JJMS **17**, No. 4, 609-634 (2024)

609

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.4.11

Modeling The Odd Moment Exponential-G Poisson Family of Distributions With Failure Time Data

Khaoula Aidi [1,*],*Farrukh Jamal*[2], *Sadaf Khan*[3],*Laba Handique* [4]

[1] Laboratory of probability and statistics LaPS, Badji Mokhtar- Annaba University 12, P.O.Box, 23000 Annaba. Algeria,
[2] Department of Statistics, Government S.A.Post-Graduate College, Dera Nawab Sahib,
[3] Department of Social and Allied Sciences, Cholistan University of Veterinary and Animal Sciences Bahawalpur, Pakistan
[4] Department of Statistics, Darrang College, Tezpur, Gauhati University, India

Received: Nov. 27, 2023 Accepted : Aug 27, 2024

Abstract: A univariate generalized family of continuous distributions, tentatively called the odd moment exponential-G Poisson family of distribution, has been introduced in this article. Among various techniques, the framework of compounding has been employed to devise the odd moment exponential-G distribution with the truncated Poisson distribution. With exponential distribution as a key model of the new family, the resultant model has been studied in lieu with theoretical and applied way. The theoretical foundation has been set up including definite mathematical expressions for shapes of density and hazard function, moments and related generating functions, process of residual life and its regeneration, ordered statistics, mechanics of material expressed in stress-strength expressions, Rnyi entropy and mean deviation among others. The estimation of the model parameters is performed by the maximum likelihood method for complete and censored scenario. A simulation study (for un-censored and censored case) is carried out under varying sample sizes to assess the efficacy of the model parameters. Three applications to the failure time data sets related to system reliability are used to showcase the extensibility of the proposed family. The postulated distribution is anticipated to be adaptable enough to model data sets in circumstances where both entire (un-censored) and partial information (censored) is accessible.

Keywords: OMEx-G family, P-G family, AIC, KS test, censoring.

2010 Mathematics Subject Classification. 60E05,62E15, 62F10.

1 Introduction

The functional efficacy of a system is undoubtedly impacted by the way its components are conceived and constructed. The model that the system engineer uses to assess the system's natural uncertainty levels, nevertheless, has an equal impact on the system's productivity. As the systems are becoming more and more complex to changing global scenarios, there still exists a need to define new models that capture the inherent uncertainties associated with failure of components. While a vacuum may be despised by nature, it is not prevalent in science.

Clearly the preceding assertion holds as we have lately noticed a huge increase in generalised families in modelling phenomena connected to many different scientific disciplines in modern literature. The exponential distribution (ED) has served as the foundation for the development, extension, and investigation of majority of these families of distributions in various scientific domains. The primary reason may be that the lifetime of a continuous process changing state is best characterized by the ED as it plays a pivotal role in estimating the potential timing of occurrence of a significant event. Additionally, it is the only distribution that holds the characteristic of bearing no after-effect, a feature commonly known as memory-less property. By expanding the traditional ED, numerous extended families of continuous distributions have lately been devised and employed to simulate a variety of phenomena. In the reference [1], Khan et al. provided a concise review of countless families which are developed due to ED with odd ratio as the fundamental functional form. Some of the prominent classes of distribution include: generalized exponential, odd generalized exponential; generalized odd generalized exponential, exponentiated extended-G distribution, modified odd Weibull family of distributions ; odd

* Corresponding author e-mail: khaoula.aidi@yahoo.fr

flexibleWeibull-H family of distributions ; new generalized odd log-logistic family of distributions ; generalized odd Lindley-G family ; extended Weibull - G family ; exponential Lindley odd log-logistic - G family, the Hjorth's IDB generator of distributions, amidst others. Interestingly, the author did propose an alternate generalised odd generalized exponential family and studied its usefulness in evaluating risk associated with financial data.

Iqbal et al. [2] emphasized the importance of moment distributions in the perspective of reliability theory among others, which are mainly the weights given to ED in particular ratio. They established a generalization of moment exponential distribution (MEx), due to Dara and Ahmed [3], employing the technique termed as Lehmann Alternative-2 and highlighted its significance using an environmental data.

Similarly, Hasnain [4] proposed exponentiated moment exponential distribution based on Lehmann Alternative-1. Hashmi et al. [5] constituted a special model, i.e. Weibull-MEx family based on famous odd Weibull-G family, initially proposed by Bourguinon et al.[6], in relation to strength data. Bhatti et al. [7] studied the BurrXII-MEx distribution by providing its utility to tax revenue and adherence limit data.

Reliability engineers frequently have to quantify the sustainability of systems with units linked in series, parallel or mix configurations as per system design. System reliability is most associated using the framework termed as compounding. For details related to compounding, the avid readers are referred to Tahir and Cordiero [8]. In the context of mix configuration (continuous- discrete) structure, Mustapha in [9] studied the Poisson odd generalized exponential (P-OGE) family. For $t > 0$, the distribution function of P-OGE is as

$$F_{poisson-OGE} = \frac{1-\exp\left(-\lambda\left(1-e^{-\alpha\left(\frac{G(x;\zeta)}{1-G(x;\zeta)}\right)}\right)^{\beta}\right)}{1-e^{-\lambda}}, \quad \text{where } \alpha,\lambda,\beta > 0$$

The author derived essential mathematical and structural properties of the proposed family, studied two special cases and applied them on two real life data set associated with component failure time data. In system reliability analysis, a standard approach for dealing with engineering data is to plot the instantaneous failure rate as a function of time. However, the failure rate behaviour of the postulated family was not addressed by the author, which somehow limited the family's potential applicability.

Ahsan-ul-Haq [10] proposed a discrete distribution viz. z viz. the Poisson-MEx distribution for over dispersed count data using the similar compounding technique. Given the significance of MEx distribution in structural analysis, this prompted us to postulate a family that is premised on MEx distribution that can account for continuous-discrete structures with sufficient flexibility to model conventional (increasing, decreasing, constant) as well as un-convential (bathtub, upside down bathtub and so forth) hazard rate function. This is achieved by introducing an extended class of distribution that contains the Poisson-G (GP) and OMEx-G distribution by adding one additional parameter, **so called Odd moment exponential-G Poisson (OMEx-GP)**, which covers some important distributions as special and related cases. The OMEx-GP family of distribution appears to be more flexible as compared to OMEx-G, which can only model an increasing failure time data when exponential distribution is chosen as baseline distribution.

Often incomplete or missing data are observed in lifetime analysis of components in system reliability analysis. Different censoring scheme are employed to observe the behavior of random events with feasible solutions. Right censoring scheme are ususally encoutured with progressive, hybrid and random a special types. In progressive censoring, data is observed at multiple stages indicating that the components are under observation and the exact time (if it occur) is unknown ; in hybrid censoring, data can be observed simultaeously or in a sequential manner ; in random censoring, the subjects are random in a sense that they are not systematically related to survivor time. The present study is unique due to the fact that OMEx-GP has the ability to model complete as well as censored data. An explanation of the paper's structure is provided herewith: in section 1, we layout the basis of OMEx-GP family, provide its physical interpretation and define the OMEx-Exponential Poisson (OMEx-EP) distribution by illustrating the behavior of density and hazard rate function; section 2 establishes the theoretical foundation of OMEX-EP with various mathematical and structural properties; section 3 comprises of inference related to complete and censored set of estimated parameters; The convergence for model parameters are validated via a simulation study for complete and censored data in section 4; section 5 provide the empirical findings when the proposed model is fitted on a failure time data for complete and right-censored set of observation in comparison with eight well-established models in literature; the whole article is summarized in section 6 with some conclusive remarks.

2 The model's inception

Let $k(t)$ be the probability density function (pdf) of a random variable, say T, where $T \in [m_0, n_1]$ for $-\infty \leq m_0 < m_1 < +\infty$. Let $W[K(z;\nabla)]$ be a function of cumulative distribution function (CdF) $K(z;\zeta)$ of a random variable, say Z

,depending on the vector parameter ∇ so that $W\left[K\left(z;\nabla\right)\right]$ is a non-decreasing differentiable function belonging to $[m_0,n_1]$ ascribed interval such that $W\left[K\left(z;\nabla\right)\right] \to m_0$ as$z \to -\infty$ and $W\left[K\left(z;\nabla\right)\right] \to m_1$ as$z \to +\infty$ Alzaatreh et al. [4], defined the CdF of the $T-X$ family of distributions by

$$K\left(z;\nabla\right) = \int_{m_0}^{W[K(z;\nabla)]} k(t)dt, \quad z \in R \tag{1}$$

Where $W\left[K\left(z;\nabla\right)\right]$ satisfies the stated conditions mentioned above. The PdF corresponding to (2) is given by

$$k\left(z;\nabla\right) = \left\{\frac{\partial}{\partial z}W\left[K\left(z;\nabla\right)\right]\right\}k\left\{W\left[K\left(z;\nabla\right)\right]\right\} \qquad z \in R. \tag{2}$$

Based on the $T-X$ family, Haq et al., [32] proposed the odd moment exponential generalized family in short $OMEx-G$ by taking $W\left[K\left(z;\nabla\right)\right] = \frac{K(z;\nabla)}{1-K(z;\nabla)}$ and $k(t)$ be the PdF of MEx distribution. The CdF of $OMEx-G$ family of distribution is defined by

$$G^{OMExG}\left(z,\beta,\nabla\right) = \int_0^{\frac{K(z;\nabla)}{1-K(z;\nabla)}} \frac{t}{\beta^2}e^{-t/\beta}dt = 1-\left\{1+\frac{1}{\beta}\frac{K\left(z;\nabla\right)}{1-K\left(z;\nabla\right)}\right\}\exp\left\{-\frac{1}{\beta}\frac{K\left(z;\nabla\right)}{1-K\left(z;\nabla\right)}\right\} \tag{3}$$

The PdF corresponding to equation (3) is

$$g^{OMExG}\left(z,\beta,\nabla\right) = \frac{k\left(z;\nabla\right)K\left(z;\nabla\right)}{\beta^2\left[1-K\left(z;\nabla\right)\right]^3}\exp\left\{-\frac{1}{\beta}\frac{K\left(z;\nabla\right)}{1-K\left(z;\nabla\right)}\right\} \tag{4}$$

where $k\left(z;\nabla\right)$ and $K\left(z;\nabla\right)$ be the **PdF** and **CdF** of any baseline distribution, and ∇ is the feasible set of parameter vector in a baseline distribution.

In the perspective of a mixed configuration of systems, the physical interpretation of the suggested model is quite intriguing. Consider a structure typically consisting of N independent modules, where N is a truncated Poisson random variable with a probability mass function (pmf)

$$p_n = \Pr\left(N=n\right) = \frac{\mu^n}{n!\left(e^{\mu}-1\right)}, \quad \mu>0; \ n=1,2,....$$

The conditional Poisson distribution has it mean $E\left(N\right) = \mu\left(1-e^{-\mu}\right)^{-1}$ with absolute squared deviation $V(N) = \left(\mu+\mu^2\right)\left(1-e^{-\mu}\right)^{-1} - \mu^2\left(1-e^{-\mu}\right)^{-2}$, correspondingly. Assume that each interdependent component's failure time conforms to the CdF specified in equation (3). Let T_i denote the time of collapse of the iˆth subsystem and Z is the period until the first of the N working subsystems malfunctions that is $X = \min\{T_1,T_2,...,T_n\}$. Then the conditional CdF of $Z|Nis$

$$F\left(z/N\right) = 1-\Pr\left(Z>z/N\right) = 1-P\left(T_i>z\right)^N = 1-\left[1-K^{OMExG}\left(z,\beta,\nabla\right)\right]^N$$

So, the unconditional CdF of X (for $z>0$) can be expressed as

$$\begin{aligned} F(x) &= \frac{1}{e^{\lambda}-1}\sum_{n=1}^{\infty}\frac{\lambda^n\left(1-\left[1-K^{OMExG}\left(z,\beta,\nabla\right)\right]^n\right)}{n!} \\ &= \frac{1-\exp\left[-\lambda K^{OMExG}\left(z,\beta,\nabla\right)\right]}{1-e^{-\lambda}} \end{aligned} \tag{5}$$

It is worth mentioning here that if we take $X = \max\left\{Z_1,Z_2,...,Z_n\right\}$ and complying as explained previously, the resultant CdF is

$$F(x) = \frac{e^{\lambda K^{OMExG}(z,\beta,\nabla)}-1}{e^{\lambda-1}} \tag{6}$$

In view of (5) and (6),the emergence of the new family of distributions unfolds as

$$F^{OMEx-GP}\left(z;\lambda,\beta,\nabla\right) = \frac{1-\exp\left[-\lambda K^{OMExG}\left(z,\beta,\nabla\right)\right]}{1-e^{-\lambda}}, \qquad \lambda \in R-\{0\};n=1,2....... \tag{7}$$

Equation (7) is termed as the distribution function of **odd moment exponential generalized Poisson family ("OMEx-GP"** for brevity) of distribution. The associated PdF and **hazard rate function (hrf)** of $OMEx-GP$ family is given by

$$f^{OMExGP}(z;\lambda,\beta,\nabla) = \left(1-e^{-\lambda}\right)^{-1} \lambda k^{OMExG}(z;\beta,\nabla) \exp\left[-\lambda K^{OMExG}(z;\beta,\nabla)\right], \lambda \in R-\{0\}; \beta > 0, \quad -\infty < z < +\infty \tag{8}$$

and

$$h^{OMEx-GP}(z;\lambda,\beta,\nabla) = \frac{\lambda k^{OMExG}(z,\beta,\nabla) \exp\left[-\lambda K^{OMExG}(z,\beta,\nabla)\right]}{\exp\left[-\lambda K^{OMEG}(z,\beta,\nabla)\right] - e^{-\lambda}} \tag{9}$$

respectively.

