p_1 + O(We^4) \tag{32}$$

$$F = F_o + We^2 F_1 + O(We^4) \tag{33}$$

Substituting equation (21) in equations (31)-(33) Following system can obtain

Zeroth Order System

$$\frac{dp_0}{dz} = \frac{1}{r}\frac{\partial}{\partial z}\left[r\frac{\partial w_0}{\partial r}\right] \tag{34}$$

The appropriate boundary conditions are

$$\frac{\partial w_0}{\partial r} = 0 at r = 0 \tag{35}$$

$$w = -1 at r = h(z) \tag{36}$$

First Order System

$$\frac{dp_1}{dz} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial w_1}{\partial r} + \frac{n-1}{2}We^2\left(\frac{\partial w_0}{\partial r}\right)^3\right)\right] \tag{37}$$

The appropriate boundary conditions are

$$\frac{\partial w_1}{\partial r} = 0 at r = 0 \tag{38}$$

$$w = 0 at r = h(z) \tag{39}$$

Zeroth Order System Solution

On solving equation (34) with boundary conditions (35)-(36), one can obtain zeroth order system solution as

$$w_0 = \left(\frac{r^2 - h^2}{4}\right)\frac{dp_0}{dz} - 1 \tag{40}$$

Volume flow rate F_0 is given by

$$F_0 = 2\int_0^h r w_0 dr \tag{41}$$

Substituting equation (40) in equation (41), one can obtain

$$\frac{dp_0}{dz} = -\frac{8(F_0 + h^2)}{h^4} \tag{42}$$

First Order System Solution

On solving equation (37) with boundary conditions (38)-(39), one can obtain First order system solution as

$$w_1 = \left(\frac{r^4 - h^4}{4}\right)\frac{dp_0}{dz} + \left(\frac{8(n-1)(r^2 - h^2)(F_0^3 + h^6 + 3F_0^2 h^2 + 3F_0 h^4)}{h^{12}}\right) \tag{43}$$

Volume flow rate F_1 is given by

$$F_1 = 2\int_0^h r w_1 dr \tag{44}$$

Substituting equation (43) in equation (44), one can obtain

$$\frac{dp_1}{dz} = -8\frac{\left(F_1 + \frac{8}{3}(n-1)\frac{(F_0^3 + h^6 + 3F_0^2 h^2 + 3F_0 h^4)}{h_6}\right)}{h_4} \tag{45}$$

Substituting the zeroth and first order system solution in equation (31)-(32), the concluding solution for velocity w and for pressure gradient $\frac{dp}{dz}$ for small values of We^2 can be written as

$$w = \left(\frac{r^2 - h^2}{4}\right)\frac{dp}{dz} - 1 + We^2\left(\frac{8(n-1)(r^4 - h^4)(F^3 + h^6 + 3F^2 h^2 + 3F h^4)}{h^{12}}\right) \tag{46}$$

$$\frac{dp}{dz} = -\frac{8F}{h^4} - \frac{8}{h^2} - We^2\left(\frac{\frac{16}{3}(n-1)(F^3+h^6+3F^2h^2+3Fh^4)}{h^{10}}\right) \tag{47}$$

Pressure rise per wavelength is given by
$\Delta P = \int_0^1 \frac{dp}{dz}dz$

$$\Delta P = \int_0^1 -\frac{8F}{h^4} - \frac{8}{h^2} - We^2\left(\frac{\frac{16}{3}(n-1)(F^3+h^6+3F^2h^2+3Fh^4)}{h^{10}}\right)dz \tag{48}$$

The nondimensionalized expression for shear stress

$$S_{rz} = \left[\left(\frac{\partial w}{\partial r} + \frac{n-1}{2}We^2\left(\frac{\partial w}{\partial r}\right)^3\right)\right] \tag{49}$$

From equation (49), we can find the expression for shear stress at wall is given as

$$S_{rz} = \left[\left(\frac{\partial w}{\partial r} + \frac{n-1}{2}We^2\left(\frac{\partial w}{\partial r}\right)^3\right)\right] at\, r = h \tag{50}$$

Equation (50), can be written as
$S_{rz} = (1+K(z))\frac{h}{2}\frac{dp}{dz} + R(z)(1+K(z)) + We^4R(z)\left(We^2 + \frac{3}{2}h\frac{dp}{dz}\right)$
where
$K(z) = We^2\frac{n-1}{2}h^2\left(\frac{dp}{dz}\right)^2$
$R(z) = We^2\left(\frac{4(n-1)(F^3+h^6+3F^2h^2+3Fh^4)}{3h^{10}}\right)$

4 Results and Discussion

This section of this manuscript deals with the influence of Weissenberg number We, power index n, non uniform parameter m on axial velocity, pressure gradient, shear stress and pressure rise.

