JJMS **17**, No. 4, 475-481 (2024)

Jordan Journal of Mathematics and Statistics.
Yarmouk University

DOI:https://doi.org/10.47013/17.4.1

Implicit Solution of Two New Models of Gohar Fractional Logistic Differential Equations

Debbouche Souheyla [1,*], *Merad Ahcene*[2]

[1] [2] Oum El Bouaghi University, Laboratory of dynamic systems and control, Algeria

Received: July 15, 2023 Accepted : June 27, 2024

Abstract: Two types of logistic fractional differential equations have been studies in this work. It also presents a new fractional derivative formula which is called the Gohar fractional derivative (GFD). The principle goal of this work is finding new solutions for each class in an implicit representation. The method used depends on the properties of the fractional derivative and some methods of functional analysis, as we will present our study with illustrative examples.

Keywords: logistic fractional differential equation; Implicit solution; Gohar Fractional Derivative.

2010 Mathematics Subject Classification. 39A50; 45K05.

1 Introduction

Due to the fact that fractional differentiation is based on the principle of generalizing the derivative of integer order to the derivative of fractional order, it has recently witnessed great development. The development of numerical analysis modelling in various softwares is considered one of the most important reasons that played a crucial role in this growth by reaching an approximation of solutions to fractional differential problems, which are difficult to be found theoretically [15], [14], [5].
The differential logistic equation is one of the differential equations is concerned by this fractional generalization, where it was studied as follows [13]:

$$^{CF}\partial_t^\alpha X(t) = X(t)\big[1 - X(t)\big],$$

with $^{CF}\partial_t^\alpha$ is the Caputo-Fabrizio fractional derivative operator. Whene, he solved this fractional version logistic differential equation by giving an implicit representation.
Also, the following logistic fractional differential equation has studied in [7]:

$$^{CF}\partial_t^\alpha X(t) = -rX(t)\big[1 - \frac{X(t)}{T}\big]\big[1 - \frac{X(t)}{K}\big],$$

with $X(t)$ is the population size at time $t \geq 0$, T is the critical threshold, K is the carrying capacity and r is called the intrinsic growth rate. We know that the constants r, T, K are positive with $T < K$, and $^{CF}\partial_t^\alpha$ is the fractional derivative in the Caputo-Fabrizio sense.
Researchers in this field have tried to find a geometric representation of the fractional derivative through the development of more accurate fractional derivative definitions for instance: Caputo derivative, Caputo-Fabrizio derivative, Riemann-Liouville fractional derivative [11],[4], [1].
The fractional derivative concept of Gohar is considered one of the new concepts that draws attention due to the presence of its geometric explanation, which is explained in the article [12].

* Corresponding author e-mail: souheyla.prof@gmail.com

This research will present a theoretical study of two new types of logistic fractional differential equations. The concept of Gohar fractional differential will be proposed as a generalization of the ordinary derivative in each proposed equation. The first problem is to find the implicit solution to the following new equation:

$$^{G}\partial_t^{\alpha}X(t)=X(t)\big[1-X(t)\big],\ t>0, \tag{1}$$

with $^{G}\partial_t^{\alpha}$ is the Gohar fractional derivative operator and $\alpha\in(0,1)$.
The second problem is the new fractional Verhults logistic differential equation with Allet effect :

$$^{G}\partial_t^{\alpha}X(t)=-rX(t)\big[1-\frac{X(t)}{\theta}\big]\big[1-\frac{X(t)}{\kappa}\big],\ t>0, \tag{2}$$

where r,θ,κ are positive constants with $\theta<\kappa$, and $^{G}\partial_t^{\alpha}$ is the operator of Gohar derivative with $\alpha\in(0,1)$.

2 Notions and preleminaries

This section recalls some needed results.

Definition 1. *[12] The Gohar fractional derivative (GFD) of $\mathscr{F}:[0,+\infty)\to\mathscr{R}$ can be expressed as*

$$^{G}\partial_t^{\alpha}\mathscr{F}(t)=lim_{s\to 0}\frac{\mathscr{F}\Big(t\big[1+ln\big(1+s\frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha+1)}t^{-\alpha}\big)\big]\Big)-\mathscr{F}(t)}{s},\ t>0,$$

with $\alpha\in(0,1)$ and $\lambda\in\mathscr{R}^{+}$.