2.1 The OME- exponential Poisson (OME-EP) distribution

In this segment, we provide a distinctive case of the $OMEx-GP$ family of distributions by choosing exponential distribution as baseline and study their main distributional characteristics. For a positive real random variable x having $CdF\ K(z) = 1-e^{-\delta z}$, *such that* $\delta > 0$, the PdF and hrf of the $OMEx-EP$ model, respectively, is given as

$$\begin{aligned} f^{OMEx-EP}(z,\lambda,\beta,\delta) = {} & \frac{\lambda\delta e^{-\delta z}(1-e^{-\delta z})}{(1-e^{-\lambda})\beta^2 e^{-3\delta z}} \exp\left\{-\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right\} \\ & \exp\left\{-\lambda\left[1-\left(1+\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right)\exp\left(-\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right)\right]\right\} \end{aligned}$$

And

$$h^{OMEx-EP}(z,\lambda,\beta,\delta) = \frac{\frac{\lambda\delta e^{-\delta z}(1-e^{-\delta z})}{\beta^2 e^{-3\delta z}} \exp\left\{-\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right\} \exp\left\{-\lambda\left[1-\left(1+\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right)\exp\left(-\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right)\right]\right\}}{\exp\left\{-\lambda\left[1-\left(1+\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right)\exp\left(-\frac{1}{\beta}\frac{1-e^{-\delta z}}{e^{-\delta z}}\right)\right]\right\} - e^{-\lambda}}$$

2.2 Shape of the pdf's and hrf's

In order to investigate the range of shapes the family can exhibit, we have shown the PdF and hrf of the $OMEx-EP$ with specific parameter values.

From the plots in Figure 1 and Figure 2 it can be seen that the proposed family is very flexible and can offer many different types of density shapes like strictly right-skewed, J-shaped or symmetric with increasing, upside down-bathtub and increasing-decreasing-increasing shape of hazard rate function. This flexibility in hrf of the proposed model makes it useful to model not just reliability data but data from distinct scientific field. For example, increasing-decreasing-increasing shape can be used to model the business cycle in economics, nomadic migration in demography, nutrient cycles in environmental studies etc.

3 Mathematical Properties

In this portion, the computational properties of the $OMEx-GP$ are explored both theoretically and empirically in order to demonstrate the appropriateness of the proposed family with exponential distribution as baseline model.

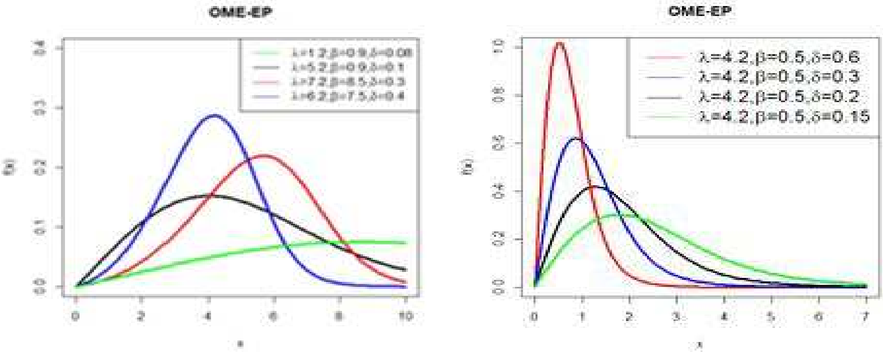

Fig. 1: Density plots of the OMEx-EP distribution for random parameter values.

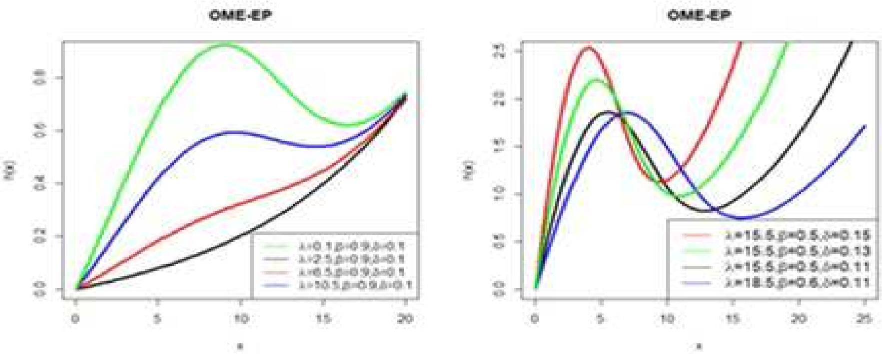

Fig. 2: Plots of hazard rate function of the OMEx-EP distribution for arbitrary .

3.1 Linear representation of density

We express (7) and (8) as infinite series expansion to show that the $OMEx-GP$ can be written as a linear combination of $OMEx-G$ as well as a linear combination of exponentiated-G distributions. These expressions will be helpful to study the mathematical characteristics of the OMEx-GP family.

Using the power series for the exponential function, we can write (8) as

$$f^{OMEx-GP}(z;\lambda,\beta,\nabla)=k^{OMEG}(z,\beta,\nabla)\sum_{i=0}^{\infty}\gamma_i\left[K^{OMExG}(z,\beta,\nabla)\right]^i \tag{10}$$

$$f^{OMEx-GP}(z;\lambda,\beta,\nabla)=\sum_{i=0}^{\infty}\gamma_i'\frac{d}{dz}\left[K^{OMExG}(z,\beta,\nabla)\right]^{i+1} \tag{11}$$

Where

$$\gamma_i'=\frac{(-1)^i\lambda^{i+1}}{(1-e^{-\lambda})(i+1)i!} \text{ and } \gamma_i=\gamma_i'(i+1)$$

By means of Taylor series expansion, the CdF in (7) can be written as

$$F^{OMEx-GP}(z;\lambda,\beta,\nabla)=\sum_{j=0}^{\infty}t_j\left[K^{OMExG}(z,\beta,\nabla)\right]^j \tag{12}$$

Where

$$t_j=\frac{(-1)^{j+1}\lambda^j}{(1-e^{-\lambda})j!}$$

3.2 Moment Generating Function

Using the findings of Section 3.1, it is straightforward to characterize the moment generating function (MgF) of the $OMEx-GP$ family in terms of those of the exponentiated $OMExG$ distribution. Employing the results of equation (10) it is evident that

$$M_Z(S) = E\left[e^{sZ}\right] = \int_{-\infty}^{+\infty} e^{sz} k^{OMEx-GP}(z;\lambda,\beta,\nabla)\,dz$$
$$= \int_{-\infty}^{+\infty} e^{sz} \sum_{i=0}^{\infty} \gamma_i' \frac{d}{dz}\left[K^{OMExG}(z,\beta,\nabla)\right]^{i+1} dz$$
$$= \sum_{i=0}^{\infty} \gamma_i' \int_{-\infty}^{+\infty} e^{sz} \frac{d}{dz}\left[K^{OMExG}(z,\beta,\nabla)\right]^{i+1} dz$$
$$= \sum_{i=0}^{\infty} \gamma_i M_Z(S)$$

where $M_Z(s)$ is the mgf of a exponentiated $OMExG$ distribution.

In Table 1 & 2, some empirical findings of $OMEx-EP$ model pertaining to the practical implications of the results due to well-known relationships of MgF have been reported. Particularly in Table 1, the numerical evaluations of descriptive statistics are scripted while in Table 2, the relative measures such as coefficient of variation (C_{var}), coefficient of skewness (C_{skw}) and coefficient of kurtosis (C_{kur}) are placed. **In quantitative research analysis, these descriptive measures are used to summarize the main characteristics such as the central value and dispersion around the central value in a dataset.** Some visualization showing the flexibility of the proposed $OMEx-EP$ in the overall shape and tail behavior are also provided in Figure 3.

Parameter Combinations				Descriptive Statistics		
λ	β	δ	μ	σ^2	B_{sk}	M_{kur}
0.5	0.5	0.5	1.1871	0.3973	0.5819	63.2375
1.5	0.5	0.5	1.0184	0.3429	0.8168	57.0930
2.0	0.5	0.5	0.9427	0.3109	0.9278	56.9108
3.0	0.5	0.5	0.8122	0.2470	1.1167	60.9978
5.0	0.5	0.5	0.6296	0.1492	1.2906	82.4686
0.5	1.0	0.5	1.8758	0.7515	0.3185	59.6596
0.5	2.0	0.5	2.7671	1.1953	0.0719	66.9887
0.5	3.0	0.5	3.3710	1.4622	-0.0586	77.2882
0.5	5.0	0.5	4.2020	1.7739	-0.2053	98.6797
1.5	5.0	0.5	3.8380	1.7339	-0.0354	77.8294
3.0	5.0	0.5	3.3631	1.5224	0.1359	69.9930
5.0	5.0	0.5	2.9006	1.2000	0.1959	77.1263
0.5	0.5	2.0	0.2967	0.0248	0.5819	307.8143
0.5	1.0	2.0	0.4689	0.0469	03185	314.9609
0.5	2.0	2.0	0.6917	0.0747	0.0719	375.6020
2.0	2.0	2.0	0.5826	0.0661	0.3497	296.2607
3.0	3.0	2.0	0.6533	0.0739	0.3189	327.1315
5.0	5.0	2.0	0.7251	0.0750	0.1959	425.6953

Table 1: Numerical values of Mean (μ) , variance (σ^2) , Bowleyskewness (B_{sk}) and Moorskurtosis of the OMEx-EP distribution with different values of λ,β and δ

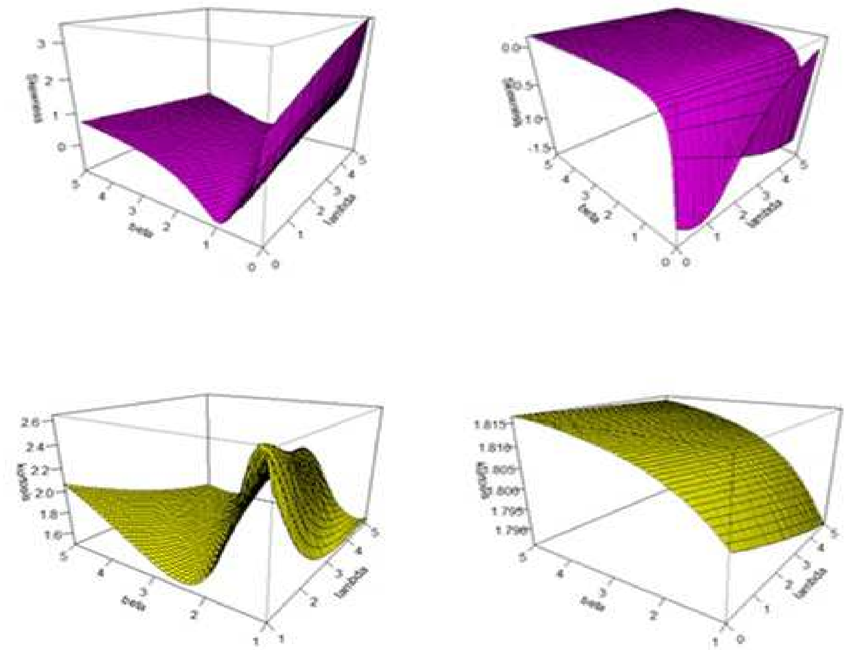

Fig. 3: dimensional plots of Bowley skewness (top row) and Moors kurtosis (bottom row) of the OMEx-EP distribution for some arbitrary parameter combinations .