The deviation in axial velocity for distinct values of non-uniform parameter m, power index n and Weissenberg number We are exhibited in Figs 2- 4. All plotted graphs are parabolic in nature also for the case of a converging tapered duct velocity gives more values as compared to a diverging and non-tapered duct in core region of the duct whereas reverse situation occurs near the wall of the duct. Fig. 2 shows in case of converging duct bile velocity increases on increasing non-uniform parameter m further opposite situation occurs in case of diverging duct. Further, velocity for non-tapered duct for distinct values of effecting parameters lies between velocity for diverging and converging ducts. Fig. 3 tells as power index increases bile flow moves fastly in ducts this gives rise raise in bile velocity in all three considered cases i.e., bile velocity is maximum when bile is considered as Newtonian fluid $(n = 1)$ as compared to Carreau fluid $(n < 1)$. Fig. 4 deals with the variation of Weissenberg number, it is noticed on increasing Weissenberg number bile velocity decreases also maximum bile velocity is noted down when bile is considered as Newtonian fluid $(We = 0)$ the reason behind this phenomenon is elastic forces increases over the viscous forces and same nature is found for all three types of ducts. Results for velocity are similar to the results of Vajravelu et al.[29].

The analysis of pressure gradient is discussed in Figs. 5- 8 for numerous values of non-uniform parameter m, power index n, Weissenberg number We and amplitude ratio ϕ.

In all plotted graphs it is noticed pressure gradient is more when bile flows through a diverging tapered duct as compared to a converging and non-tapered duct.

The influence of non-uniform parameter is carried out in Fig. 5. As three cases are considered here i.e., non-tapered duct, converging duct and diverging duct. As non-tapered duct is free from m , hence no change in pressure gradient. When non-uniform parameter increases in case of converging duct this leads to make the duct narrower hence less pressure gradient is essential to flow whereas reverse pattern is detected in case of diverging duct as increasing non-uniform parameter make duct wider. Fig. 6 illustrates flux is more for Carreau fluid as compared to Newtonian fluid $(n = 1)$. Fig. 7 demonstrates that Weissenberg number We has a positive impact on pressure gradient i.e., on increasing We pressure gradient also increasing as Weissenberg number is inversely proportional to the duct thickness resulting smaller the duct width higher

the pressure gradient is needed to maintain the flux throughout the duct also same behaviour is captured in all considered cases of duct. Fig. 8 tells pressure gradient increases as amplitude ratio increases.
Figs. 9- 11 convey the behaviour of wall shear stress for different values of non-uniform parameter m, power index n and Weissenberg number We. from all plotted graphs it is noticed converging duct requires more shear stress S_{rz} as compared to diverging and non-tapered duct. Fig. 9 expresses the variation of wall shear stress S_{rz} on varying non-uniform parameter m and it is found non-uniform parameter has an ability to rise and down the wall shear stress S_{rz} as three cases are taken into account i.e., non-tapered duct, converging duct and diverging duct. As non-tapered duct is free from m , hence no change in shear stress. Two different patterns are captured, in the case of a converging duct, shear stress S_{rz} increase on increasing m whereas in the case of a diverging duct shear stress S_{rz} decreasing on increasing m. It is detected in Fig. 10 as power index n increases wall shear stress S_{rz} decreases and it is least when bile is considered as Newtonian fluid ($n = 1$). Fig. 11 exhibits when Weissenberg number We increases, shear stress S_{rz} also boosts up in all three types of ducts as Weissenberg number is the proportion of stress relaxation time and specific process time. So, it is very clear when We increases, there will be an increment in stress relaxation time and when relaxation time increases flow can pass through quickly resulting increment in shear stress S_{rz}.
Figs. 12-14 relates the nature of pressure rise vs average flux Θ for different values of non-uniform parameter m, power index n and Weissenberg number We. Pressure rise is found to be increased in case of converging duct. The pumping phenomenon can be classified into three regions, where the deviation in pressure rise carried out. The region where $\Delta P < 0$ is known as pumping region, and also known as positive pumping for $\Theta > 0$. The region where $\Delta P < 0$ is known as co-pumping region and region where $\Delta P = 0$ is known as pumping free region. To see the influence of non-uniform parameter on pressure rise $\Delta P = 0$, Fig. 12 is carried out and is explained that two patterns are observed, in case of converging duct ΔP increases while in case of diverging duct $DeltaP$ decreases on increasing m, in all three regions of pumping. Fig. 13 is constructed to present the impact of power index n on ΔP. Figure tells relation is linear for Newtonian bile ($n = 1$), further, the curves for Carreau fluid ($n < 1$) lies below the curve of ($n = 1$) in pumping region, and lies above in co-pumping region. Same nature is found in case of all three ducts. Fig. 14 deals with impact of We on ΔP, it is noticed curves are linear for $We = 0$ while non-linear behaviour is found for other values of We. It is noticed, in pumping region ΔP decreases on increasing We whereas opposite trend is found in case of co-pumping region.