Definition 2. *[12] The Gohar fractional integral (GFI) of a function $\mathscr{F}:(0,t]\to\mathscr{R}$, is given by*

$$^{G}I_t^{\alpha}\mathscr{F}(t)=\frac{\Gamma(\lambda-\alpha+1)}{\Gamma(\lambda)}\int_0^t\frac{\mathscr{F}(s)}{s^{1-\alpha}}ds,\ t>0,$$

with $\alpha\in(0,1)$ and $\lambda\in\mathscr{R}^{+}$.

Theorem 1. *[12] Let $\mathscr{F}:[0,+\infty)\to\mathscr{R}$, $t>0$, and $\alpha\in(0,1)$. If $\mathscr{F}$ is Gohar integrable function, then:*

$$^{G}\partial_t^{\alpha}\big(^{G}I_t^{\alpha}\mathscr{F}(t)\big)=\mathscr{F}(t),$$

and

$$^{G}I_t^{\alpha}\big(^{G}\partial_t^{\alpha}\mathscr{F}(t)\big)=\mathscr{F}(t)-\mathscr{F}(0).$$

Corollary 1. *Let $\mathscr{F}:[0,+\infty)\to\mathscr{R}$, $t>0$, and $\alpha\in(0,1)$. If $\mathscr{F}$ is Gohar integrable function, then:*

$$\frac{d}{dt}\big(^{G}I_t^{\alpha}(\mathscr{F}(t)\big)=\frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}\mathscr{F}(t),\ t>0,$$

with $\lambda\in\mathscr{R}^{+}$.

Proof. If we denote by

$$\mathscr{P}(s)=\int\frac{\mathscr{F}(s)}{s^{1-\alpha}}ds,\ s>0,$$

we have :

$$\frac{d}{dt}\big(^{G}I_t^{\alpha}(\mathscr{F}(t)\big)=\frac{d}{dt}\big(\frac{\Gamma(\lambda-\alpha+1)}{\Gamma(\lambda)}\int_0^t\frac{\mathscr{F}(s)}{s^{1-\alpha}}ds\big)$$

$$=\frac{\Gamma(\lambda-\alpha+1)}{\Gamma(\lambda)}\frac{d}{dt}\big(\mathscr{P}(t)-\mathscr{P}(0)\big)$$

$$=\frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}\mathscr{F}(t).$$

Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

The local fractional derivative in Gohar sense is considered one of the modern fractional derivatives which obeys classical properties including: linearity, product rule, quotient rule, power rule, chain rule, vanishing derivatives for constant functions, the Rolle's Theorem and the Mean Value Theorem. In the next element, we will show some of them.

Theorem 2. *[12]* (1) *The GFD is a fractal derivative or a generalization of the* $q-$*derivative.*

(2) *Let* $\alpha \in (0,1], t \geq 0, \lambda \in \mathscr{R}^+$ *and* $\mathscr{F}$ *a differentiable function in the Gohar sense with* $\mathscr{F} : [0,+\infty) \to \mathscr{R}$*. Then :*

$$^{G}\partial_t^{\alpha}\mathscr{F}(t) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha+1)} t^{1-\alpha}\frac{d\mathscr{F}(t)}{dt}.$$

(3) *Let* $t > 0$ *and* $\alpha \in (0,1)$*. If* $\mathscr{F}$ *is a* $G_\alpha-$ *differentiable function in an open interval* $(0,\delta), \delta > 0$*, and* $lim_{t\to 0^+}(^{G}\partial_t^{\alpha}\mathscr{F}(t))$ *exists, then :*

$$^{G}\partial_t^{\alpha}\mathscr{F}(0) = lim_{t\to 0^+}(^{G}\partial_t^{\alpha}\mathscr{F}(t)).$$

(4) *If* $\alpha = 1$*, the GFD of* $\mathscr{F}$ *reduces to the classical derivative of* f*.*

(5) *The GFD of* $\mathscr{F}$ *at* $t_0 \geq 0$ *is the slope of the Gohar fractional curve at* $(t_0, \mathscr{F}(t_0))$ *which is given by the following expression*

$$\mathscr{F}(t) = \big(\mathscr{F}^{\alpha}(t_0) + \big(\frac{\mathscr{F}^{\alpha-1}(t_0)(t^{\alpha}-t_0^{\alpha})}{t_0^{\alpha-1}}\big)^{G}\partial_t^{\alpha}\mathscr{F}(t_0)\big)^{\frac{1}{\alpha}},$$

with $\alpha \in (0,1]$.