Random parameter Combinations				Moments related measures		
λ	β	δ	μ	σ^2	B_{sk}	M_{kur}
0.5	0.5	0.5	1.1871	0.3973	0.5819	63.2375
1.5	0.5	0.5	1.0184	0.3429	0.8168	57.0930
2.0	0.5	0.5	0.9427	0.3109	0.9278	56.9108
3.0	0.5	0.5	0.8122	0.2470	1.1167	60.9978
5.0	0.5	0.5	0.6296	0.1492	1.2906	82.4686
0.5	1.0	0.5	1.8758	0.7515	0.3185	59.6596
0.5	2.0	0.5	2.7671	1.1953	0.0719	66.9887
0.5	3.0	0.5	3.3710	1.4622	-0.0586	77.2882
0.5	5.0	0.5	4.2020	1.7739	-0.2053	98.6797
1.5	5.0	0.5	3.8380	1.7339	-0.0354	77.8294
3.0	5.0	0.5	3.3631	1.5224	0.1359	69.9930
5.0	5.0	0.5	2.9006	1.2000	0.1959	77.1263
0.5	0.5	2.0	0.2967	0.0248	0.5819	307.8143
0.5	1.0	2.0	0.4689	0.0469	03185	314.9609
0.5	2.0	2.0	0.6917	0.0747	0.0719	375.6020
2.0	2.0	2.0	0.5826	0.0661	0.3497	296.2607
3.0	3.0	2.0	0.6533	0.0739	0.3189	327.1315
5.0	5.0	2.0	0.7251	0.0750	0.1959	425.6953

Table 2: Moments related relative measures results of C_{var},C_{skw} and C_{kur} for OMEx-EP model with random combinations of λ ,β and δ

3.3 Residual Life and Reversed Residual Life

For gauging a unit's ageing trait, the well-known reliability metrics mean residual life (MRL) and mean average reversed residual life (ARRL) functions are frequently employed. They are indispensable to reliability studies and survival studies. The MRL is a crucial factor to consider when assessing an item's optimum burn-in period while ARRL is the period of time from the failure time $Z \sim OMExEP(\lambda,\beta,\delta)$ to the observed time t, presuming that the failure took place sooner than anticipated time. The waiting times, for instance, can be interesting when discussing various maintenance tactics. The p^{th} moment of the residual life, say $m_p(t) = E\left[(Z-t)^P / Z > t\right]$, $p = 1,2,...$uniquely determines $F(z)$.The p^{th} moment of the residual life of X is given by

$$m_p(t) = \frac{1}{1-F(t)} \int_t^{\infty} (z-t)^p dF(z)$$

$$m_p(t) = \frac{1}{1-F(t)} \sum_{i=0}^{\infty} \gamma_i^* \int_t^{\infty} z^r \left[K^{OMExG}(z,\beta,\nabla)\right]^i k^{OMExG}(z,\beta,\nabla)\, dz$$

where

$$\gamma_i^* = \gamma_i (-1)^p \sum_{r=0}^{p} \binom{p}{r} t^{p-r}$$

By setting $p = 1$ in equation (12), the MRL of Z can be obtained which represents the expected additional life length for a unit which is alive at age t. Following similar pattern by setting $p =$1in equation (13), the ARRL can easily be obtained which represents the waiting time elapsed since the failure of an item on the condition that this failure had occurred in $(0,t)$.

3.4 Distribution of Order Statistics

Consider a random sample $Z_1, Z_2, ..., Z_n$ from any $OMEx-GP$ distribution. Let $Z_{r:n}$ denote the r^{th} order statistic. The PdF of $Z_{r:n}$ can be expressed as

$$\begin{aligned} f_{r;n}(z) &= \frac{n!}{(r-1)!(n-r)!} f^{OMExGP}(z) F^{OMExGP}(z)^{r-1} \left\{1-F^{OMExGP}(z)\right\}^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} f^{OMExGP}(z) \left\{1-F^{OMExGP}(z)\right\}^{m+r-1} \end{aligned}$$

The PdF of the r^{th} th order statistic for of the $OMEx-GP$ can be derived by using the expansion of the PdF and CdF as

$$\begin{aligned} f_{r;n}(z) &= \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} \sum_{i=0}^{\infty} \gamma_i \left[K^{OMExG}(z;\beta,\nabla)\right]^i k^{OMExG}(z;\beta,\nabla) \\ &\quad \times \left\{\sum_{j=0}^{\infty} t_j \left[K^{OMExG}(z;\beta,\nabla)\right]^j\right\}^{m+r-1} \quad \textit{where } \gamma_i \textit{ and } t_j \textit{ are defined above.} \end{aligned}$$

Using power series raised for positive power integer $n(\geq 1)$ (see Gradshteyn and Ryzhik, [25]) $\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i} u^i$, where the for $i = 1,2,...$ are easily obtained from the recurrence equation

$$c_{n,i} = (ia_0)^{-1} \sum_{m=0}^{1} [m(n+1)-i] a_m c_{n,i-m} \text{ with } c_{n,0} = a_0^n$$

Now

$$\left\{\sum_{j=0}^{\infty} t_j \left[K^{OMExG}(z;\beta,\nabla)\right]^j\right\}^{m+r-1} = \sum_{j=0}^{\infty} d_{m+r-1,j} \left[K^{OMExG}(z;\beta,\nabla)\right]^j$$

Therefore, the density function of the r^{th} order statistics of $OMEx-GP$ distribution can be expressed as

$$
\begin{aligned}
f_{r;n}(z) &= \frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_i d_{m+r-1,j}\left[K^{OMExG}(z;\beta,\nabla)\right]^{i+j}k^{OMExG}(z;\beta,\nabla) \qquad (13)\\
&= \sum_{i,j}^{\infty}\mu_{i,j}\left[K^{OMExG}(z;\beta,\nabla)\right]^{i+j}k^{OMExG}(z;\beta,\nabla)\\
&= \sum_{i,j}^{\infty}\frac{\mu_{i,j}}{(i+j+1)}\frac{d}{dz}\left[K^{OMExG}(z;\beta,\nabla)\right]^{i+j+1}
\end{aligned}
$$

where

$$\mu_{i,j} = \frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\gamma_i d_{m+r-1,j}$$

3.5 Probability Weighted Moments

The $(p,q,r)^{th}$ probability weighted moments $(PWMs)$ of Z is defined by $\Gamma_{p,q,r} = \int_{-\infty}^{+\infty} z^p\left[F(z)\right]^q\left[1-F(z)\right]^r f'z)dz$ (Greenwood et al., [24]). From equation (9), the s^{th} moment of X can instantly be articulated as

$$
\begin{aligned}
E(Z^s) &= \int_{-\infty}^{+\infty} z^s f^{OMEx-GP}(z;\lambda,\beta,\nabla)dz\\
&= \sum_{i=0}^{\infty}\gamma_i\int_{-\infty}^{+\infty} z^s\left[K^{OMExG}(z;\beta,\nabla)\right]^i k^{OMExG}(z;\beta,\nabla)dz\\
&= \sum_{i=0}^{\infty}\gamma_i\Gamma_{s,i,0}
\end{aligned}
$$

where γ_i is defined in section 3.1. As a result, the PWMs of the weighted sum of the OMExG distributions can be used to describe the moments of the OMEx-GP.

Performing alike actions, we are able to define s^{th} moment of the r^{th} order statistic $Z_{(r:n)}$ in a random sample of size n from OMEx-GP on using equation (14) as $E(X^s_{r;n}) = \sum_{i=0}^{\infty}\mu_{ij}\Gamma_{s,i+j,0}$, where μ_{ij} defined as above.

3.6 Mean Deviation

Let Z be the OMEx-GP chance variable with mean $\mu = E(Z$ and median $M = "Median"(Z)) = Q^{OMEx-EP}(0.5)$. The mean deviation from the mean $[\delta_\mu(Z) = E(|Z-\mu|)]$ and that from the median $[\delta_M(Z) = E(|Z-M|)]$ can be stated as

$$\delta_\mu(Z) = \int_{-\infty}^{\infty}|Z-\mu|f(z)dz = \int_{-\infty}^{\mu}(\mu-Z)f(z)dz + \int_{\mu}^{\infty}(Z-\mu)f(z)dz = 2\mu F(\mu) - 2\Psi(\mu) \qquad (14)$$

and

$$\delta_M(Z) = \int_{-\infty}^{\infty}|Z-M|f(z)dz = \int_{-\infty}^{M}(M-Z)f(z)dz + \int_{M}^{\infty}(Z-M)f(z)dz = M - 2\Psi(M) \qquad (15)$$

apiece, where $F(.)$ is the CdF of the OMEx-GP distribution, and $\Psi(t) = \int_{-\infty}^{t} zf(z)dz$. By computing $\Psi(t)$ as follows:

$$\Psi(t) = \sum_{i=0}^{\infty}\gamma_i\int_{-\infty}^{t} z\left[K^{OMExG}(z;\beta,\nabla)\right]^i k^{OMExG}(z;\beta,\nabla)dz$$

where γ_i defined in Section 3.

3.7 Stress-Strength System Reliability

The likelihood that the component can withstand the stress is known as stress strength system reliability (SSSR). SSSR has become a distinctive tool primarily in civil, automotive, and aviation sector. In stress-strength modeling $R = P(Z_1 < Z_2)$ is a measure of component reliability of the system with random stress $Z_1$and strength Z_2. It measures the probability that the systems strength X_2 is greater than environmental stress X_1,applied on that system. The probability of failure of a system is based on the probability of stress exceeding strength, whereas, the reliability of the system is the reversed probability. the system reliability is given by

$$R = P(Z_1 < Z_2) == P(Strengt > Stress) = \int_0^\infty f_{strengt}(z)F_{stress}(z)dz$$

Let Z_1 and Z_2 be two independent arbitrary variables such that $OMEx-GP(z;\lambda_1,\beta_1)$ and $OMEx-GP(z;\lambda_2,\beta_2)$distributions, respectively. Then SSSR is defined as

$$R = \int_0^\infty f_2(z;\lambda_2,\beta_2)F_1(z;\lambda_1,\beta_1)dz$$

where $f_2(.)$ and $F_1(.)$ are the PdF and CdF of the $OMEx-GP$ random variables $Z_1$and Z_2 respectively.
Note that the PdF and CdF of Z_1 and Z_2 are given by

$$f^{OMEx-GP}(z;\lambda_1,\beta_1) = k^{OMExG}(z;\beta_1)\sum_{i=0}^{\infty}\gamma_i^1\left[K^{OMExG}(z;\beta_1)\right]^i$$

and

$$F^{OMEx-GP}(z;\lambda_2,\beta_2) = \sum_{j=0}^{\infty}t_j^2\left[K^{OMExG}(z;\beta_2)\right]^j$$

Thus

$$R = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_i^1 t_j^2\int_0^\infty k^{OMExG}(z;\beta_1)\left[K^{OMExG}(z;\beta_1)\right]^i\left[K^{OMExG}(z;\beta_2)\right]^j dz$$

where

$$\gamma_i^1 = \frac{(-1)^i\lambda_1^{i+1}}{(1-e^{-\lambda_1})i!} \text{ and } t_j^2 = \frac{(-1)^{j+1}\lambda_2^j}{(1-e^{-\lambda_2})j!}$$