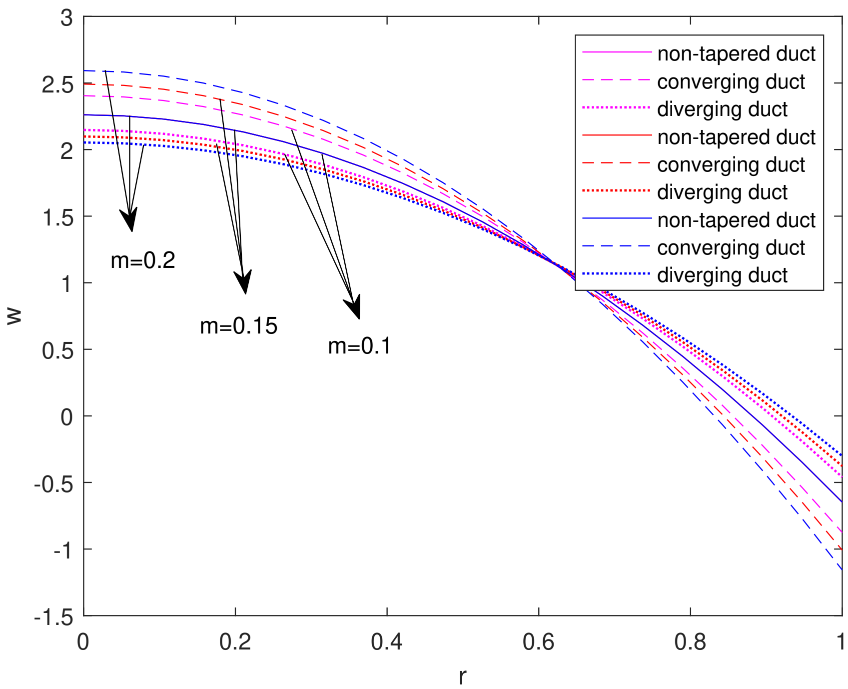

Fig. 2: Plot of velocity w with r for different values of m with $n = 0.398, \Theta = 0.5$ and $We = 0.1$

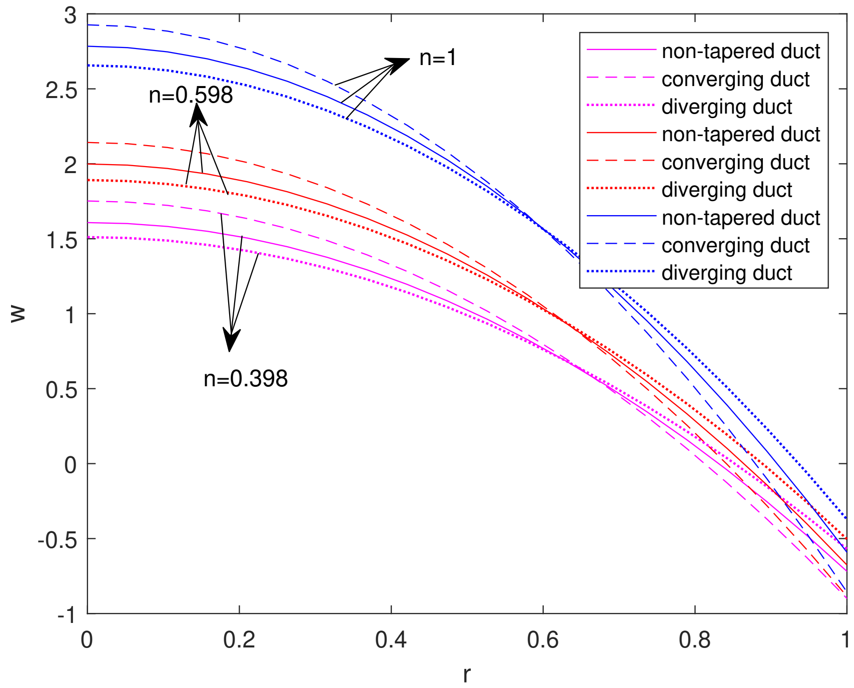

Fig. 3: Plot of velocity w with r for different values of n with $We = 0.1, \Theta = 0.5$ and $m = 0.1$

