Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.4.12

On Classes of Surfaces with a Common Special Curve

Azam Etemad Dehkordy[1,*]

Department of Mathematical Sciences,
Isfahan University of Technology
Isfahan, IRAN.

Received: Oct. 7, 2022 Accepted : March 17, 2024

Abstract: In recent years, many researchers have studied surfaces with common geodesic or common line of curvature in different spaces. In this paper we give necessary and sufficient conditions for some classes of surfaces in the three dimensional Euclidean space $\mathbb{E}^3$ to have a curve as common geodesic or common line of curvature. Precisely, for a given curve, we suggest a family of surfaces which have it as their common line of curvature or common geodesic. Moreover, we illustrate the results by examples.

Keywords: geodesic; line of curvature; classes of surfaces.

2010 Mathematics Subject Classification. 53A04; 53A05; 53A17..

1 Introduction

From differential geometry, we know that a geodesic is a locally length-minimizing curve ([4]). Thus, the problem of geodesics on a surface is one of the important topics not only for differential geometry in geometric design and surface analysis, but also in many branches of sciences, such as physics, mechanics and computer design. We can deal with this problem in either of two aspects. One aspect is finding the geodesics on a given surface (for example see [3], [7]). This problem has a classical approach with solving a certain second order system of ordinary differential equations. It is, of course, not possible to have an analytical solution for all such systems of ODE.
The second aspect is determining surfaces with a given curve as common geodesic. Maybe the motivation for this aspect is stemmed from various industrial applications.
We use an example in shoe design to illustrate potential applications of this problem. A model of a women's shoe, which is usually represented by one or several free-form B-spline surfaces. On the shoe, there is one important characteristic curve called the girth, which more or less measures the width and height of the shoe. Given a particular model and the nominal size of the shoe, the girth is usually fixed, while the shape of the shoe changes frequently to suit various design intents, e.g. the fashion. A common practice in shoe manufacturing industry is to require that, when the shoe's surface is flattened to the plane, the girth should be mapped to a straight or near-straight line with minimum flattening distortion. This implies that the girth is preferred to be a geodesic on the shoe's surface ([10]).
Rcently, the later aspect has been considered by some of researchers. For example Gülnur Saffak and Emin Kasap in [13] analyzed the problem of finding a surfaces family through a null curve with Cartan frame. Moreover, surfaces with a common geodesic curve according to Bishop frame is subject to a research by them ([14]). Zühal Kücükarslan Yüzbasi and Mehmet Bektas studied surface family with common geodesic in Galilean Space G_3 in sequel ([8]). In [12], constructing a family of parametric surfaces from a given pseudo null curve using the Frenet trihedron frame of the curve in the Minkowski 3-space is also studied by Ufuk Ozturk. Furthermore, surface family with a common natural geodesic lift is title of [5] by Evren Ergün and Ergin Bayram.
On the other hand, there are papers that study surfaces with common certain line of curvature. Line of curvature is a tool which applied in surface analysis for exhibiting variations of the principal direction ([9], [15]). The corresponding principal curvature and principal directions are also important attributes on the smooth surfaces ([10]).
Analyzing the problem of constructing a surface pencil from a given spacelike (timelike) line of curvature ([6]) and

* Corresponding author e-mail: ae110mat@iut.ac.ir

finding the parametric presentation of a surface family with a common line of curvature in 3-dimensional Euclidean $\mathbb{E}^3$ and Galilean spaces G_3 ([1],[2] respectively) are examples of such papers. The relation between lines of curvature and particular points such as umbilical points is also one of the interesting topics ([11]).
In this paper, at first, we focus on translational surfaces in three dimensional Euclidean space. In detail, we find that for special curve, one can construct a class of translational surfaces with common geodesic and common line of curvature.
Plane curves are bases of a branch of geometry, named plane geometry. One of important applications of plane curves is simplifying algebraic problems. We can also find applications of the plane curve in algebraic geometry and complex analysis. In the sequel, we introduce some new classes of surfaces with plane curves as common lines of curvature or common geodesics. Some examples are also given as illustration of our main results.