3.8 Rényi entropy

After taking into consideration a system's observable macroscopic elements like design, structure, or temperature etc., the entropy is a measure of volatility and instability associated with the system. Entropy, which is sometimes viewed as a measure of "disorder," is a statistical mechanics term that describes the number of possible arrangements of a structure. The higher the entropy, the greater the level of uncertainty it is with increased risk portfolio. Entropy is precisely a logarithmic measurement of the number of system states with a significant chance of being occupied. Entropy approach provides profound insight into the direction of spontaneous change for many banal processes. Rényi entropy is the most frequently utilized approach to measure information while maintaining additivity for independent incidents. It is defined as follows

$$I_R(\Delta) = (1-\Delta)^{-1}\log\left(\int_{-\infty}^{\infty}f(z)^\Delta dz\right), \qquad \Delta > 0 \text{ and } \Delta \neq 1$$

Using power series exponential function in equation (9) we can write

$$f^{OMEx-GP}(z;\lambda,\beta,\nabla)^\Delta = k^{OMExG}(z;\beta,\nabla)^\Delta\sum_{m=0}^{\infty}\eta_m\left[K^{OMExG}(z;\beta,\nabla)\right]^{m\Delta}$$

Thus

$$I_R(\Delta) = (1-\Delta)^{-1}\log\left(\int_0^\infty k^{OMExG}(z;\beta,\nabla)^\Delta \sum_{m=0}^{\infty}\eta_m\left[K^{OMExG}(z;\beta,\nabla)\right]^{m\Delta}dz\right)$$
$$= (1-\Delta)^{-1}\log\left(\sum_{m=0}^{\infty}\eta_m\int_0^\infty k^{OMExG}(z;\beta,\nabla)^\Delta\left[K^{OMExG}(z;\beta,\nabla)\right]^{m\Delta}dz\right)$$

where

$$\eta_m = \frac{(-1)^m\lambda^{\Delta(m+1)}}{(1-e^{-\lambda})^\Delta m!}$$

Parameter			Δ					
λ	β	δ	0.1	0.6	1.5	2	3	5
0.5	0.5	0.5	1.4199	1.0024	0.8292	0.7824	0.7233	0.6605
1.5	0.5	0.5	1.3917	0.9174	0.7105	0.6550	0.5858	0.5139
2.5	0.5	0.5	1.3566	0.8091	0.5657	0.5028	0.4264	0.3493
5.0	0.5	0.5	1.2481	0.5098	0.2253	0.1600	0.0831	0.0069
0.5	1.5	0.5	1.8153	1.4937	1.3409	1.2969	0.2401	1.1786
0.5	5.0	0.5	2.1199	1.7995	1.61774	1.5655	1.4992	1.4296
2.0	5.0	0.5	2.1050	1.7773	1.5933	1.5398	1.4716	1.3996
5.0	5.0	0.5	2.0400	1.6239	1.4150	1.3585	1.2881	1.2151
2.0	2.0	2.0	0.4863	0.1381	−0.0347	−0.0843	−0.1477	−0.2155
5.0	5.0	2.0	0.6537	0.2376	0.0284	−0.0277	−0.0981	−0.1711

Table 3: Numericalcomputation of Rényi entropy of OMEx-EP distribution for arbitrary combinations of model parameters λ ,β andδ

4 Estimation Method

This section is devoted to the estimation of the OMEx-GP model parameters via the maximum likelihood estimation (MLE) method. The complete data and censored data cases are investigated separately. Also devised is the criteria test for right censored data with estimated matrices for complete and censored data.

4.1 vvML estimation for complete data

Let $z=(z_1,z_2,...,z_n)$ be a random sample of size n from $OMEx-GP$ with parameter vector $\rho=(\lambda,\beta,\nabla)$ where $\nabla=(\nabla_1,\nabla_2,...\nabla_n)$ is the parameter vector of K. The log-likelihood function is written as

$$l(\rho) = n\log\lambda - n\log\left(1-e^{-\lambda}\right)+\sum_{i=1}^{n}\log\left(k(z_i,\nabla)\right)+\sum_{i=1}^{n}\log\left(K(z_i,\nabla)\right)-2n\log\beta-3\sum_{i=1}^{n}\log\left(1-K(z_i,\nabla)\right)$$
$$-\frac{1}{\beta}\sum_{i=1}^{n}\frac{K(z_i,\nabla)}{1-K(z_i,\nabla)}-\lambda\sum_{i=1}^{n}\left[1-\left\{1+\frac{K(z_i,\nabla)}{[1-K(z_i,\nabla)]\beta}\right\}\exp\left\{-\frac{K(z_i,\nabla)}{[1-K(z_i,\nabla)]\beta}\right\}\right]$$

Due to its complex nature, this log-likelihood function cannot be solved analytically, but it can be computationally optimized by using R's global optimization techniques. By taking the partial derivatives of the log-likelihood function with respect to λ and β , we can acquire the components of the score vector $U_\rho=(U_\lambda,U_\beta,U_\nabla)$

The asymptotic dispersion matrix of the ML estimation of parameters can be attained by flipping the Fisher information matrix which can be derived, empirically, using the second partial derivatives of the log-likelihood function with respect to each parameter. The ij^{th} elements of $I_n(\rho)$ are given by

$$I_{ij} = -E\left[\partial^2 l(\rho)/\partial \rho_i \rho_j\right], \quad i,j = 1,2,q$$

It could be challenging to assess the aforementioned expectations precisely. In practice, one can estimate $I_n(\rho)$ by the observed Fisher's information matrix $\hat{I}_n(\hat{\rho}) = \left(\hat{I}_{i,j}\right)$ defined as

$$\hat{I}_{i,j} \approx \left(-\partial^2 l(\rho)/\partial \rho_i \rho_j\right)_{\eta=\hat{\eta}}, i,j = 1,2+q.$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}(\hat{\rho}-\rho)$ is $N_k(0,V_n)$ where $V_n = (v_{jj}) = I_n^{-1}(\rho)$. The asymptotic behavior remains valid if V_n is substituted by $\hat{V}_n = I_n^{-1}(\rho)$. Using this result, large sample standard errors of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{jj}}$

4.2 ML estimation for partial data

Let us consider $Z = (Z_1, Z_2, ..., Z_n)^T$ to be a subset from the $OMEx-EP$ distribution with the parameter vector $\rho = (\lambda, \beta, \delta)^T$ which can contain right censored data with stable censoring time τ. Each Z_i can be written as $Z_i = (z_i, d_i)$ where

The right censoring is anticipated to be non-informative, so the log-likelihood function can be written as:

$$L_n(\rho) = \sum_{i=1}^{n} d_i \ln f(z_i,\rho) + \sum_{i=1}^{n} (1-d_i) \ln S(z_i,\rho)$$

we obtain

$$L_n(\rho) = \sum_{i=1}^{n} d_i \left[\begin{matrix} \ln(\lambda\delta) + 2\delta x_i - \ln(\omega_i) - \ln\left(1-e^{-\lambda}\right) \\ -2\ln(\beta) - u_i - \lambda\varphi_i \end{matrix}\right] + \sum_{i=1}^{n} (1-d_i)\left[\ln\left(e^{-\lambda\varphi_i} - e^{-\lambda}\right) - \ln(1-e^{-\lambda})\right]$$

We suppose $u_i = \frac{1-e^{-\delta z_i}}{\beta e^{-\delta z_i}}, \quad \varphi_i = 1-(1+u_i)e^{-u_i}, \quad \omega_i = 1-e^{-\delta z_i}$

The ML estimators$\hat{\lambda}$, $\hat{\beta}$ and $\hat{\delta}$ of the unknown parameters $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\delta}$ are derived from the nonlinear following score equations

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} d_i \left[\frac{1}{\lambda} - \frac{e^{-\lambda}}{1-e^{-\lambda}} - \varphi_i\right] - \delta \sum_{i=1}^{n} (1-d_i)\left[\frac{e^{-\lambda} - \varphi_i e^{-\lambda\varphi_i}}{e^{-\lambda\varphi_i} - e^{-\lambda}} - \frac{e^{-\lambda}}{1-e^{-\lambda}}\right]$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} d_i \left[\frac{-2}{\beta} + \frac{\omega_i}{\beta^2 e^{-\delta z_i}} + \frac{\lambda u_i e^{-u_i}}{\beta}\right] + \sum_{i=1}^{n} (1-d_i) \frac{\lambda u_i e^{-u_i} e^{-\lambda\varphi_i}}{\beta\left(e^{-\lambda\varphi_i} - e^{-\lambda}\right)}$$

$$\frac{\partial L}{\partial \delta} = \sum_{i=1}^{n} d_i \left[\frac{1}{\delta} + 2z_i - \frac{z_i}{\beta} + \frac{z_i e^{-\delta z_i}}{\omega_i} - \frac{z_i \omega_i}{\beta e^{-2\delta z_i}} - \frac{\lambda z_i \omega_i e^{-u_i}}{\beta^2 e^{-2\delta z_i}}\right] - \sum_{i=1}^{n} (1-d_i) \frac{\lambda z_i \omega_i e^{-u_i} e^{-\lambda\varphi_i}}{\beta^2 e^{-2\delta z_i}\left(e^{-\lambda\varphi_i} - e^{-\lambda}\right)}$$

Since the precise form of $\hat{\lambda}$, $\hat{\beta}$ and $\hat{\delta}$ cannot be obtained, so we avail numerical methods.

5 Test statistic for right censored data

Let $Z = (Z_1, Z_2, ..., Z_n)^{\mathbf{T}}$ be n i.i.d. chance variables assembled into r classes I_j. To determine whether a parametric model is acceptable F_0

$$H_0: \quad P(z \geq Z_i \mid H_0) = F_0(z, \rho), \qquad z \geq 0 \text{ and } \rho = (\rho_1, ..., \rho_s)^T,$$

When data are right censored and the parameter vector p is unknown, Bagdonavicius and Nikulin [5,6] proposed a statistic test Y^2 based on the vector

$$Z_j = \tfrac{1}{\sqrt{n}}(\varpi_j - e_j), \qquad j = 1, 2, ..., r \quad, \text{ with } r \succ s,$$

This one embodies the deviations between observed and probable numbers of failures (ϖ_j and e_j) to fall into these grouping intervals $I_j = (a_{j-1}, a_j]$ with $a_0 = 0$, $a_r = \tau$ where τ is a predetermined time. The authors considered ρ_j as random data functions such as the r intervals chosen have equal expected numbers of failures .

The statistic test Y^2 is defined by

$$Y^2 = M^T \widehat{\Sigma}^- M = \sum_{j=1}^{r} \frac{(\varpi_j - e_j)^2}{\varpi_j} + Q$$

where $M = (M^1, ..., M^2)^T$ and $\widehat{\Sigma}^-$ is a generalized inverse of the covariance matrix $\widehat{\Sigma}$ and

$$Q = W^T \widehat{G}^- W \qquad \widehat{A}_j = \varpi_j / n, \qquad \varpi_j = \sum_{i: Z_i \in I_j} \delta_i,$$

$$\widehat{G} = [\widehat{g}_{ll'}]_{sxs}, \quad \widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{C}_{l'j} \widehat{A}_j^{-1}, \quad \widehat{C}_{lj} = \frac{1}{n} \sum_{i: x_i \in I_j} d_i \frac{\partial}{\partial \widehat{\rho}} \ln h(z_i, \widehat{\rho}),$$

$$\widehat{i}_{ll'} = \frac{1}{n} \sum_{i=1}^{n} d_i \frac{\partial \ln h(z_i, \widehat{\rho})}{\partial \widehat{\rho}_l} \frac{\partial \ln h(z_i, \widehat{\rho})}{\partial \widehat{\rho}_{l'}}, \qquad \widehat{W}_l = \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{A}_j^{-1} Z_j, \qquad l, l' = 1, ..., s.$$

$\hat{\rho}$ is the maximum likelihood estimator of ρ on initial non-grouped data.

Under the H_0, the limit distribution of the statistic Y^2 is a chi-square with $r = rank(\Sigma)$ degrees of independence. Modified chi-square tests are outlined and their usefulness is addressed in Voinov et al. [29].