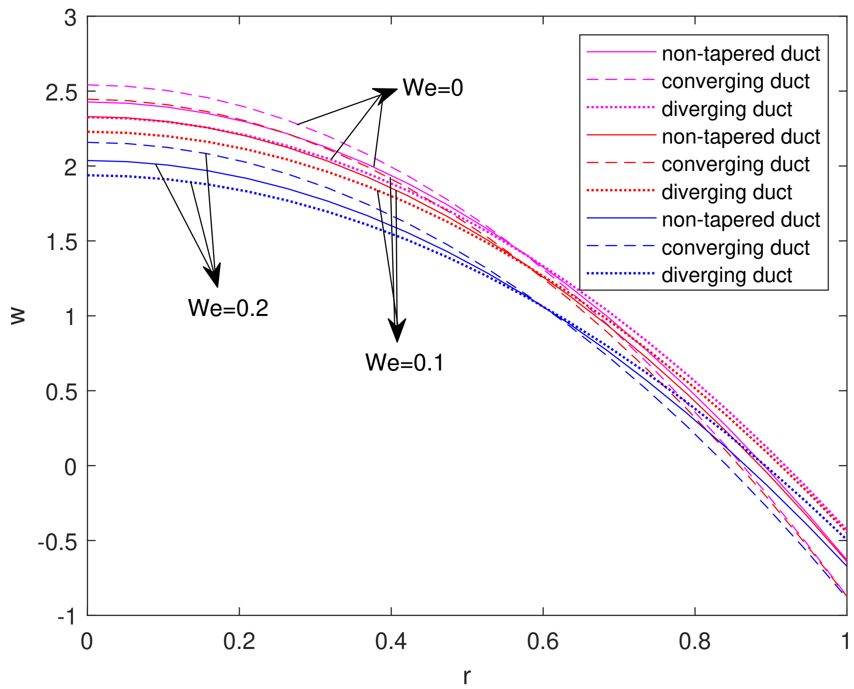

Fig. 4: Plot of velocity w with r for different values of We with $n = 0.398, \Theta = 0.5$ and $m = 0.1$

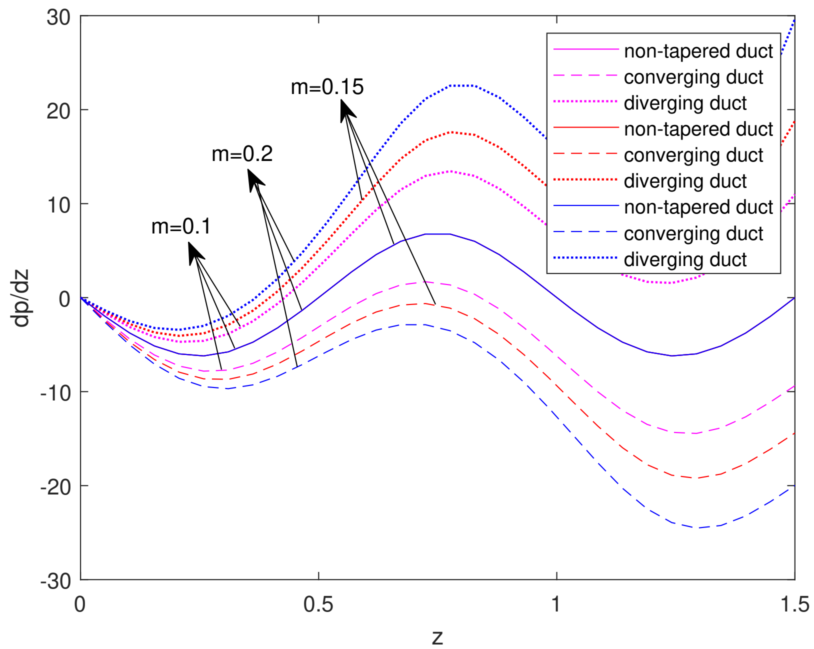

Fig. 5: Plot of pressure gradient $\frac{dp}{dz}$ with z for different values of m with $n = 0.398, \Theta = 0.5, \phi = 0.1$ and $We = 0.1$