2 Preliminaries

In this section, we recall some notions from differential geometry.
A regular curve is a curve that admits a tangent vector field. In this paper, we assume that all curves are regular. Let $\alpha : I \to \mathbb{E}^3$ be a parametric curve with parameter u, where I is a closed interval of $\mathbb{R}$. In the sequel, let prime denote the derivative of single variable function with respect to its variable.
So, the Frenet-Serret frame along $\alpha(u)$ is $\{T(u), N(u), B(u)\}$, where

$$\begin{aligned} T(u) &= \frac{\alpha'(u)}{||\alpha'(u)||}, \\ N(u) &= \frac{(\alpha'(u)\times\alpha''(u))\times\alpha'(u)}{||\alpha'(u)\times\alpha''(u)||\ ||\alpha'(u)||}, \\ B(u) &= \frac{\alpha'(u)\times\alpha''(u)}{||\alpha'(u)\times\alpha''(u)||}. \end{aligned}$$

are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(u)$, respectively. We have also

$$\begin{pmatrix} T'(u) \\ N'(u) \\ B'(u) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(u)||\alpha'(u)|| & 0 \\ -\kappa(u)||\alpha'(u)|| & 0 & \tau(u)||\alpha'(u)|| \\ 0 & -\tau(u)||\alpha'(u)|| & 0 \end{pmatrix} \begin{pmatrix} T(u) \\ N(u) \\ B(u) \end{pmatrix}$$

where $\kappa(u) = \frac{||\alpha'(u)\times\alpha''(u)||}{||\alpha'(u)||^3}$ and $\tau(u) = \frac{(\alpha'(u)\times\alpha''(u)).\alpha'''(u)}{||\alpha'(u)\times\alpha''(u)||^2}$ are defined curvature and torsion at the point $\alpha(u)$ respectively.
The set $\{T, N, B, \kappa, \tau\}$ is called the Frenet-Serret apparatus of α.
Definition of a planar curve in three dimensional Euclidean space is one that lie in a plan. Here for simplification, we consider curves in planes orthogonal to coordinate axes. Also we assume several possibles parameterization for a plane curve in the three dimensional space.
An isoparametric curve of a surface M is a curve obtained by varying one of the two surface parameters of M, while leaving the value of the other parameter unchanged. A rational surface (a rational curve) is a surface(curve) that has a parametrization by rational functions.

3 Translational Surface

A translational surface is a rational surface generated from two rational space curves by translating one curve along the other curve. Without loss of generality we will parameterize a translational surface M as follows.

$$\mathbf{x}(u,v) = (u, v, \varphi(u) + \psi(v)), \qquad -a \le u \le a, \quad -b \le v \le b \qquad (3.1)$$

where, $\alpha(u) = (u, 0, \varphi(u))$ with $-a \le u \le a$ and $\beta(v) = (0, v, \psi(v))$ with $-b \le v \le b$ are curves that generate the translational surface. Thus the elements of Frenet-Serret apparatus of α are:

$$\begin{aligned} T(u) &= \tfrac{1}{A}(1, 0, \varphi'(u)), \\ N(u) &= \tfrac{1}{A}(-\varphi'(u), 0, 1), \\ B(u) &= (0, -1, 0), \\ \kappa(u) &= \frac{\varphi''(u)}{A^3}, \\ \tau(u) &= 0, \end{aligned}$$

where $A = \sqrt{1+\varphi'^2(u)}$. So as desired, α is a plan curve. On the other hand, for M we have,

$$\begin{aligned}
&\mathbf{x}_u(u,v) = (1,0,\varphi'(u)),\\
&\mathbf{x}_v(u,v) = (0,1,\psi'(v)),\\
&\eta(u,v) = \tfrac{\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)}{||\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)||} = \tfrac{1}{B}(-\varphi'(u),-\psi'(v),1)
\end{aligned}$$

where $B = \sqrt{1+\varphi'^2(u)+\psi'(v)}$. Therefore we get the following theorem.

Theorem 1. *Let M be a translational surface with parametrization as in (3.1). If $\psi(0)=0$ then α is a geodesic on M if and only if $\psi'(0)=0$.*

Proof. At first, by assumption we have, $\mathbf{x}(u,0) = (u,0,\varphi(u)) = \alpha(u)$. Thus α is a curve on M. By classical geometry, α is a geodesic on M, if and only if $\eta(u,0)$ and $N(u)$ are parallel. But the calculations before this theorem, implies that this is equivalent to $\psi'(0)=0$. ■

We conclude from Theorem 1 that at each point of α, the principal normal vector and unit normal vector of the translational surface M are congruent in this case.
We recall that a curve α is a line of curvature on a surface M if at each point of α, the unit tangent vecor T of α and the unit normal vector η to the surface satisfy in the following relation,

$$\frac{d\eta}{du} = \lambda T$$

where $\lambda : [-a,a] \to \mathbb{R}$ is a suitable function.
Using the above calculation, we can deduce the next proposition about line of curvature for a translational surface.