The interval limits a_j for grouping data into j classes I_j are considered as data functions and defined by

$$\hat{a}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H(z_l, \widehat{\rho})}{n - i + 1}, \widehat{z}\right), \qquad \hat{a}_r = \max\left(Z_{(n),} \tau\right)$$

such as the expected failure times e_j to fall into these intervals are $e_j = \frac{E_r}{r}$ for any j, with $E_r = \sum_{i=1}^{n} H(z_i, \rho)$. The distribution of this statistic test Y_n^2 is chi-square (see Voinov et al., []).

5.1 Criteria test for OMEx-EP distribution

n order to validate via the null hypothesis (H_0) that the data belong to the $OMEx - EP$, a modified chi-squared type goodness-of-fit test based on the statistic Y^2 has been conceptualized. Assume that τ is a finite time, and observed data are grouped into $r > s$ sub-intervals $I_j = (a_{j-1}, a_j]$ of $[0, \tau]$. The limit intervals a_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures e_j are obtained as

$$E_j = \frac{-j}{r-1} \sum_{i=1}^{n} \ln\left(e^{-\lambda \varphi_i} - e^{-\lambda}\right) - \ln\left(1 - e^{-\lambda}\right), \quad j = 1, ..r - 1$$

5.2 Estimated matrix $\hat{W}$

The modules of the estimated matrix $\widehat{W}$ are derived from the estimated matrix $\widehat{C}$ which is given by:

$$\hat{C}_{1j} = \frac{1}{n}\sum_{i:z_i\in I_j}^{n} d_i \left[\frac{1}{\lambda} - \varphi_i - \frac{e^{-\lambda} - \varphi_i e^{-\lambda\varphi_i}}{e^{-\lambda\varphi_i} - e^{-\lambda}}\right]$$

$$\hat{C}_{2j} = \frac{1}{n}\sum_{i:z_i\in I_j}^{n} d_i \left[\frac{-2}{\beta} + \frac{\omega_i}{\beta^2 e^{-\delta z_i}} + \frac{\lambda u_i e^{-u_i}}{\beta} - \frac{\lambda u_i e^{-u_i} e^{-\lambda\varphi_i}}{\beta\left(e^{-\lambda\varphi_i} - e^{-\lambda}\right)}\right]$$

$$\hat{C}_{3j} = \frac{1}{n}\sum_{i:z_i\in I_j}^{n} d_i \left[\frac{1}{\delta} + 2z_i - \frac{z_i}{\beta} + \frac{z_i e^{-\delta z_i}}{\omega_i} - \frac{z_i\omega_i}{\beta e^{-2\delta z_i}} - \frac{\lambda z_i \omega_i e^{-u_i}}{\beta^2 e^{-2\delta z_i}} + \frac{\lambda z_i \omega_i e^{-u_i} e^{-\lambda\varphi_i}}{\beta^2 e^{-2\delta z_i}\left(e^{-\lambda\varphi_i} - e^{-\lambda}\right)}\right]$$

and

$$\widehat{W}_l = \sum_{j=1}^{r} \widehat{C}_{lj}\widehat{A}_j^{-1} Z_j, \quad l,l' = 1,2,3 \quad j = 1,\ldots,r$$

5.3 Estimated Matrix $\widehat{G}$

The estimated matrix $\widehat{G} = [\widehat{g}_{ll'}]_{sxs}$is defined by

$$\widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{C}_{lj}\widehat{C}_{l'j}\hat{A}_j^{-1}$$

where

$$\widehat{i}_{ll'} = \frac{1}{n}\sum_{i=1}^{n} d_i \frac{\partial \ln h(z_i,\widehat{\rho})}{\partial \widehat{\rho}_l}\frac{\partial \ln h(z_i,\widehat{\rho})}{\partial \widehat{\rho}_{l'}}, \quad l,l' = 1,2,3$$

Therefore the quadratic form of the test statistic can be obtained easily:

$$Y_n^2(\widehat{\rho}) = \sum_{j=1}^{r} \frac{(\varpi_j - e_j)^2}{\varpi_j} + \hat{W}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lj}\hat{C}_{l'j}\hat{A}_j^{-1}\right]^{-1} \hat{W}$$

6 Simulation study

Multiple probability simulation, commonly known as Monte Carlo simulation (MCs) technique, is a mathematical technique which is used to estimate the possible outcomes of an uncertain event. Using the result defined in section 4.1 and 4.2, the precision of the method of estimations is appraised using the MLEs of OMEX-EP distribution parameters via MCs, both empirically and graphically, in this subsection.

6.1 Simulations for complete data

In this section, both tabular and graphical Monte Carlo simulation study is conducted to compare the performance of the different estimators of the unknown parameters for the $OMEx-EP(\lambda,\beta,\delta)$ distribution. All the computations in this section are done by R program. We generate $N = 1000$ samples of size $n = 10$ to 100 from $OMEx-EP(\lambda,\beta,\delta)$ distribution with true parameters estimates. The initial parameter combinations (PCs) for each $\rho = (\lambda,\beta,\delta)$ were selected completely arbitrarily. The bias and MSE are calculated following the defined approach as

$$Bias_h = \frac{1}{N}\sum_{i=1}^{N}(\hat{\rho}_i - \rho) \text{ and } MSE_\rho = \frac{1}{N}(\hat{\rho}_i - \rho)^2; \quad \rho = \lambda,\beta,\delta$$

The empirical findings of the bias and mean square error (MSE) by the MLEs are put forward in Table 4 and 5, respectively. Additionally, **a** graphical simulation was also performed with results delivered in Figure 4 and Figure 5. We perceive that when the sample size increases, the empirical biases and It is better to say that the MSE values decrease.

skip

Sample sizes	PC-I λ=2.5,β=1.5,δ=2.1			PC-II λ=1.5,β=0.5,δ=0.5			PC-III λ=2.0,β=0.8,δ=1.8			PC-IV λ=0.5,β=0.2,δ=1.0		
n	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs
	2.822	0.256	1.632	2.029	0.337	1.780	2.213	0.708	2.046	0.777	0.118	2.16
10	1.985	0.183	0.112	0.663	-0.201	0.157	0.595	0.191	0.202	0.235	-0.051	0.442
	1.991	-0.127	0.716	0.99	-0.119	0.962	2.015	0.292	0.624	1.288	0.132	0.789
	2.794	0.182	1.093	1.702	0.314	1.384	2.175	0.553	1.484	0.635	0.101	1.291
20	1.893	0.118	0.092	0.525	-0.167	0.116	0.662	0.151	0.151	0.228	0.064	0.333
	2.164	0.113	0.499	0.87	0.019	0.374	1.913	0.173	0.443	1.124	0.121	0.501
	2.672	0.029	0.701	1.636	0.220	1.248	2.077	0.209	1.223	0.589	0.066	1.205
50	1.497	0.066	0.073	0.513	-0.125	0.104	0.835	0.019	0.097	0.208	-0.004	0.275
	2.087	-0.002	0.335	0.994	0.012	0.286	1.834	0.111	0.315	1.105	0.078	0.367
	2.592	0.010	0.414	1.502	0.117	1.164	1.997	0.131	1.112	0.521	0.031	1.201
100	1.495	0.059	0.069	0.509	-0.014	0.101	0.805	0.006	0.094	0.206	-0.001	0.243
	2.043	0.005	0.331	1.994	0.010	0.281	1.804	0.108	0.233	1.099	0.042	0.232

Table 4: ML estimates, MSEs with average Bias of OMEx-EP distribution for arbitrary PCs

skip

Sample sizes	PC-I λ=1.6,β=0.72,δ=0.73			PC-II λ=1.2,β=0.04,δ=0.14			PC-III λ=1.7,β=0.2,δ=1.54			PC-IV λ =0.12,β =1.17,δ=1.31		
n	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs	Estimates $\left[\hat{\lambda},\hat{\delta},\hat{\beta}\right]$	Bias	MSEs
	2.567	0.253	4.295	2.037	0.332	4.060	2.222	0.706	3.546	0.238	0.115	3.060
10	1.730	0.180	0.137	0.672	−0.206	0.814	0.604	0.189	1.702	1.291	−0.054	1.342
	1.736	−0.130	1.229	0.999	−0.124	2.424	2.024	0.290	2.124	1.291	0.129	1.689
	2.262	0.179	4.245	1.733	0.311	3.910	2.217	0.550	3.446	0.238	0.098	2.960
20	1.425	0.115	0.087	0.367	−0.170	0.664	0.599	0.148	1.602	1.291	0.061	1.242
	1.431	0.110	1.179	0.694	0.016	2.274	2.019	0.170	2.024	1.276	0.118	1.589
	1.757	0.026	3.995	1.428	0.217	3.860	2.088	0.206	3.146	0.133	0.063	2.660
50	0.920	0.063	−0.163	0.062	−0.128	0.614	0.470	0.016	1.302	1.186	−0.007	0.942
	0.926	−0.005	0.929	0.389	0.009	2.224	1.890	0.108	1.724	1.151	0.075	1.289
	1.557	0.016	3.990	1.178	0.067	3.850	1.735	0.193	2.796	0.118	0.040	2.310
100	0.720	0.053	−0.008	0.037	0.027	0.604	0.117	0.003	0.952	1.171	−0.030	0.592
	0.726	−0.015	0.874	0.139	0.008	2.214	1.537	0.095	1.374	1.306	0.052	0.939

Table 5: ML estimates, MSEs with average Bias of OMEx-EP distribution for arbitrary PCs

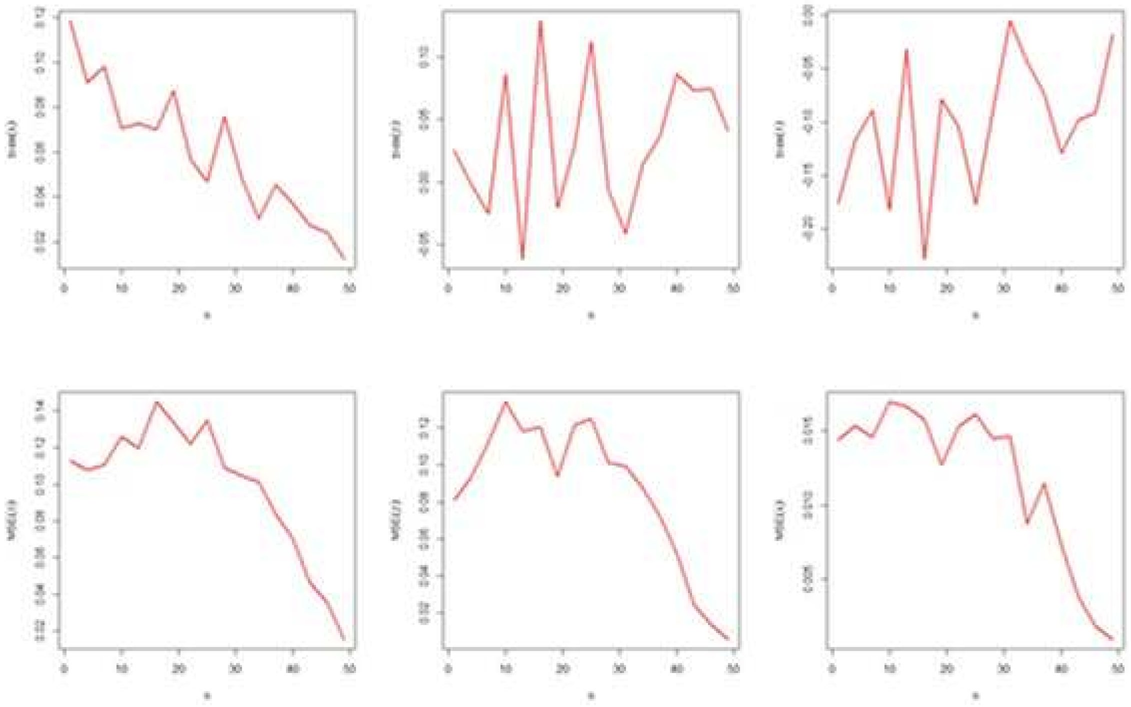

Fig. 4: The Biases (toprow) and MSEs (bottomrow)of the parameter values $\lambda = 0.5, \beta = 0.3, \delta = 0.4$ for OMEx-EP distribution.