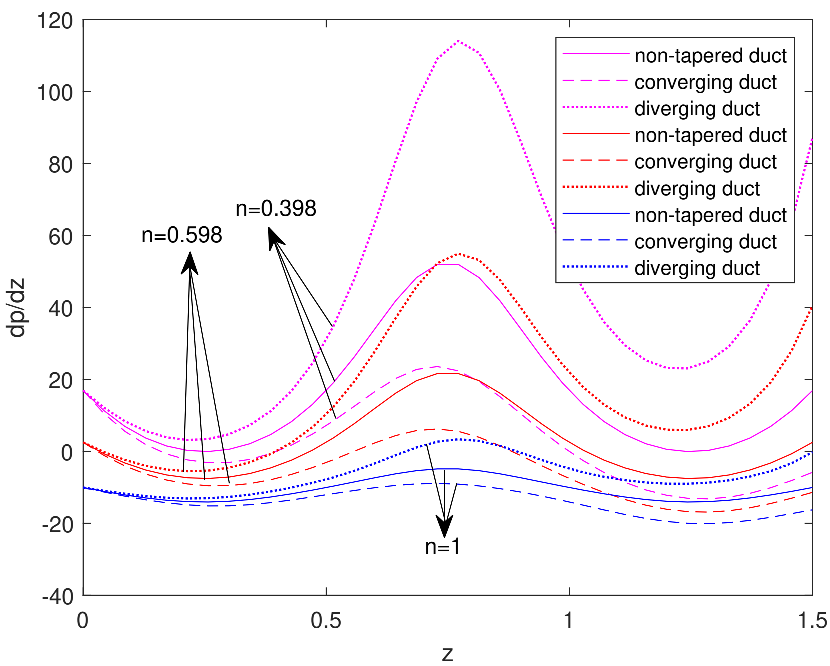

Fig. 6: Plot of pressure gradient $\frac{dp}{dz}$ with z for different values of n with $m = 0.1, \Theta = 0.5, \phi = 0.1$ and $We = 0.1$

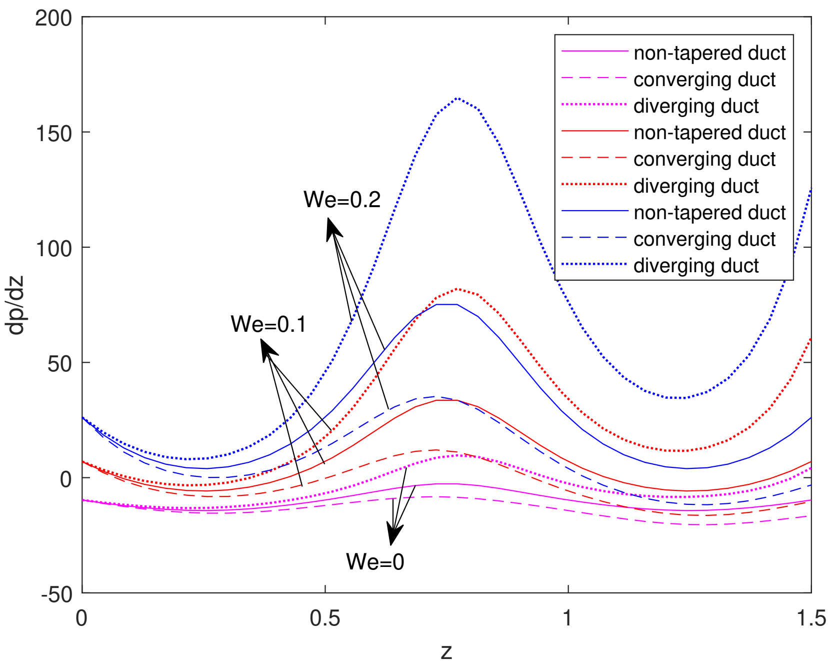

Fig. 7: Plot of pressure gradient $\frac{dp}{dz}$ with z for different values of We with $n = 0.398, \Theta = 0.5, \phi = 0.1$ and $m = 0.1$

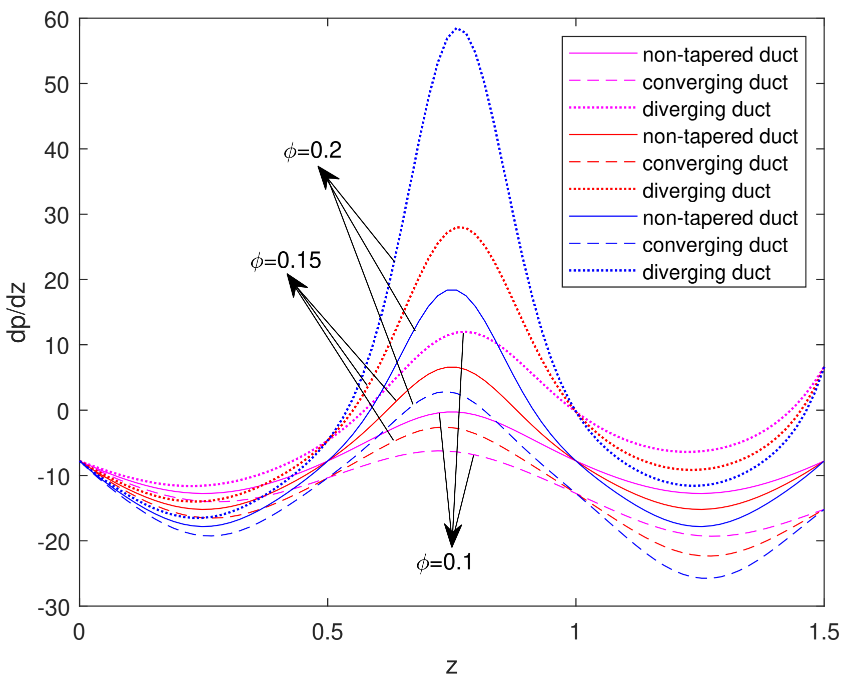

Fig. 8: Plot of pressure gradient $\frac{dp}{dz}$ with z for different values of ϕ with $n = 0.398, \Theta = 0.5, m = 0.1$ and $We = 0.1$

