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# $QMLE$ OF THE GENERAL PERIODIC $GARCH$ MODELS

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Abstract. In this article, we study the necessary and sufficient conditions that guarantee the strict stationarity of general periodic generalized autoregressive conditional heteroskedasticity models (in the periodic sense). We also obtain conditions for the existence of finite higher-order moments under general and tractable assumptions. We propose the quasi-maximum likelihood estimation of general periodic generalized autoregressive conditional heteroskedasticity parameters and derive their asymptotic properties. We demonstrate the strong consistency and asymptotic normality of the quasi-maximum likelihood estimation in special cases.

## 1. Introduction

In this article, we aim to study the generalization of periodic generalized autoregressive conditional heteroskedasticity ($PGARCH$) models as proposed by Bollerslev and Ghysels ([6], 1996). These models focus on the conditional variance based on the past information. We consider the process $(\varepsilon_t)_{t\in\mathbb{Z}}$, which satisfies $\varepsilon_t = \sigma_t e_t$, where $(e_t)_{t\in\mathbb{Z}}$ is a process independent of $(\sigma_t)_{t\in\mathbb{Z}}$ and is i.i.d. with $E\{e_t\} = 0$, $E\{e_t^2\} = 1$. The volatility process $(\sigma_t)_{t\in\mathbb{Z}}$ is assumed to satisfy

$$(1.1)\quad \begin{cases} \sigma_t^2 = h_t^{-1} \\ h_t := h(\sigma_t^2) = \beta_0(s_t) + \sum_{i=1}^{p} \beta_i(s_t)\, g_{s_t}(e_{t-i}) + \sum_{j=1}^{p} \alpha_j(s_t)\, f_{s_t}(e_{t-j})\, h_{t-j} \end{cases},$$

where $(s_t)_t$ is a sequence of positive integers with a finite state space $\mathbb{P} = \{1, ..., s\}$. The coefficients $\alpha_j(.)$ and $\beta_i(.)$ are defined on $\mathbb{P}$ and take values in $\mathbb{R}^+$ with $\beta_0(.) > 0$. Additionally, $h_t$ is defined on $\mathbb{R}^+$ with values in $\mathbb{R}^+$ as an invertible function, while

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$h$, $g_{.}$ and $f_{.}$ are real-valued and measurable functions. When $s \geq 2$, the processes $(\varepsilon_t)_t$ are nonstationary but exhibit periodic stationarity, making them attractive for investigating volatility and distinct (seasonal) patterns. To rewrite model (1.1), let's consider $s_t = \sum_{v=1}^{s} v\mathbb{I}_{\Delta(v)}(t)$, where $\Delta(v) := \{sn+v,\ (v,n) \in \mathbb{P} \times \mathbb{Z}\}$. Consequently, the model (1.1) is equivalent to

$$\begin{cases} \varepsilon_{sn+v} = \sigma_{sn+v} e_{sn+v} \\ h_{sn+v} := h\left(\sigma^2_{sn+v}\right) = \beta_0(v) + \sum_{i=1}^{p} \beta_i(v)\, g_v(e_{sn+v-i}) \\ \qquad\qquad + \sum_{j=1}^{p} \alpha_j(v)\, f_v(e_{sn+v-j})\, h_{sn+v-j} \end{cases} \tag{1.2}$$

with $\sigma^2_{sn+v} = h^{-1}_{sn+v}$, where $\alpha_1(v), ..., \alpha_p(v), \beta_1(v), ..., \beta_p(v)$ are the coefficients at regime $v$. We can derive several models from Eq (1.1) or (1.2),