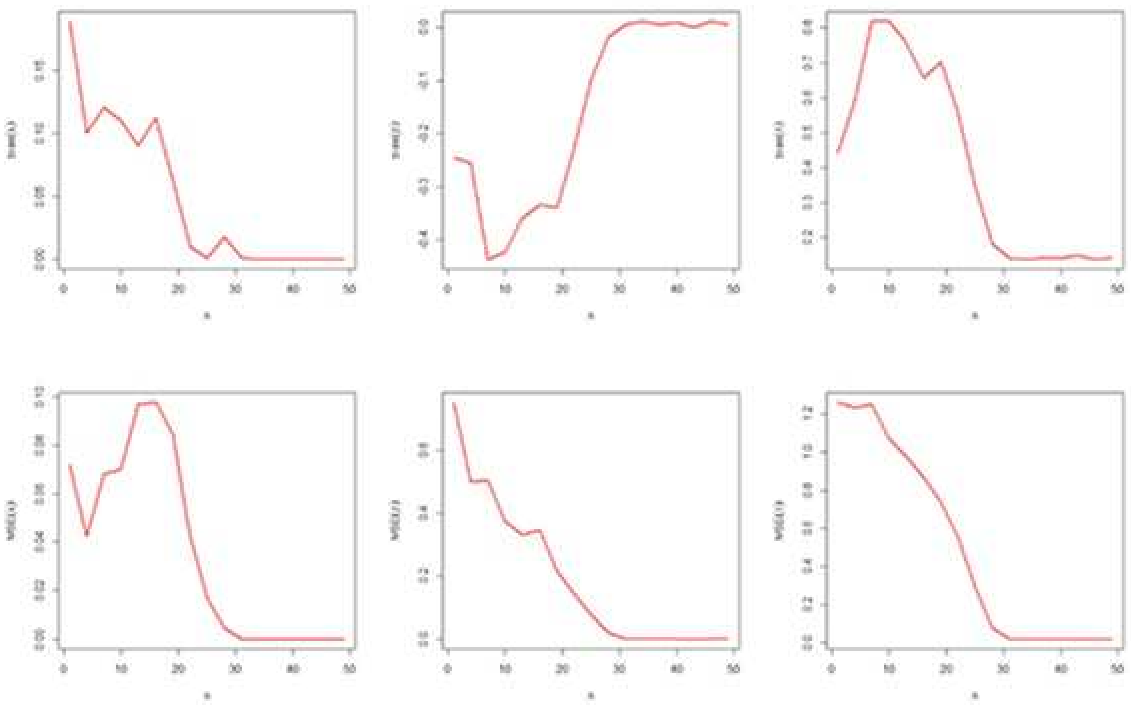

Fig. 5: The Biases and MSEs of the parameter values $\lambda = 1.5, \beta = 1.8, \delta = 0.5$ for OMEx-EP distribution .

6.2 Simulations for censored data

We generated $N = 10,000$ right censored samples with different sizes $(n = 25, 50, 150, 350, 500)$ from the $OMEx - EP$ model with parameters $\lambda = 2$, $\delta = 0.6$ and $\beta = 0.5$. Using R statistical software and the Barzilai-Borwein (BB) algorithm (Ravi, [36]), we calculate the ML estimators of the unknown parameters and MSE. The empirical findings are reported in Table 6.

$N = 10,000$	Sample sizes (n_i)				
	$n_1 = 25$	$n_2 = 50$	$n_3 = 150$	$n_4 = 350$	$n_5 = 500$
$\hat{\lambda}$	1.9856	1.9896	2.9905	2.9953	2.9989
$M.E.S$	0.0119	0.0092	0.0076	0.0048	0.0035
$\hat{\delta}$	07186	0.7086	0.6925	0.6925	0.6053
$M.E.S$	0.0196	0.0172	0.0136	0.0096	0.0083
$\hat{\beta}$	0.5962	0.5784	0.5458	0.5263	0.5023
$M.E.S$	0.0162	0.0143	0.0103	0.0086	0.0076

Table 6: Mean simulated values of MLEs $\hat{p}$their corresponding square mean errors

The ML estimated parameter values, presented in Table 6, tend to align with the real parameter values.

6.3 Criteria test Y_n^2

For testing the H_0 that right censored data become from $OMEx - EP$ model, we take into account the qualifying statistic $Y_n^2(\rho)$ as defined above for $10,000$ simulated samples from the hypothezised distribution with different sizes (30, 50,150, 350, 500).Then, we evaluate empirical levels of statistical significance, when $Y^2 > \chi_\varepsilon^2$, analogous to theoretical echelons of significance $(\varepsilon = 0.10, \varepsilon = 0.05, \varepsilon = 0.01)$, We choose $r = 5$. The outcomes are reported in Table 7.

$N = 10,000$	Sample sizes (n_i)				
	$n_1 = 25$	$n_2 = 50$	$n_3 = 150$	$n_4 = 350$	$n_5 = 500$
$\hat{\lambda}$	1.9856	1.9896	2.9905	2.9953	2.9989
$M.E.S$	0.0119	0.0092	0.0076	0.0048	0.0035
$\hat{\delta}$	07186	0.7086	0.6925	0.6925	0.6053
$M.E.S$	0.0196	0.0172	0.0136	0.0096	0.0083
$\hat{\beta}$	0.5962	0.5784	0.5458	0.5263	0.5023
$M.E.S$	0.0162	0.0143	0.0103	0.0086	0.0076

Table 7: Mean simulated values of MLEs p their corresponding square mean errors

The H_0 for which simulated samples are fitted by $OMEx - EP$ distribution is broadly validated for the different significance levels. Consequently, the test proposed in this work, can be used to fit data from this new distribution.

7 Applications

This section comprises of three real life data sets being employed to authenticate the expediency of the proposed OMEx-EP. In the first sub-section, two complete failure time data sets (FD1 & FD2) have been applied befitting the OMEx-EP distribution using standard well known criterions while a censored data has been employed in the second part of this section with relevant inference.

7.1 The Co-rival models

The considered well-established models which have been compared with the proposed OMEx-EP include exponential (Exp), moment exponential (MEx), Marshall-Olkin exponential (MO-E) (Marshall and Olkin, [34]), generalized Marshall-Olkin exponential (GMO-E) (Jayakumar and Mathew, [33]), Kumaraswamy exponential (Kw-E) (Cordeiro and de Castro, [12]), Beta exponential (BE) (Eugene et al., [21]), Marshall-Olkin Kumaraswamy exponential (MOKw-E) (Handique et al., [30]) and Kumaraswamy Marshall-Olkin exponential (KwMO-E) (Alizadeh et al., [11]) and Kumaraswamy Poisson exponential (KwP-E) (Chakraborty et al., [18]) distribution. At this point, we consider adapting the failure time data sets in view of clinical data (s), to show that the distributions from the proposed OMEx-GP family provide exceedingly superior results than the corresponding distributions.

7.2 Complete failure time data sets: An outline

A period that has transpired cumulatively since a failure commenced is generally termed as failure time data. Failure time may include period of time between the spread of a virus and the emergence of a symptom in an individual, duration of a diagnostic trial's death from detection, component malfunctioning in a configuration system etc. Failure analysis seeks to discover the root cause of a failure (i.e., its most fundamental cause), presumably with the goal of eradicating it and acquiring solutions that will avoid future occurrences. The first data (denoted as FD1), extracted from Klein and Moschberger [38], represents the time of exposure period for those who are confirmed to be carriers of sexually transmitted virus (STV). The data was recorded in months and is given as:

$129, 103, 129, 125, 103, 111, 149, 115, 131, 106, 102, 138, 141, 140, 155, 149, 106, 132, 137, 118, 136, 127, 166, 179, 171, 178, 166, 164, 144, 148, 145, 145, 166, 153, 153, 173, 184, 203, 170, 177, 124, 157, 207, 168, 192, 182, 191, 193, 195, 194, 156, 267, 180, 145, 207, 159, 149, 172$

Recently, Khosa et al applied the data understudy (termed as FD2) on the newly proposed family of distributions defined as the New Extended-F (NE-F). The data concerns the failure time of guinea pigs infected with virulent tubercle bacilli, initially observed and reported by Bjerkedal [15]. The descriptive statistics related to FD1 & FD2 are summarized in Table 8.

Data Sets	n	Z_{min}	$\mu_{\bar{Z}}$	μ_Z	SD	β_1	β_2	Q_1	Q_2	Z_{max}
FD1	58	102	155.261	153	31.899	0.607	1.069	122.915	221.0674	267
FD2	72	0.100	1.851	1.560	1.200	1.788	4.157	1.080	2.303	7

Table 8: Descriptive Statistics to FD1 and FD2

Both FD1 & FD2 show the tendency of right skewed distributions with heavier tails making it suitable to fit OMEx-EP model to the data. Supplementarily, the behaviour of the total time on test (TTT) plot (see Aarset, [39]) was also considered in order to observe the associated failure of the datasets being considered. The TTT plot is a method for obtaining details about the hazard function's shape. A convex (concave) curvature denotes decreasing (increasing) hazard, whilst a straight linear line signifies constant hazard for the data set. The TTT and estimated hrf plots for FD1 & FD2 in Figure 6 and Figure 7 indicate that the data set also has an increasing and a slight increasing-decreasing-increasing hazard rate, equivalent to the hrf shapes of the anticipated model.

7.3 Complete failure time data sets: The output

We use the MLE approach to estimate the distribution parameters for each dataset. We also evaluate the following model validation measures (MVMs), which are based on parameter estimates such as Akaike's information criterion (MVM-I), Bayesian information criterion (MVM-III), Corrected Akaike's information criterion (MVM-II) and Hannan-Quinn information criterion (MVM-IV). The Anderson-Darling (MVM-V), Cramér-Von Mises (MVM-VI), and Kolmogorov Smirnov (K-S) statistics (MVM-VII), along with their p-values, are used to contrast the OMEx-EP distribution's fits to

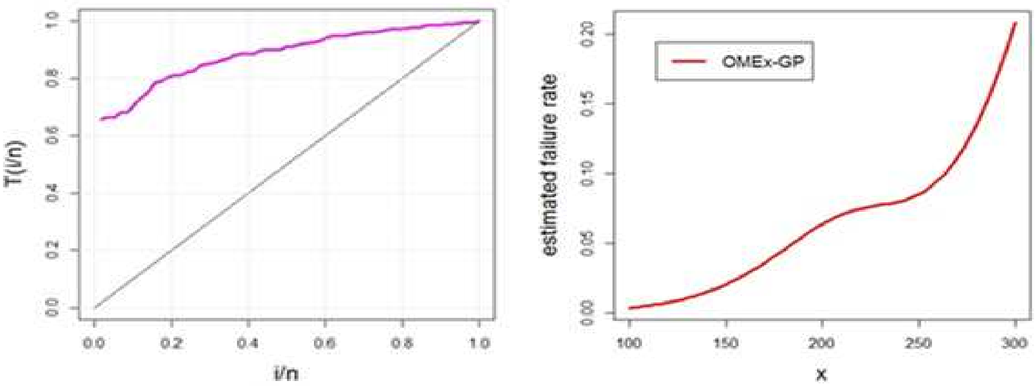

Fig. 6: TTT and estimated hrf (left) plots for FD1

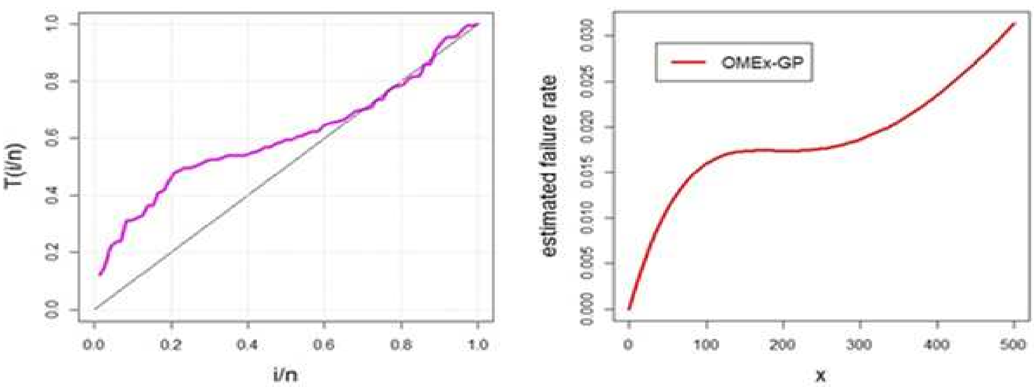

Fig. 7: TTT and estimated hrf (left) plots for FD2 (right)& FD2 (left).

other competing models. In general, the best model for matching the data has lower values for these measures and a higher p-value for K-S statistics. We have also provided the asymptotic standard errors and confidence intervals of the MLEs of the parameters for each competing model.