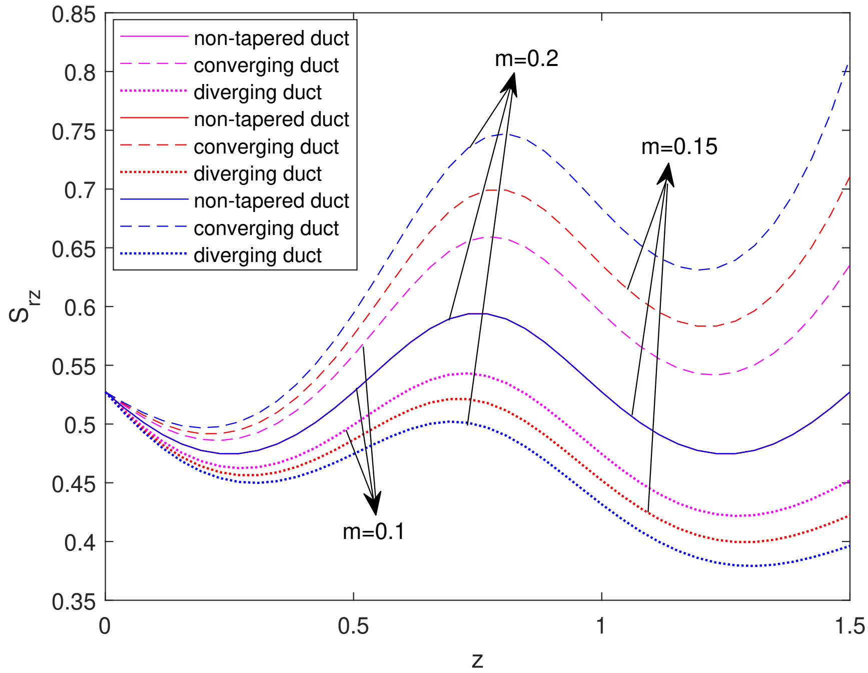

Fig. 9: Plot of wall shear stress S_{rz} with z for different values of m with $n = 0.398, \Theta = 0.5, \phi = 0.1$ and $We = 0.1$

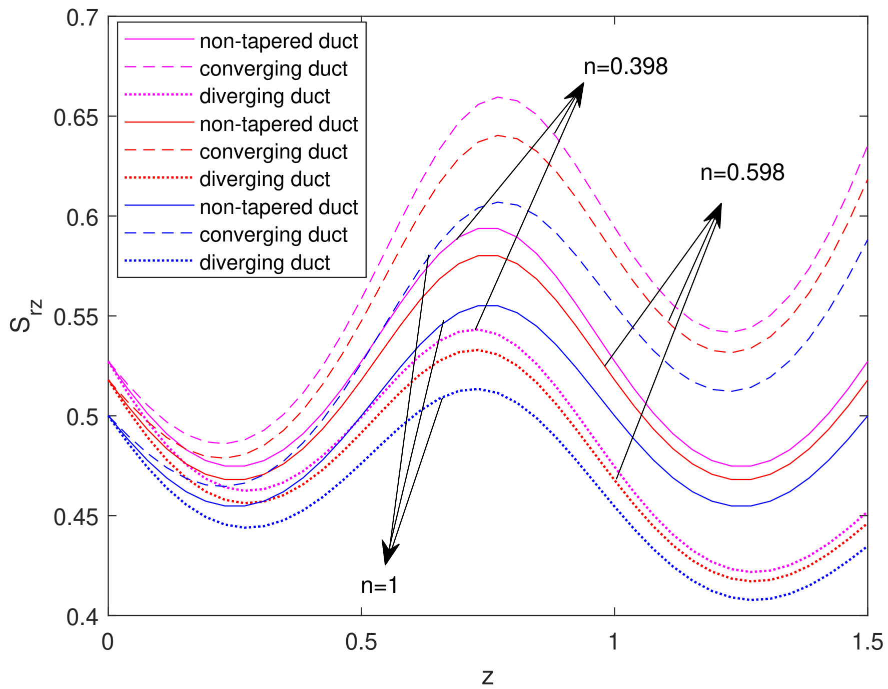

Fig. 10: Plot of wall shear stress S_{rz} with z for different values of n with $m = 0.1, \Theta = 0.5, \phi = 0.1$ and $We = 0.1$

5 conclusion

A manuscript is detailed here to explore the analysis of bile flow as Carreau fluid model in tapered duct. Model deals with long wavelength and low Reynolds number assumptions with suitable boundary conditions, also perturbation approach is