Theorem 2. *Suppose that M is a translational surface which parameterized as (3.1) and $\psi(0)=\psi'(0)=0$. Then α is a line of curvature.*

Proof. By definition, α is a line of curvature for M if $S_\alpha(\alpha') = \lambda\alpha'$, where S is the shape operator of M and λ is a real valued function on $[-a,a]$. But $S_\alpha(\alpha') = -D\eta_\alpha(\alpha') = -\frac{d\eta}{du}$.
Now by assumption we have,

$$\begin{aligned}
\frac{d\eta}{du}(u,v) &= \tfrac{-\varphi'(u)\varphi''(u)}{A^3}(-v'(u),-\psi'(v),1) + \tfrac{1}{A}(-\varphi''(u),0,0)\\
&= \tfrac{1}{A^3}(-\varphi''(u),\varphi'(u)\varphi''(u)\psi(v),-\varphi'(u)\varphi''(u))\\
&= -\tfrac{\varphi''(u)}{A^3}(1,-\varphi'(u)\psi'(v),\varphi'(u))
\end{aligned}$$

Therefore, since $\psi'(0)=0$, $\kappa(u) = \frac{\varphi''(u)}{A^3}$ and $T(u) = (1,0,\varphi'(u))$, we obtain the result by taking $\lambda(u) = -\kappa(u)$ on $[-a,a]$.■

Example 1. let $\psi(v) = r_nv^n + r_{n-1}v^{n-1} + \cdots + r_2v^2$ be a polynomial of degree $n \geq 2$ with real coefficients, $r_n,\ldots,r_2$. Then Theorems 1 and 2 imply that , the translational surface M with parametrization $\mathbf{x}(u,v) = (u,v,\varphi(u)+\psi(v))$ has a line of curvature that is also a geodesic for every curve $\varphi : [-a,a] \to \mathbb{R}^3$, where a is a positive real number.
For instance, let φ be defined by $\varphi(u) = \ln(3-u)$ and $\psi(v) = v^3 - 3v^2$ on $[-\frac{5}{2},\frac{5}{2}]$. So we have a member of translational surface family with common geodesic and line of curvature which is parameterized by $x(u,v) = (u,v,\ln(3-u)+v^3-3v^2)$.

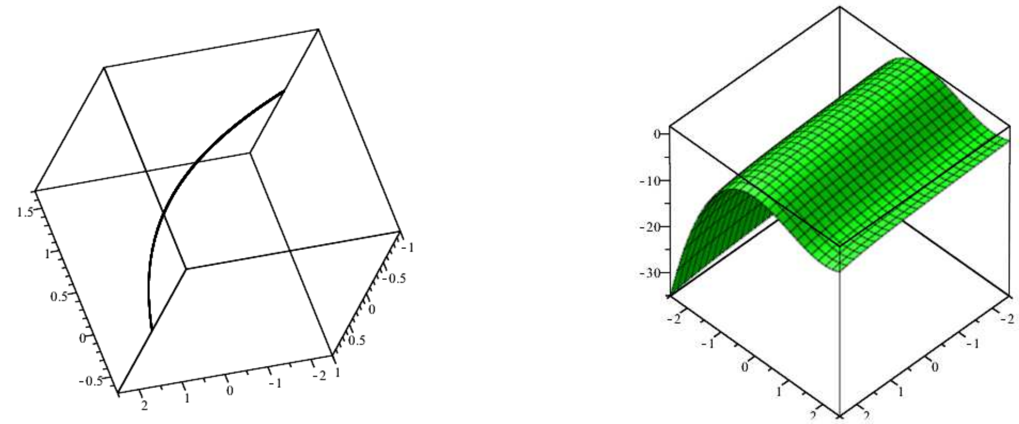

Fig. 1: (a)Image of α (b)Image of surface $\mathbf{x}(u,v)$

4 Some New Surfaces

In this section, we aim to generalize some results of section 3 to new classes of surfaces other than translational surfaces. Let β be a curve in the plane $z=b$ with the following parameterization:

$$\beta(u)=(u,a\varphi(u),b), \qquad u\in I$$

where a and b are real numbers, $I\subset\mathbb{R}$ is a closed interval and φ is a real valued (at least C^2-)differentiable function. Suppose that for a real closed interval J that contains a, $\psi: J\to\mathbb{R}$ is a C^2-differentiable function with $\psi(a)=b$. Now, we define the surface M which is parameterized by,

$$\mathbf{x}(u,v)=(u,v\varphi(u),\psi(v)) \tag{4.1}$$

So we have $\mathbf{x}(u,a)=(u,a\varphi(u),b)=\beta(u)$, i.e. β is an isoparametric curve on M. This surface can be considered as a kind of generalization of Whitney umbrella with parametrization $\mathbf{x}(u,v)=(u,uv,v^2)$.
We can trace the Whitney umbrella in many mathematical studies, mechanics and physical applications. For example in Hamiltonian mechanics, Whitney umbrella has very important type of singularities, i.e. it's singularities are not avoidable by small perturbation.
The idea for line of curvature and geodesic of surfaces in this section, can be also applied to many branches of mathematics, mechanics and physics.

Theorem 3. *The curve $\beta(u)=(u,a\varphi(u),b),\ u\in I$ with above assumptions is a line of curvature on the surface defined in (4.1) if and only if $\psi'(a)=1$.*

Proof. The Frenet–Serret frame of β is as follows:

$$\begin{aligned} T(u)&=\tfrac{1}{C}(1,a\varphi'(u),0),\\ N(u)&=\tfrac{1}{C}(-a\varphi'(u),1,0),\\ B(u)&=(0,0,1), \end{aligned}$$

where $C=C(u,v)=\sqrt{1+a^2\varphi'^2(u)}$. Furthermore, the unit normal vector field on M given by,

$$\eta(u,v)=\frac{\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)}{||\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)||}=\frac{1}{D}(v\varphi'(u)\psi'(v),-\psi'(v),\varphi(u))$$

where $D=D(u,v)=\sqrt{v^2\varphi'^2(u)\psi'^2(v)+\psi'^2(v)+\varphi^2(u)}$.
Suppose that $\psi(a)=b,\ \psi'(a)=1$. If we set $\overline{D}=\sqrt{C^2+\varphi^2(u)}$, then for $v=a$, we have.

$$\eta(u,a)=(\frac{-C}{\overline{D}})N(u)+(\frac{\varphi(u)}{\overline{D}})B(u)$$

Let $\mathbf{n} = \cos\theta N + \sin\theta B$ be a unit vector field orthogonal to β, where $\theta = \theta(u)$ is the angle between N and $\mathbf{n}$. But the curve $\beta(u)$ is a line of curvature on the surface $\mathbf{x}(u,v)$ if and only if $\mathbf{x}(u,v)$ is developable and $\mathbf{n}(u)$ is parallel to the normal vector field $\eta(u,a)$ of the surface. Obviously the condition $n||N$ is satisfied if and only if $\frac{-C}{D} = \lambda\cos\theta$ and $\frac{\varphi}{D} = \lambda\sin\theta$, where λ is a non-zero real valued function on I. But we have $(\frac{-C}{D})^2 + (\frac{\varphi}{D})^2 = 1$. Thus if we put $\lambda = 1$, we get the result.■

The functions $\lambda(u)$ and $\theta(u)$ in Theorem 3, are called controlling functions.

Example 2. Let $\beta(u) = (u, 2e^u, 1)$ and $\psi(v) = \frac{1}{3-v}$ be defined on $[-\frac{5}{2}, \frac{5}{2}]$. Therefore, by Theorem 3, β is a line of curvature on the surface given by $\mathbf{x}(u,v) = (u, ve^u, \frac{1}{3-v})$(Figure 2).