Visual comparisons of the fitted density, CdF, sf and probability plots (PPP) for the two datasets are presented in Figure 8 & 9, respectively. These plots reveal that the proposed distributions provide a good fit to this data when compared to co-rival models. Moreover, for FD1 & FD2, a correlation matrix is provided as an input for other complex analyses such as exploratory factor analysis and structural equation models. For FD1 (right) & FD2 (left), it is given as

$$\begin{pmatrix} 1 & -0.389 & 0.791 \\ . & 1 & 0.213 \\ . & . & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.052 & 0.623 \\ . & 1 & 0.705 \\ . & . & 1 \end{pmatrix}$$

Next, the covariance among each combination of components in a given stochastic process is represented by the square matrix known as the dispersion matrix. The covariance matrix logically broadens the concept of variance to alternate realities. For FD1(right) & FD2 (left), it is computed below

$$\begin{pmatrix} 5.269*10^{-06} & -0.002 & 0.027 \\ . & 1.015 & 7.277 \\ . & . & 211.264 \end{pmatrix}$$

$$\begin{pmatrix} 8.494*10^{-08} & 0.00002 & 1.151*10^{-5} \\ . & 1.015 & 4.505*10^{-02} \\ . & . & 4.024*10^{-03} \end{pmatrix}$$

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\delta}$
Ex (δ)	—	—	—	—	0.493 0.085 $[0.17, 0.65]$
ME (δ)	—	—	—	—	1.747 0.193 $[1.37, 2.13]$
$MO-E$ (β,δ)	—	5.553 2.940 $[0, 11.41]$	—	—	1.379 0.193 $[1.00, 1.75]$
$GMO-E$ (λ,β,δ)	0.728 0.111 $[0.51, 0.95]$	26.575 13.761 $[0, 53.547]$	—	—	4.465 1.327 $[1.86, 7.07]$
$Kw-E$ a,b,δ	—	—	4.119 1.362 $[1.449, 6.789]$	0.977 0.437 $[0.120, 1.834]$	2.333 0.424 $[1.502, 3.164]$
$B-E$ (a,b,δ)	—	—	22.318 10.545 $[1.828, 42.808]$	7.677 3.299 $[1.211, 14.143]$	1.543 0.388 $[0.783, 2.303]$
$MOKw-E$ (β,a,b,δ)	—	0.119 0.035 $[0.050, 0.188]$	2.244 1.253 $[0.389, 5.230]$	1.273 0.369 $[0.550, 1.996]$	0.112 0.024 $[0.065, 0.159]$
$KwMO-E$ (β,a,b,δ)	—	0.873 0.217 $[0.448, 1.298]$	9.119 3.56 $[2.128, 16.110]$	2.887 0.939 $[1.047, 4.727]$	0.164 0.187 $[0, 0.531]$
$KwP-E$ (λ,a,b,δ)	2.705 3.007 $[0, 8.599]$	—	5.898 2.242 $[1.504, 10.292]$	3.332 1.289 $[0.806, 5.858]$	0.869 0.555 $[0, 1.957]$
$OMEx-EP$ (λ,β,δ)	6.029 2.346 $[1.431, 10.623]$	37.531 14.535 $[9.042, 66.020]$	—	—	0.020 0.002 $[0.012, 0.024]$

Table 9: MLEs,standarderrors with 95% sureness intervals(asides)values for FD1

Models	MVM-I	MVM-II	MVM-III	MVM-IV	MVM-V	MVM-VI	$MVM-VII$ $p-value$
Ex (δ)	613.76	612.39	613.36	612.54	0.563	0.22	0.19 (0.07)
ME (δ)	602.24	600.37	601.72	602.11	0.505	0.25	014 (0.13)
$MO-E$ (β,δ)	589.38	596.32	597.67	597.98	0.83	0.21	0.10 (0.47)
$GMO-E$ (λ,β,δ)	579.56	578.75	579.07	580.61	0.43	0.086	0.092 (0.77)
$Kw-E$ a,b,δ	590.16	593.85	591.29	593.46	0.53	0.132	0.098 (0.75)
$B-E$ (a,b,δ)	593.78	591.22	590.57	592.25	0.51	0.131	0.108 (0.73)
$MOKw-E$ (β,a,b,δ)	588.52	587.28	588.37	587.98	0.39	0.127	0.10 (0.74)
$KwMO-E$ (β,a,b,δ)	579.56	580.39	579.09	581.64	0.47	0.109	0.10 (0.78)
$KwP-E$ (λ,a,b,δ)	580.03	582.66	579.75	581.83	0.48	0.073	0.097 (0.77)
$OMEx-EP$ (λ,β,δ)	574.10	580.27	574.54	576.50	0.40	0.051	0.084 (0.80)

Table 10: Log-likelihood, MVM-I, MVM-II, MVM-III, MVM-IV,MVM-IV, MVM-V, MVM-VI,MVM-VII (p-value) values for FD1

Models	$\hat{\lambda}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\delta}$
Ex (δ)	—	—	—	—	0.540 0.063 $[0.42, 0.66]$
ME (δ)	—	—	—	—	0.925 0.077 $[0.62, 108]$
$MO-E$ (β,δ)	—	8.778 3.555 $[1.81, 15.74]$	—	—	1.379 0.193 $[1.00, 1.75]$
$GMO-E$ (λ,β,δ)	0.179 0.070 $[0.04, 0.32]$	47.635 44.901 $[0, 135.64]$	—	—	4.465 1.327 $[1.86, 7.07]$
$Kw-E$ a,b,δ	—	—	3.304 1.106 $[1.13, 5.47]$	1.100 0.764 $[0, 2.59]$	1.037 0.614 $[0, 2.24]$
$B-E$ (a,b,δ)	—	—	0.807 0.696 $[0, 2.17]$	3.461 1.003 $[1.49, 5.42]$	1.331 0.855 $[0, 3.01]$
$MOKw-E$ (β,a,b,δ)	—	0.008 0.002 $[0.004, 0.01]$	2.716 1.316 $[0.14, 5.29]$	1.986 0.784 $[0.449, 3.52]$	0.099 0.048 $[0, 0.19]$
$KwMO-E$ (β,a,b,δ)	—	0.373 0.136 $[0.11, 0.64]$	3.478 0.861 $[1.79, 5.17]$	3.306 0.779 $[1.78, 4.83]$	0.299 1.112 $[0, 2.48]$
$KwP-E$ (λ,a,b,δ)	4.001 5.670 $[0, 15.11]$	—	3.265 0.991 $[1.32, 5.21]$	2.658 1.984 $[0, 6.55]$	0.177 0.226 $[0, 0.62]$
$OMEx-EP$ (λ,β,δ)	4.232 0.365 $[3.52, 4.95]$	0.569 0.026 $[0.52, 0.62]$	—	—	0.208 0.017 $[0.17, 0.24]$

Table 11: MLEs, standard errors, sureness intervals (asides) values for FD2

Models	MVM-I	MVM-II	MVM-III	MVM-IV	MVM-V	MVM-VI	$MVM-VII$ $p-value$
Ex (δ)	234.63	236.91	234.68	235.54	6.53	1.25	0.27 (0.06)
MEx (δ)	210.40	212.68	210.45	211.30	1.52	0.25	0.14 (0.13)
$MO-E$ (β,δ)	210.36	214.92	210.53	212.16	1.18	0.17	0.10 (0.43)
$GMO-E$ (λ,β,δ)	210.54	217.38	210.89	213.24	1.02	0.16	0.09 (0.51)
$Kw-E$ a,b,δ	209.42	216.24	209.77	212.12	0.74	0.11	0.08 (0.50)
$B-E$ (a,b,δ)	207.38	214.22	207.73	210.08	0.98	0.15	0.11 (0.34)
$MOKw-E$ (β,a,b,δ)	209.44	218.56	210.04	213.04	0.79	0.12	0.10 (0.44)
$KwMO-E$ (β,a,b,δ)	207.82	216.94	208.42	211.42	0.61	0.11	0.08 (0.73)
$KwP-E$ (λ,a,b,δ)	206.63	215.74	207.23	210.26	0.48	0.07	0.09 (0.79)
$OMEx-EP$ (λ,β,δ)	205.42	212.23	205.77	208.12	0.41	0.05	0.08 (0.84)

Table 12: Log-likelihood, MVM-I, MVM-II, MVM-III, MVM-IV,MVM-V, MVM-VI,MVM-VII (p-value) values for FD2

Table 9 and Table 11 for FD1 and FD2, respectively, list the MLEs of the parameters in conjunction with the associated standard errors in parenthesis for every estimated model. Further, Table 10 and Table 12 list the multiple MVMs for the models that were suited to the FD1 & FD2, respectively. The OMEx-EP is identified to be a superior choice than the various analogies of exponential models for the entirety of the data relying on these findings based on the lowest values of several considerations. Figure 8 also exhibits a graphic representation of the fitted density's agreement with the observed histogram, estimated CdF, estimated survival function (sf), and PP plot of the data set I. These findings imply that the proposed distributions produce a considerably better fit to the particular set of data.

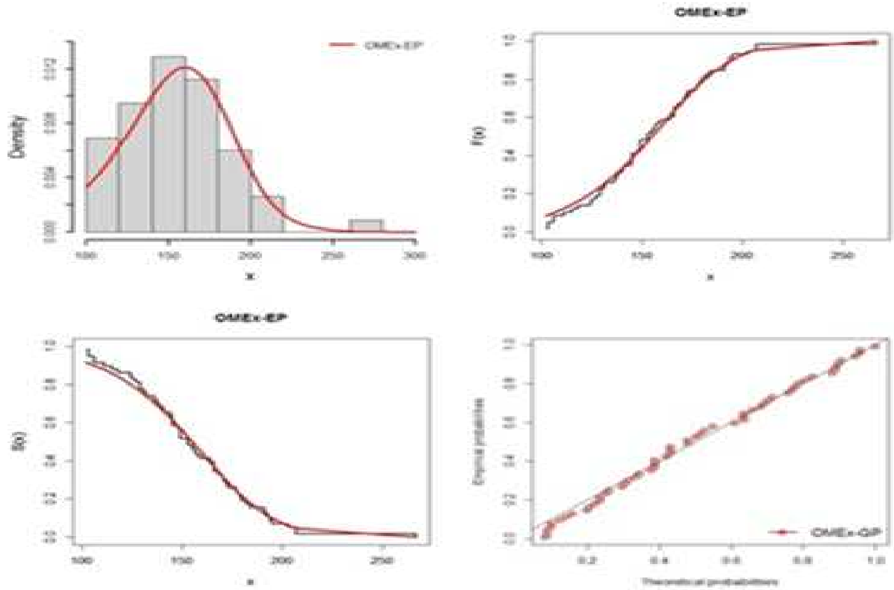

Fig. 8: Plots of the estimated PdF (Top-Left),CdF(Top-Right),sf(Bottom-Left) and P-Pplots(Bottom-right) of the OMEx-EP model for FD1.

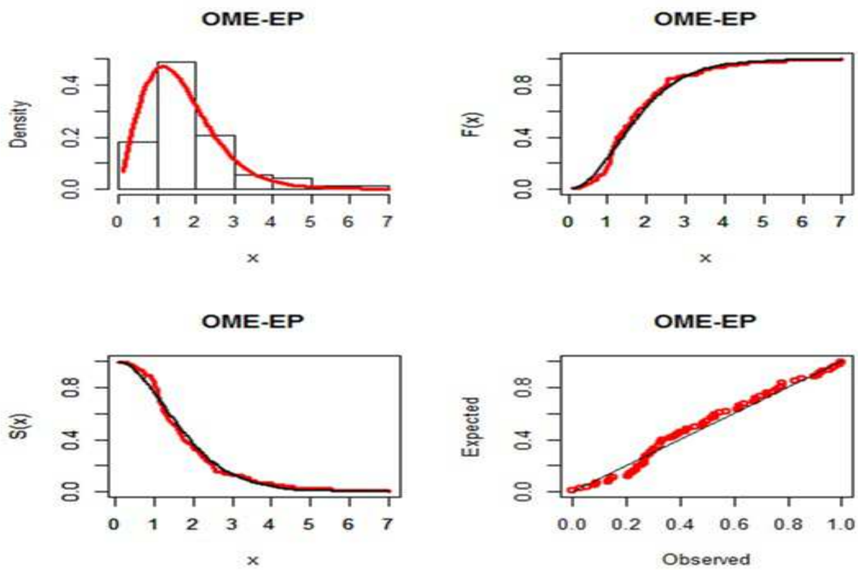

Fig. 9: Plots of the estimated PdF (Top-Left),CdF(Top-Right),sf(Bottom-Left) and P-Pplots(Bottom-right) of the OMEx-EP model for FD2.