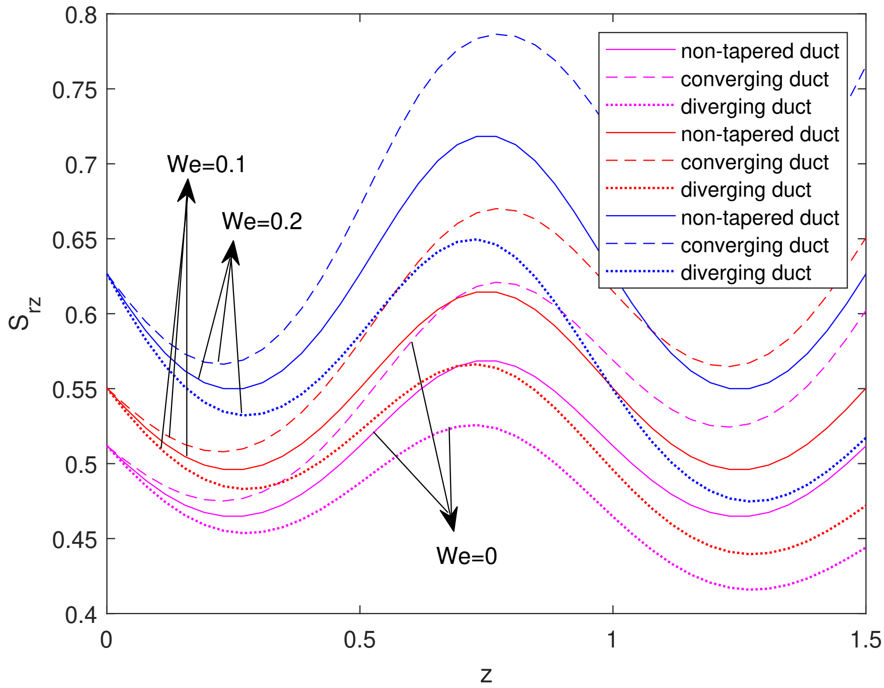

Fig. 11: Plot of wall shear stress S_{rz} with z for different values of We with $m = 0.1, \Theta = 0.5, \phi = 0.1$ and $n = 0.398$

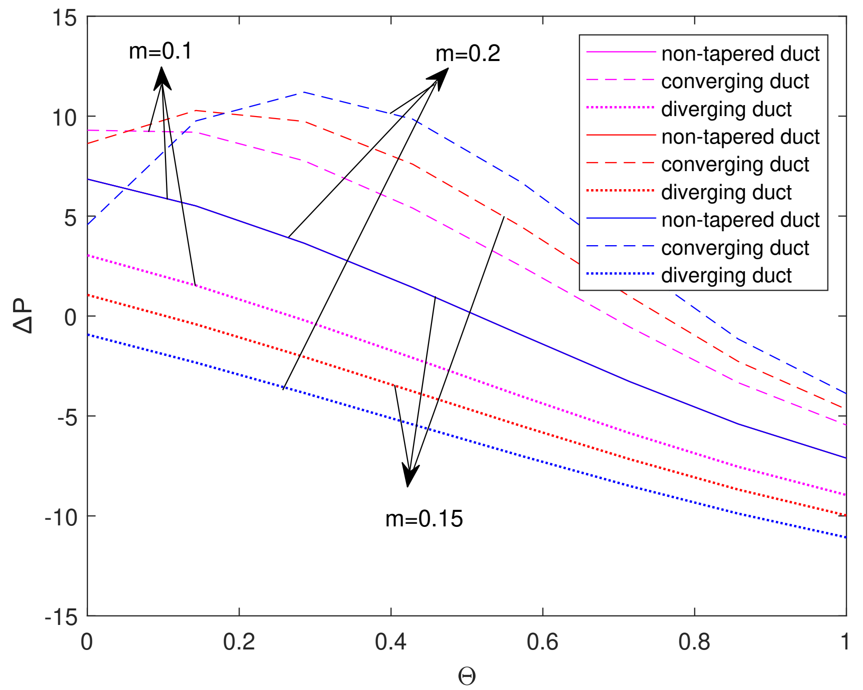

Fig. 12: Plot of pressure rise ΔP with Θ for different values of m with $n = 0.398, \phi = 0.1$ and $We = 0.1$