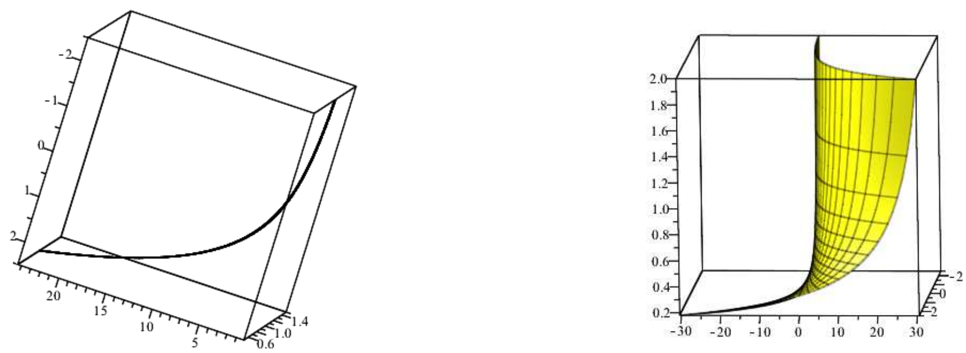

Fig. 2: (a)Image of β (b)Image of surface $\mathbf{x}(u,v)$

Example 3. Let $\gamma(u) = (u, 3u, \frac{3}{2})$ and $\psi(v) = \frac{v^2}{6}$ defined on $[-\frac{5}{2}, \frac{5}{2}]$. Therefore, by Theorem 3, γ is a line of curvature on the Whitney umbrella that given by $\mathbf{y}(u,v) = (u, vu, \frac{v^2}{6})$(Figure 3).

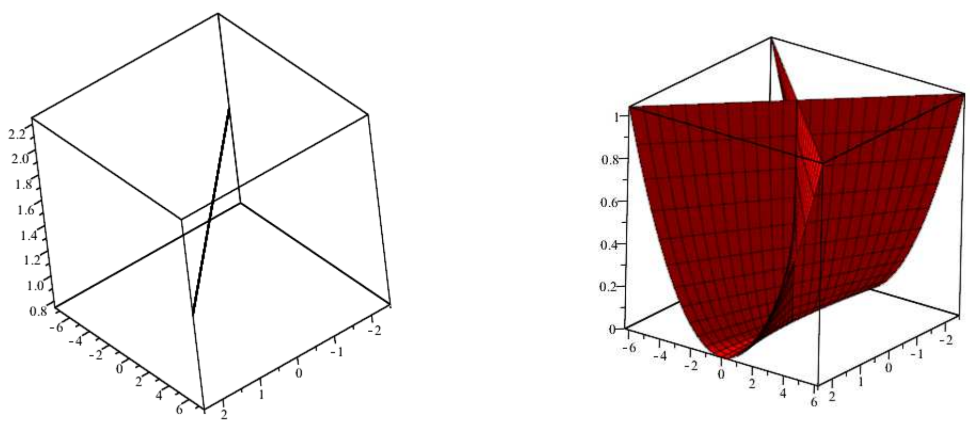

Fig. 3: (a)Image of γ (b)Image of surface $\mathbf{y}(u,v)$

Now let $\delta(v) = (d, c\psi(v), v)$ where c and d are real numbers, ψ is a real valued (at list C^2-)differentiable function and $v \in J$ for a closed interval J of $\mathbb{R}$. Consider a real interval I that contains c and let $\varphi : I \to \mathbb{R}$ be at least a C^2-differentiable with $\varphi(c) = d$. If a surface $\widehat{M}$ is defined by the parametrization,

$$\mathbf{z}(u,v) = (\varphi(u), u\psi(v), v)$$

then $\mathbf{z}(c,v) = (d, c\psi(v), v) = \delta(v)$, i.e. δ is isoparametric on $\widehat{M}$. We also have the following Frenet–Serret frame for δ.

$$T(v) = \tfrac{1}{E}(0, c\psi'(u), 1),\ N(v) = \tfrac{1}{E}(0, 1, -c\psi'(u)),\ B(v) = (-1, 0, 0)$$

where $E = E(u,v) = \sqrt{1 + c^2\psi'^2(v)}$. Moreover, if

$$F = F(u,v) = \sqrt{u^2\varphi'^2(u)\psi'^2(v) + \varphi'^2(u) + \psi^2(v)}$$

then

$$\eta(u,v) = \frac{1}{F}(\psi(v), -\varphi'(u), u\varphi'(u)\psi'(v))$$

is the unit normal vector field to $\widehat{M}$. Hence by the same way in Theorem 3, we obtain the following theorem.