(6) *We have :* $^{G}\partial_t^{\alpha}\mathscr{F}(c) = 0$*, with c is a real constant.*

(7) *Let* $\mathscr{F}(t) = t^n$*, then* $^{G}\partial_t^{\alpha}t^n = ^{C}\partial_t^{\alpha}t^n$*, with* $^{C}\partial_t^{\alpha}$ *is the derivative in the Caputo sense.*

2.1 Solution of the Gohar fractional logistic differential equation (GFLDE)

The logistic equation is one of the well-known ordinary differential nonlinear equations, which was developed by Pierre François Verhulst [17], which was as follows:

$$\partial_t X(t) = X(t)\big[1 - X(t)\big].$$

This equation is a model of the population evolution according to certain data. In this section the normal derivative in the previous equation will be replaced by the new fractional derivative, which is called Gohar fractional derivative (GFD). At the end of this section, a solution to this new fractional logistic differential equation, which was presented earlier in (1).

Theorem 3. *Let* $X : [0,+\infty) \to \mathscr{R}$*,* $t > 0$*, and* $\alpha \in (0,1)$*. The Gohar fractional logistic equation (1) have a solution given by an implicit form :*

$$\frac{X(t)}{1-X(t)} = ke^{\frac{\Gamma(\lambda-\alpha+1)}{\alpha\Gamma(\lambda)}t^{-\alpha}}, \tag{3}$$

with $\lambda \in \mathscr{R}^+$ *and* $k \in \mathscr{R}$*.*

Proof. We first appliying the Gohar integral $\big(^{G}I_{\alpha}\big)$ to (1), we obtain

$$^{G}I_t^{\alpha}\big(^{G}\partial_t^{\alpha}X(t)\big) = \big(^{G}I_t^{\alpha}\mathscr{Y}(t)\big), \tag{4}$$

with

$$\mathscr{Y}(t) = X(t)\big[1 - X(t)\big].$$

By using theorem (1), we have

$$X(t) - X(0) = \frac{\Gamma(\lambda-\alpha+1)}{\Gamma(\lambda)}\int_0^t \frac{\mathscr{Y}(s)}{s^{1-\alpha}}ds. \tag{5}$$

Taking the first derivative of equation (5), we obtain

$$X'(t) = \frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}\mathscr{Y}(t). \tag{6}$$

Replacing $\mathscr{Y}(t)$ in (6), then

$$X'(t) = \frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}X(t)(1-X(t)),$$

then

$$\frac{X'(t)}{X(t)(1-X(t))} = \frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)},$$

which refers to

$$\frac{X'(t)}{X(t)} + \frac{X'(t)}{1-X(t)} = \frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}. \tag{7}$$

Integrating both sides of equation (7), we have

$$ln|X(t)| - ln|1-X(t)| = \frac{\Gamma(\lambda-\alpha+1)}{\alpha t^{\alpha}\Gamma(\lambda)} + c,$$

then

$$ln\frac{|X(t)|}{|1-X(t)|} = \frac{\Gamma(\lambda-\alpha+1)}{\alpha t^{\alpha}\Gamma(\lambda)} + c. \tag{8}$$

Hence, we have obtained the solution (3).

Example. We fixed the parameters :

$$\lambda = \frac{1}{2},\ \alpha = \frac{1}{2}.$$

We rewrite (1) become as follows :

$${}^{G}\partial_t^{\frac{1}{2}}X(t) = X(t)\big[1-X(t)\big], \tag{9}$$

Then

$$\frac{\Gamma(\lambda-\alpha+1)}{\alpha\Gamma(\lambda)} = \frac{\Gamma(1)}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{2}{\sqrt{\pi}}$$

then, we obtain the solution of GLFDE (9) as follows :

$$\frac{X(t)}{1-X(t)} = ke^{\frac{2}{\sqrt{\pi t}}}. \tag{10}$$

2.2 Solution of the Gohar fractional logistic differential equation with Allet effect

The logistic differential equation with Allet effect is named after the scientist Walter Clyde Allee, who proved that the positive growth of a population is directly related to the number of its elements, for some examples see [9], [18], [20], [16]. This principle has been modeled in the following differential logistic equation :

$$\partial_t X(t) = -rX(t)\big[1-\frac{X(t)}{\theta}\big]\big[1-\frac{X(t)}{\kappa}\big].$$