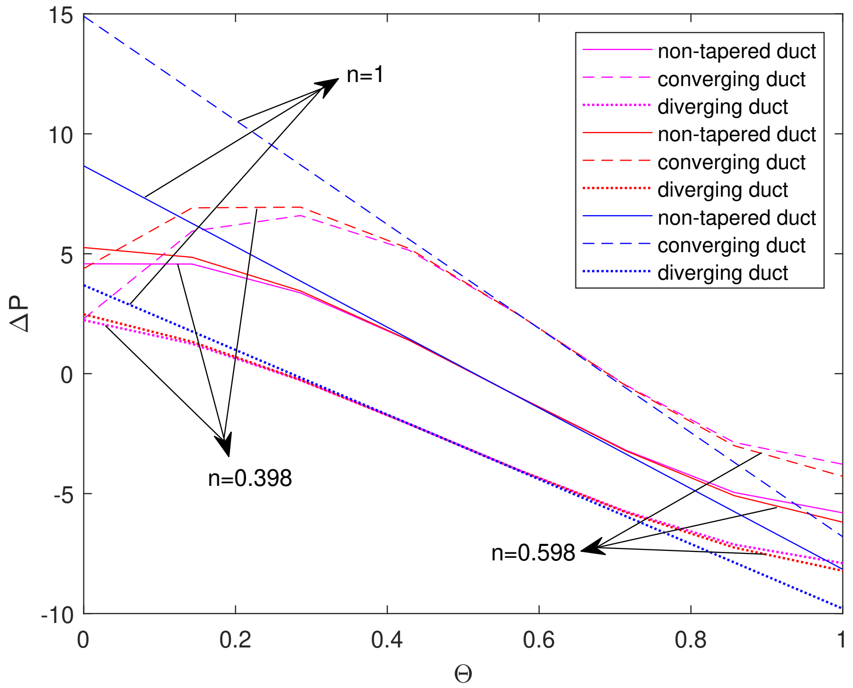

Fig. 13: Plot of pressure rise ΔP with Θ for different values of n with $m = 0.1, \phi = 0.1$ and $We = 0.1$

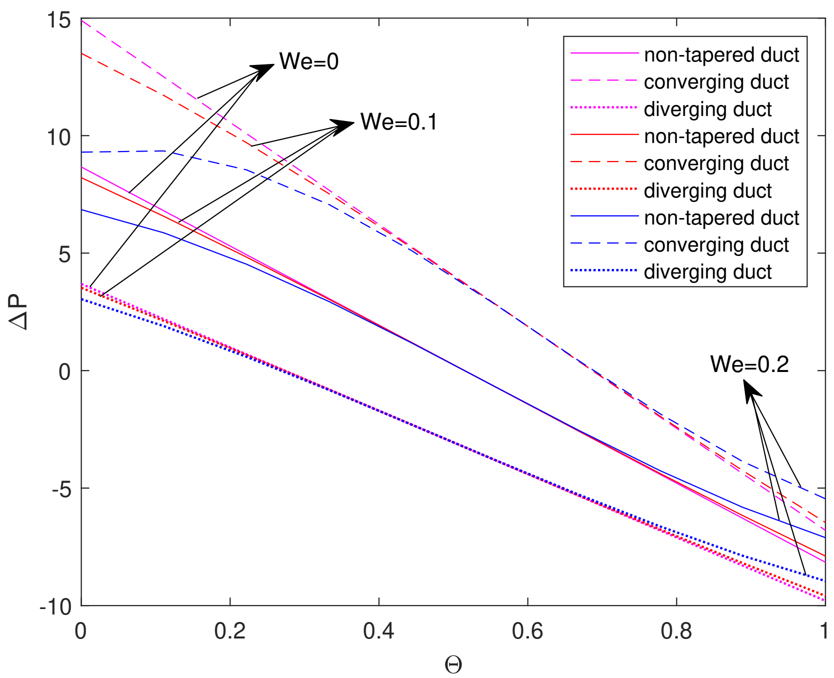

Fig. 14: Plot of pressure rise ΔP with Θ for different values of We with $n = 0.398, \phi = 0.1$ and $m = 0.1$

taken out to govern the analytical expressions for axial velocity, pressure gradient and shear stress. The impact of effecting parameters of the problem is stretched out. The main points of this problem are listed below as follow.

–Bile velocity is captured maximum in case of converging duct. Also, for Newtonian bile i.e., $n = 1$ or $We = 0$ bile reaches to its maximum velocity.
–Maximum velocity is found in the central part of the ducts.
–Interestingly here it is found if bile flows through wider duct, it requires more pressure gradient to flow also more $\frac{dp}{dz}$ is needed for more magnitude of We and m where as converse tendency is gained in case of n.
–Through obtained graph it is observed wall shear stress S_{rz} increases with increasing We and m.
–When bile is considered as Newtonian fluid ($n = 1$ or $We = 0$) less amount of wall shear stress S_{rz} is noticed.

Declarations

Competing interests: We declare that we have no competing interest related to this research article.
Authors' contributions: First two authors developed the mathematical model according to the physical problem. First author solved this model by analytical approach. First and Second authors anaylzed the results by plotting graphs in MATLAB R2018b software. All authors reviewed and approved the final version of the manuscript.
Funding: There is no financial support for the research by any organizations or institutions.
Availability of data and materials: The governing equations of continuity and motion are available in literature.
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