7.4 Analysis of right censored data

When there is partial knowledge of a subject's event time but not the definite occurrence time, the phenomena of censoring unfolds. Well before research reaches its end, a participant doesn't somehow experience the incident will be termed as right censoring. Pike [35] documented the facts from a lab experiment in which rats' vaginas were coated with the carcinogenic agent DMBA, and the length of t days it took for cancer to manifest was observed. The information below pertains to a 19-rat group (referred to as Group 1 in Pike's study) with the observations marked with an exclamatory sign, being censored:

$$143, 164, 188, 188, 190, 192, 206, 213, 216, 216^{*}, 220, 227, 230, 234, 244^{*}, 246, 265, 304$$

To assess if these data are matched by the OMEx-EP distribution, we apply the statistic test described in section 3.3 specifically. To that purpose, we first compute the maximum likelihood estimators of the unknown parameters.

$$\rho = (\lambda, \delta, \beta)^T = (2.435, 0.695, 0.4263)^T$$

Data are clustered into $r = 4$ intervals I_j We give the indispensable calculus in the following Table 13.

a_j	191.53	218.63	241.63	304
ε_j	5	6	4	4
e_j	0.4895	0.4895	0.4895	0.4895
$\hat{C}_{1j}$	0.3624	0.4587	-1.6352	0.7485
$\hat{C}_{2j}$	1.5236	-1.2536	0.9748	-0.9568
$\hat{C}_{3j}$	-0.5196	0.9415	1.0236	0.2396

Table 13: values of $a_j, e_j, \varepsilon_j, \hat{C}_{1j}, \hat{C}_{2j}, \hat{C}_{3j}$.

Then we obtain the value of the statistic test Y_n^2 :

$$Y^2 = \sum_{j=1}^{r} \frac{(\varpi_j - e_j)^2}{\varpi_j} + Q = 4.9365 + 2.6543 = 7.5908$$

For significance level $U3b5 = 0.05$, the critical value $\chi_5^2 = 9.4877$ is superior than the value of $Y_n^2 = 7.5908$, so we can say that the proposed model OMEx-EP fit these data set.

8 Conclusion

The OMEx-G and Poisson-G families of distributions are amalgamated into a new extended class of distribution. Mathematical features and its crucial special situations are researched. Simulation is used to explore and assess the MLE for parameter estimation. Two real-world data sets fitting the application yield positive results in support of the distributions from the suggested class. Because of the additional parameter and increased flexibility, this novel family is anticipated to supplement the literature on continuous distributions and significantly boost model fitting.

Conflict of interest: No authors claim to have any potential conflicts of interest.

Abbreviations
ED ***exponentialdistribution***
MEx ***momentexponentialdistribution***
P − OGE ***Poissonoddgeneralizedexponential***
GP ***Poisson − G***
OMEx − GP ***oddmomentexponential − GPoisson***
OMEx − EP ***OMEx − Poissonexponential***
PdF‘ ***probabilitydensityfunction***
CdF ***cumulativedistributionfunction***
pmf ***probabilitymassfunction***
hrf ***hazardratefunction***
sf ***survivalfunction***
Cvar ***coefficientofvariation***
Cskw ***coefficientofskewness***
Ckur ***coefficientofkurtosis***
MRL ***meanresiduallifefunction***

$ARRL$	$meanaveragereversedresiduallifefunction$
$PWMs$	$probabilityweightedmoments$
MLE	$maximumlikelihoodestimation$
MCs	$MonteCarlosimulation$
PCs	$initialparametercombinations$
MSE	$meansquareerror$
BB	$Barzilai-Borweinalgorithm$
$FD1$	$failuretimedata-1$
$FD2$	$failuretimedata-2$
STV	$sexuallytransmittedvirus$
$NT-F$	$NewExtended-F$
$MV-M$	$modelvalidationmeasures$

Declarations

Competing interests: The authors declare no competing interests.

Authors' contributions: All authors equally contributed.

Funding: The authors declare that this research did not receive any research grant.

Availability of data and materials: The data used in this article are freely available in the mentioned references.

Acknowledgments: The authors would like to thank the Editor-in-Chief, and the referees for constructive comments which greatly improved the paper.

References

[1] Khan, S.; Balogun, O.S.;Tahir, M.H.; Almutiry,W.; Alahmadi,A.A. An Alternate Generalized Odd Generalized Exponential Family with Applications to Premium Data.Symmetry 2021, 13, 2064.
[2] Iqbal, Z.; Hasnain, S. A.; Salman, M.; Ahmad, M. and Hamedani, G.G., "Generalized Exponentiated Moment Exponential Distribution" (2014). Mathematics, Statistics and Computer Science Faculty Research and Publications. 269.
[3] Dara, S.T. and Ahmed, M. Recent Advances in Moment Distribution and Their Hazard Rates. Lap Lambert Academic Publishing GmbH KG (2012).
[4] Hasnain, Syed Iqbal, Zafar Ahmad, Munir. On Exponentiated moment exponential distribution. Pakistan Journal of Statistics. 31. 267-280.(2015).
[5] Hashmi, S.; Haq, M.A.U.; Usman, R.M.; Ozel, G. The Weibull-moment exponential distribution: Properties, characterizations and applications. J. Reliab. Stat. Stud. 12, 122,2019.
[6] Bourguignon, Marcelo Silva, Rodrigo Cordeiro, Gauss. (2021). The Weibull-G Family of Probability Distributions. Journal of Data Science. 12. 53-68,(2014).
[7] Bhatti, F.A.; Hamedani, G.G.; Korkmaz, M..; Sheng, W.; Ali, A. On the Burr XII-moment exponential distribution. PLoS ONE 16, e0246935,2021.
[8] Tahir, M. H., Cordeiro, G. M. Compounding Probability Distributions: A Survey and New Results. Journal of Statistical Distributions and Applications, 3(1)(2016).
[9] Mustapha, A. B. (2019). Poisson Odd Generalized Exponential Family: Properties and Applications in Reliability. Communications in Statistics - Theory and Methods, 48(12), 3032-3045.
[10] Ahsanullah, Mohammad Nevzorov, Valery. Records of Discrete Distributions. 123-142.(2015).
[11] Alizadeh, M., Tahir, M.H., Cordeiro, G.M., Zubai, M. and Hamedani, G.G. The Kumaraswamy Marshal-Olkin family of distributions. Journal of the Egyptian Mathematical Society, 23, 546-557 (2015).
[12] Alzaatreh, A., Lee, C. and Famoye, F. A new method for generating families of continuous distributions. Metron 71, 63-79 (2013).
[13] Bagdonaviius, V. and Nikulin, M. Chi-squared goodness-of-fit test for right censored data, International Journal of Applied Mathematics and Statistics, 24, 30-50 (2011).
[14] Bagdonaviius, V., Levuliene, R.J. and Nikulin, M. Chi-squared goodness-of-fit tests for parametric accelerated failure time models, Communications in Statistics- Theory and Methods, 15, 2768-2785 (2013).

[15] Bjerkedal, T. Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle bacilli. American Journal of Hygiene, 72, 130-148 (1960).
[16] Chakraborty, S. and Handique, L. The generalized Marshall-Olkin-Kumaraswamy-G family of distributions. Journal of Data Science, 15, 391-422 (2017).
[17] Chakraborty, S. and Handique, L. Properties and data modelling applications of the Kumaraswamy generalized Marshall-Olkin-G family of distributions. Journal of Data Science 16, 605-620 (2018).
[18] Chakraborty, S., Handique, L. and Farrukh, J. The Kumaraswamy Poisson-G family of Distribution: its properties and applications. Annals of Data Science (2020) (Accepted).
[19] Chakraborty, S., Handique, L. and Ali, M.M. A new family which integrates beta Marshall-Olkin-G and Marshall-Olkin-Kumaraswamy-G families of distributions. Journal of Probability and Statistical Science, 16, 81-101 (2018).
[20] Cordeiro, G.M. and De Castro, M. A new family of generalized distributions. J. Stat. Comput. Simul. 81, 883- 893 (2011).
[21] Eugene, N., Lee, C. and Famoye, F. Beta-normal distribution and its applications. Commun Statist Theor Meth, 31: 497-512 (2002).
[22] Gokarna, R.A., Sher, B.C., Hongwei, L. and Alfred, A.A. On the Beta-G Poisson family. Annals of Data Science. 18-0176 (2018).
[23] Gokarna, R.A. and Haitham, M.Y. The exponentiated generalized-G Poisson family of distributions. Stochastics and Quality Control, 32, 7-23 (2017).
[24] Greenwood, J.A., Landwehr, J.M., Matalas, N.C. and Wallis, J.R. Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form. Water Resour Res, 15, 1049- 1054 (1979).
[25] Gradshteyn, I.S. and Ryzhik, I.M. Tables of Integrals, Series, and Products, AcademicPress, New York (2007).
[26] Handique, L. and Chakraborty, S. A new beta generated Kumaraswamy Marshall-Olkin-G family of distributions with applications. Malaysian Journal of Science, 36, 157-174 (2017a).
[27] Handique, L. and Chakraborty, S. The Beta generalized Marshall-Olkin Kumaraswamy-G family of distributions with applications. Int. J. Agricult. Stat. Sci. 13, 721-733 (2017b).
[28] Handique L, Chakraborty S, Hamedani GG (2017a) The Marshall-Olkin-Kumaraswamy-G family of distributions. Journal of Statistical Theory and Applications, 16: 427-447.
[29] Handique, L., Chakraborty, S. and Ali, M.M. The Beta-Kumaraswamy-G family of distributions. Pakistan Journal of Statistics, 33, 467-490 (2017).
[30] Handique, L., Chakraborty, S. and Thiago, A.N. The exponentiated generalized Marshall-Olkin family of distributions: Its properties and applications. Annals of Data Science, 1-21 (2018).
[31] Handique, L., Ahsan, A.L. and Chakraborty, S. Generalized Modified exponential-G family of distributions: its properties and applications. International Journal of Mathematics and Statistics. 21, 1-17. (2020).
[32] Haq, M.A., Handique, L. and Chakraborty, S. The odd moment exponential family of distributions: Its properties and applications. International Journal of Applied Mathematics and Statistics 57, 47-62 (2018).
[33] Jayakumar, K. and Mathew, T. On a generalization to Marshall-Olkin scheme and its application to Burr type XII distribution. Stat Pap, 49, 421-439 (2008).
[34] Marshall, A. and Olkin, I. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika, 84, 641-652 (1997).
[35] Pike M. C. A method of analysis of a certain class of experiments in carcinogenesis, Biometrics 22, 142-161(1966).
[36] Ravi, V. and Gilbert, P. D. An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function, J. Statist. Software, 32(4) (2009).
[37] Voinov, V., Nikulin, M., and Balakrishnan, N. Chi-Squared Goodness of Fit Tests with Applications, Academic Press, Elsevier (2013).
[38] John P. Klein and Melvin L. Moschberger, Survival Analysis: Techniques for Censored and Truncated Data, Springer (2nd edition), New York (2003).
[39] Aarset, M.V. How to Identify a Bathtub Hazard Rate. IEEE Trans. Reliab. 36, 106108,(1987).
[40] Voinov , V. and Nikulin , M. Chi-square goodness-of-fit tests: Drawbacks and improvements . In International Encyclopedia of Statistical Science , Edited by: Lovric , M. Berlin : Springer, (2011) .