In this section, we will study the new Gohar fractional differential version of the previous equation, which we presented in (2).
The following result, which will be needed later, will be first included.
Lemma 1. [7] For all constants θ and κ, we have

$$\frac{1}{X(t)\big[1-\frac{X(t)}{\theta}\big]\big[1-\frac{X(t)}{\kappa}\big]} = \frac{1}{X(t)} + \frac{\theta\beta}{\theta-X(t)} + \frac{\kappa\zeta}{\kappa-X(t)},$$

with

$$\beta = \frac{\kappa}{\kappa\theta - \theta^2},$$

and

$$\zeta = \frac{1}{\kappa} + \frac{1}{\theta} - \frac{\kappa}{\kappa\theta - \theta^2}.$$

Therefore, the following theorem has been concluded:

Theorem 4. *For all r, θ, κ positive constants with $\theta < \kappa$, the Gohar fractional logistic differential equation with Allee effect (2) have a implicit solution given by :*

$$X(t)\big(\theta - X(t)\big)^{-\theta\beta}\big(\kappa - X(t)\big)^{-\kappa\zeta} = ke^{\frac{\Gamma(\lambda-\alpha+1)}{\alpha\Gamma(\lambda)}t^{-\alpha}},\ t > 0, \tag{11}$$

with $\lambda \in \mathscr{R}^+$ and $k \in \mathscr{R}$.

Proof. The Gohar fractional integral $\left({}^{G}I_{\alpha}\right)$ has first been applied to (2), we obtain

$$ {}^{G}I_t^{\alpha}\left({}^{G}\partial_t^{\alpha}X(t)\right) = \left({}^{G}I_t^{\alpha}\mathscr{V}(t)\right), \tag{12}$$

with

$$\mathscr{V}(t) = -rX(t)\big[1 - \frac{X(t)}{\theta}\big]\big[1 - \frac{X(t)}{\kappa}\big].$$

By using theorem (3), we obtain

$$X(t) - X(0) = \frac{\Gamma(\lambda-\alpha+1)}{\Gamma(\lambda)}\int_0^t \frac{\mathscr{V}(s)}{s^{1-\alpha}}ds. \tag{13}$$

Taking the first derivative of equation (13), we have

$$X'(t) = \frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}\mathscr{V}(t). \tag{14}$$

Replacing $\mathscr{V}(t)$ in (14), then

$$X'(t) = \frac{-r\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}X(t)\big[1 - \frac{X(t)}{\theta}\big]\big[1 - \frac{X(t)}{\kappa}\big],$$

then

$$\frac{X'(t)}{X(t)\big[1 - \frac{X(t)}{\theta}\big]\big[1 - \frac{X(t)}{\kappa}\big].} = \frac{-r\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)},$$

We use the lemma (1), we get

$$\frac{X'(t)}{X(t)} + \frac{\theta\beta X'(t)}{\theta - X(t)} + \frac{\kappa\zeta X'(t)}{\kappa - X(t)}, = \frac{\Gamma(\lambda-\alpha+1)}{t^{1-\alpha}\Gamma(\lambda)}. \tag{15}$$

Integrating both sides of (15), we have

$$ln|X(t)| - \theta\beta ln|\theta - X(t)| - \kappa\zeta ln|\kappa - X(t)| = \frac{\Gamma(\lambda-\alpha+1)}{\alpha t^{\alpha}\Gamma(\lambda)} + c,$$

then

$$e^{ln|X(t)| - \theta\beta ln|\theta - X(t)| - \kappa\zeta ln|\kappa - X(t)|} = ke^{\frac{\Gamma(\lambda-\alpha+1)}{\alpha\Gamma(\lambda)}t^{-\alpha}}, \tag{16}$$

finally, we get the implicit solution formula (11).

Example. Let :

$$r = 2,\ \theta = 1,\ \kappa = 2,\ \alpha = \frac{1}{2}, \lambda = \frac{1}{2}.$$

Then, we have the following equation :

$$ {}^{G}\partial_t^{\frac{1}{2}}X(t) = -2X(t)\big[1 - X(t)\big]\big[1 - \frac{X(t)}{2}\big], \tag{17}$$

Here

$$\beta = 2,\ \zeta = \frac{-1}{2}.$$

The implicit solution of Gohar fractional logistic equation (17) is:

$$X(t)\big(1 - X(t)\big)^{-2}\big(2 - X(t)\big)^{-1} = ke^{\frac{2}{\sqrt{\pi t}}},$$

then

$$\frac{X(t)}{\big(2 - X(t)\big)\big(1 - X(t)\big)^{2}} = ke^{\frac{2}{\sqrt{\pi t}}},\ k \in \mathscr{R}.$$

3 Conclusion

This work presented the first application of Gohar's fractional definition (GFD) in differential equations, especially the logistic differential equations, which are characterized by their importance in modeling different population changes. Based on functional methods, the solutions of the proposed new Gohar fractional logistic equations (GFLDE) has been reached throughout this work. A numerical study of these new equations has been proposed as future works.