Theorem 4. *Let $\varphi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ be differentiable functions at least of class C^2, where I, J are closed subsets of $\mathbb{R}$. Then for real numbers $c \in I$ and d, the curve $\delta(v) = (d, c\psi(v), v)$ is a line of curvature on the surface $\widehat{M}$ with parameterization $\mathbf{z}(u,v) = (\varphi(u), u\psi(v), v)$ if and only if $\varphi(c) = d$ and $\varphi'(c) = 1$.*

Example 4. Let $\delta(v) = (0, \frac{\pi}{2}\cosh v, v)$ and $\varphi(u) = \cos u$ defined on $[-\frac{7}{2}, \frac{7}{2}]$. Hence, by Theorem 4, δ is a line of curvature on the surface with parameterization $\mathbf{z}(u,v) = (\cos u, u\cosh v, v)$(Figure 4).

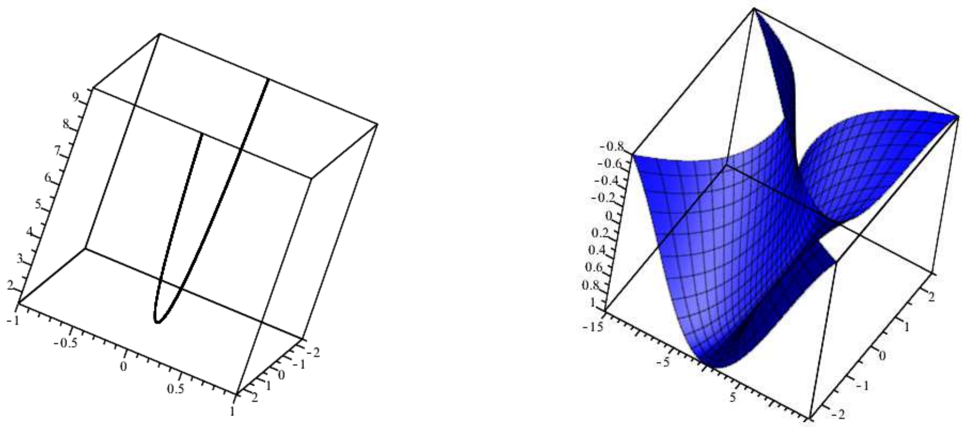

Fig. 4: (a)Image of δ (b)Image of surface $\mathbf{z}(u,v)$

We finally introduce a new surface family with common geodesic. Let e and h represent real numbers and I is a closed subset of $\mathbb{R}$. Assume that $\lambda : I \to \mathbb{R}^3$ is a curve defined by $\lambda(u) = (\varphi_1(u), e, \varphi_2(u) + h)$, where $\varphi_1, \varphi_2 : I \to \mathbb{R}$ are at least C^2-differentiable functions. Then we have the following theorem about λ.

Theorem 5. *Suppose that λ is defined as above, J is a closed interval in $\mathbb{R}$ and $\psi_1, \psi_2 : J \to \mathbb{R}$ are at least C^2-differentiable functions with $\psi_1(v_0) = e$ and $\psi_2(v_0) = h$ for $v_0 \in J$. Then λ is a geodesic on the surface $\bar{M}$ defined by*

$$\mathbf{w}(u,v) = (\varphi_1(u), \psi_1(v), \varphi_2(u) + \psi_2(v))$$

if and only if $\psi_1'(v_0) = 1$ and $\psi_2'(v_0) = 0$.

Proof. We have $\mathbf{w}(u, v_0) = (\varphi_1(u), e, \varphi_2(u) + h) = \lambda(u)$, so λ is an isoparametric curve on the $\bar{M}$. Furthermore

$$T(u) = \frac{1}{G}(\varphi_1'(u), 0, \varphi_2'(u)) \qquad N(u) = \frac{1}{G}(\varphi_2'(u), 0, -\varphi_1'(u)) \qquad B(u) = (0, -1, 0)$$

where $G = G(u,v) = \sqrt{\varphi_1'(u) + \varphi_2'(u)}$ and $\{T, N, B\}$ is the Frenet-Serret frame of λ.
Now if we set η for the unit normal vector field on $\bar{M}$, then

$$\eta(u,v) = \frac{1}{H}(-\varphi_2'(u)\psi_1'(v), -\varphi_1'(u)\psi_2'(v), \varphi_1'(u)\psi_1'(v))$$

where $H = H(u,v) = \sqrt{(\varphi_2'(u)\psi_1'(v))^2 + (\varphi_1'(u)\psi_2'(v))^2 + (\varphi_1'(u)\psi_1'(v))^2}$. But

$$\eta(u,v_0) = \frac{1}{H}(-\varphi_2'(u)\psi_1'(v_0), -\varphi_1'(u)\psi_2'(v_0), \varphi_1'(u)\psi_1'(v_0))$$

and $\eta(u,v_0)$ is parallel to $N(u)$ if and only if $\psi_1'(v_0) = 1$ and $\psi_2'(v_0) = 0$.■

*Example 5.*Let $\lambda(u) = (u^3, -1, u+1)$. Suppose that $\psi_1(v) = v$ and $\psi_2(v) = \cos v$ are defined on $[-\frac{7}{2}, \frac{7}{2}]$. If we assume the surface $\bar{M}$ is parametrized by $\mathbf{w}(u,v) = (u^3, v-1, u+\cos v)$, then since $\lambda(u) = \mathbf{w}(u,0)$, $\psi_1'(0) = 1$ and $\psi_2'(0) = 0$, we have λ is a geodesic on $\bar{M}$ by Theorem 5. (Figure 5).

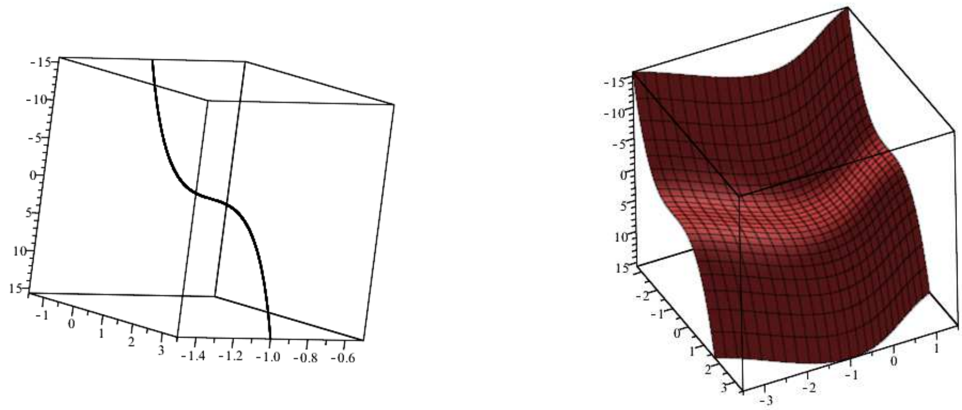

Fig. 5: (a)Image of λ (b)Image of surface $\mathbf{w}(u,v)$

By changing the roles of φ_1, φ_2 with ψ_1, ψ_2, we have the following similar theorem.

Theorem 6. *Let I and J be closed intervals in* $\mathbb{R}$, $\lambda : J \to \mathbb{R}^3$ *be a curve defined by* $\lambda(v) = (e, \psi_1(v), \psi_2(v) + h)$, *where* $\psi_1, \psi_2 : J \to \mathbb{R}$ *are at least* C^2*-differentiable functions and* $\varphi_1, \varphi_2 : I \to \mathbb{R}$ *be at least* C^2*-differentiable functions with* $\varphi_1(u_0) = e$ *and* $\varphi_2(u_0) = h$ *for* $u_0 \in I$ *and* $c, d \in \mathbb{R}$. *Then* λ *is a geodesic on the surface* $\bar{M}$ *parametrized by* $\mathbf{w}(u,v) = (\varphi_1(u), \psi_1(v), \varphi_2(u) + \psi_2(v))$ *if and only if* $\varphi_1'(u_0) = 1$ *and* $\varphi_2'(u_0) = 0$.

Example 6. Let $\lambda(v) = (1, \ln(8+v^2), -\frac{2}{5-v} + 4)$. Suppose that $\psi_1(v) = v$ and $\psi_2(v) = \cos v$ are defined on $[-\frac{7}{2}, \frac{7}{2}]$. If we assume the surface M is parametrized by $\mathbf{w}(u,v) = (\frac{1}{u^2+1} + u, \ln(8+v^2), 4\cosh u - \frac{2}{5-v})$, then since $\lambda(v) = \mathbf{w}(0,v)$, $\varphi_1'(0) = 1$ and $\varphi_2'(0) = 0$, we have λ is a geodesic on M by Theorem 6. (Figure 6).

Declarations

Competing interests: Art, architecture, religious.
Authors' contributions: AED wrote all the manuscript.
Funding: No funding was received.
Availability of data and materials: All data analyzed during this study are included in this article.
Acknowledgments: The author would like to thank the editor and the referees for their valuable comments and suggestions which improved the presentation of the paper.