Declarations

Competing interests: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Authors' contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.
Funding: The authors declare that they have no known competing financial interests.
Availability of data and materials: The data that supports the findings of this study are available within the article.
Acknowledgments: The authors would like to thank the editors and referees for their constructive and invaluable comments and suggestions for improving the manuscript.

References

[1] MS Abdo, W Shammakh, HZ Alzumi. New Existence and Stability Results for Caputo-Fabrizio Fractional Nonlocal Implicit Problems. *Journal of Mathematics*,**V 7**, (2023)(3152502, 10 pages).
[2] Area, I, Losada, J, and Nieto, J. J. A note on the fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*. **444**, (2016), 182– 187.
[3] Arshad. M. S, Baleanu. D, Riaz. M. B, and Abbas, M. A novel 2-stage fractional Rungekutta method for a time-fractional logistic growth model. *Discrete Dynamics in Nature and Society*, **Vol 2020**, (2020), 1–8.
[4] Caputo. M and Fabrizio. M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ*.**1(2)**(2015), 73–85.
[5] Das, S. *Functional fractional calculus*. Berlin: Springer, (2011).
[6] Debbouche. N, Ouannas. A, Momani. S, D Cafagna, VT Pham. Fractional-order biological system: chaos, multistability and coexisting attractors. *The European Physical Journal Special Topics*, **V:231**, 1061–1070, (2021).
[7] Debbouche.S, Implicit solution for logistic Caputo-Fabrizio fractional differential equation with Allee effect. *Journal of Fractional Calculus and Nonlinear Systems*, **4(1)**, (2023), 1–17.
[8] Debbouche, S. Merad, A. A study of N-dimensional fractional differential problem with integral conditions. *Nonlinear Studies*, **30(3)**, (2023), 795–805.
[9] Dennis, B. Allee effects: population growth, critical density, and the chance of extinction. *Natural Resource Modeling*, **3(4)**, (1989), 481–538.
[10] El-Sayed, A. M. A, El-Mesiry, A. E. M, and ElSaka, H. A. A. On the fractional-order logisticequation. *Applied Mathematics Letters*, **20(7)**, (2007), 817–823.
[11] Hilfer, R. (Ed.). *Applications of fractional calculus in physics*. World scientific, (2000).
[12] Gohar. A, Younes. M and Doma. S. Gohar Fractional Derivative: Theory and Applications. *Journal of Fractional Calculus and Nonlinear Systems*, **4(1)**, (2023), 17–34.

[13] Juan J. Nieto. Solution of a fractional logistic ordinary differential equation. *Applied Mathematics Letters*.**20(7)** (2022),817– 823.
[14] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J, *Theory and Application of the Fractional Diferential Equations*. Elsevier, Netherlands(2006).
[15] Podlubny. I, *Fractional Differential Equations*, Academic Press, San Diego, (1999).
[16] Schreiber, S. J. (2003). Allee effects, extinctions, and chaotic transients in simple population models. *Theoretical population biology*, **64(2)**, (2020), 201-209.
[17] Strogatz.S. H. *Nonlinear Dynamics and Chaos*. Levant Books, Kolkata, Indiça, (2007).
[18] Suryanto. A, Darti. I and Anam. S. Stability analysis of a fractional order modified Leslie-Gower model with additive Allee effect. *International Journalof Mathematics and Mathematical Sciences*, **Vol:2017**, (2017).
[19] Tarasov, V. E. Exact solutions of Bernoulli and logistic fractional differential equations with power law coefficients. *Mathematics*, **8(12)**,(2020), 22–31.
[20] Vortkamp. I, Schreiber.S. J, Hastings. A and Hilker, F. M. Multiple attractors and long transients in spatially structured populations with an Allee effect. *Bulletin of mathematical biology*, 82:1-21.
[21] West, B. J. Exact solution to fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*. **429**, (2015), 103–108.