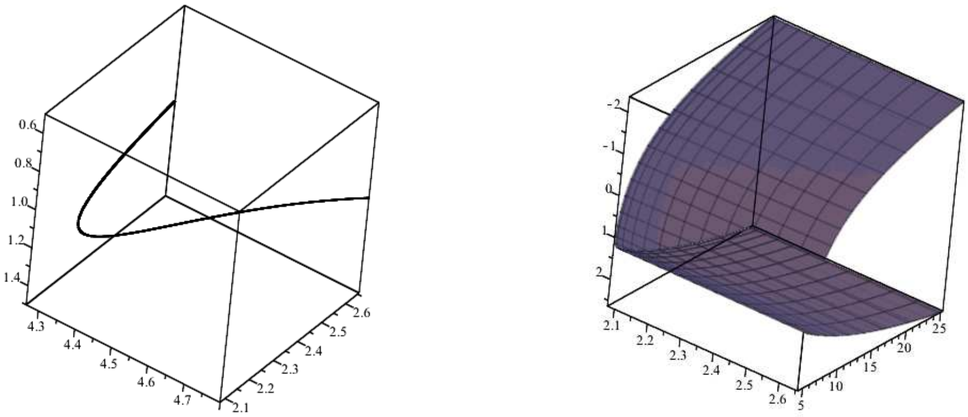

Fig. 6: (a)Image of λ (b)Image of surface $\mathbf{w}(u,v)$

References

[1] R. A. Abdel-Baky and N., Alluhaibi, Surfaces family with a common geodesic curve in Euclidean 3-space $\mathbb{E}^3$, *International Journal of Mathematical Analysis*,**13(9)**(2013), 433–447.

[2] M. Altin and İ. Ünal, Surface family with common line of curvature in 3-dimensional Galilean space, *Facta Universitatis (NIŠ) Ser. Math. Inform.*, **35(5)**(2020), 1315–1325.

[3] K. Crane, M. Livesu, E. Puppo and Y. Qin, A survey of algorithms for geodesic paths and distances, *arXiv preprint*, arXiv:2007.10430 (2020).

[4] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Facta Universitatis (NIŠ) Englewood Cliffs: Prentice Hall, (1976).

[5] E. Ergün and E. Bayram, Surface family with a common natural geodesic lift, *International J.Math. Combin.*, **1**(2016), 34–41.

[6] E. Ergün, E. Bayram and E. Kasap, Surface pencil with a common line of curvature in Minkowski 3-space, *Surface pencil with a common line of curvature in Minkowski 3-space*, **30(12)**(2013), 2103-2118.

[7] R. Kimmel and J.A. Sethian, Computing geodesic paths on manifolds, *Proceedings of the National Academy of Sciences*, **95(15)**(1998), 8431–8435.

[8] Z. Kücükarslan Yüzbasi and M. Bektas , On the construction of a surface family with common geodesic in Galilean space G_3, *Open Phys.*, **14**(2016), 360–363.

[9] C.Y. Li, R.H. Wang, and C.G. Zhu, An approach for designing a developable surface through a given line of curvature, *Comput. Aided Des.* , **45(11)**(2013), 621–627.

[10] C.Y. Li, R.H. Wang, and C.G. Zhu, Parametric representation of a surface pencil with a common line of curvature, *Comput. Aided Design*, **43(9)**(2011), 1110–1117.

[11] T. Maekawa, F.E. Wolter, and N.M. Patrikalakis, Umbilics and lines of curvature for shape interrogation, *Comput. Aided Geom. Design*, **13**(1996), 133–161‘.

[12] U. Ozturk, On surfaces with common pesudo null geodesic in Minkowski 3-space, *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.*, **66(1)**(2017), 229–241.

[13] G. Saffak and E. Kasap, Family of surface with a common null geodesic, *International Journal of Physical Sciences*, **4(8)**(2009), 426–433.

[14] G. Saffak Atalay and E. Kasapr, *Surfaces with a common geodesic curve according to Bishop frame*, XII. Geometry Symposium, Haziran, Bilecik, (2014).

[15] X.P. Zhang, W.J. Che, and J.C. Paul, Computing lines of curvature for implicit surfaces, *Computing lines of curvature for implicit surfaces*, **26(9)**(2009), 923–940.