| Model | The volatility process $(\sigma_t)$ |
|---|---|
| Periodic $GARCH$ | $\sigma_t^2 = \beta_0(s_t) + \sum_{i=1}^{q} \beta_i(s_t)\, \varepsilon^2_{t-i} + \sum_{j=1}^{p} \alpha_j(s_t)\, \sigma^2_{t-j}$ |
| Periodic $LGARCH$ | $\sigma_t = \beta_0(s_t) + \sum_{i=1}^{q} \beta_i(s_t)\, \varepsilon_{t-i} + \sum_{j=1}^{p} \alpha_j(s_t)\, \sigma_{t-j}$ |
| Periodic exponential $GARCH$ | $\log \sigma_t^2 = \beta_0(s_t) + \sum_{i=1}^{q} (\beta_i(s_t)\, e_{t-i} + \gamma_i(s_t) \times (\lvert e_{t-i}\rvert - E\{\lvert e_{t-i}\rvert\})) + \sum_{j=1}^{p} \alpha_j(s_t) \log \sigma^2_{t-j}$ |
| Periodic multiplicative $GARCH$ | $\log \sigma_t^2 = \beta_0(s_t) + \sum_{i=1}^{q} \beta_i(s_t) \log e^2_{t-i} + \sum_{j=1}^{p} \alpha_j(s_t) \log \sigma^2_{t-j}$ |
| Periodic $\log GARCH$ | $\log \sigma_t^2 = \beta_0(s_t) + \sum_{i=1}^{q} \beta_i(s_t) \log \varepsilon^2_{t-i} + \sum_{j=1}^{p} \alpha_j(s_t) \log \sigma^2_{t-j}$ |
| Periodic asym-power $GARCH$ | $\sigma_t^\delta = \beta_0(s_t) + \sum_{i=1}^{q} \beta_i(s_t) (\lvert\varepsilon_{t-i}\rvert + \gamma(s_t)\, \varepsilon_{t-i})^\delta + \sum_{j=1}^{p} \alpha_j(s_t)\, \sigma^\delta_{t-j}$ |

Table 1 : General $PGARCH$ specification.

Continuation of Table 1

| | |
|---|---|
| Periodic threshold $GARCH$ | $\sigma_t = \beta_0(s_t) + \sum_{i=1}^{q}\left(\beta_i(s_t)\,\varepsilon_{t-i}^{-} + \gamma_i(s_t)\,\varepsilon_{t-i}^{+}\right) + \sum_{j=1}^{p}\alpha_j(s_t)\,\sigma_{t-j}$ |
| Periodic $GJR$ power $GARCH$ | $\sigma_t^{\delta} = \beta_0(s_t) + \sum_{i=1}^{q}\left(\beta_i(s_t) + \gamma_i(s_t)\,\mathbb{I}_{\{\varepsilon_{t-i}>0\}}\right)\varepsilon_{t-i}^{\delta} + \sum_{j=1}^{p}\alpha_j(s_t)\,\sigma_{t-j}^{\delta}$ |
| Periodic absolute value $GARCH$ | $\sigma_t = \beta_0(s_t) + \sum_{i=1}^{q}\beta_i(s_t)\,\lvert\varepsilon_{t-i}\rvert + \sum_{j=1}^{p}\alpha_j(s_t)\,\sigma_{t-j}$ |
| Periodic nonlinear $GARCH$ | $\sigma_t^{\delta} = \beta_0(s_t) + \sum_{i=1}^{q}\beta_i(s_t)\,\lvert\varepsilon_{t-i}\rvert^{\delta} + \sum_{j=1}^{p}\alpha_j(s_t)\,\sigma_{t-j}^{\delta}$ |
| Periodic $VGARCH$ | $\sigma_t^2 = \beta_0(s_t) + \sum_{i=1}^{q}\beta_i(s_t)\left(e_{t-i} - \gamma_i(s_t)\right)^2 + \sum_{j=1}^{p}\alpha_j(s_t)\,\sigma_{t-j}^2$ |

In this article, we present some properties of the general $PGARCH$ process. In Section 3, we propose the Quasi-Maximum Likelihood Estimation ($QMLE$) of the general $PGARCH$ parameters and analyze their asymptotic properties. In Section 4, we demonstrate the strong consistency ($SC$) and asymptotic normality ($AN$) of the $QMLE$ in specific states.

Some notations used throughout the article are defined as follows:

- The identity matrix is denoted by $I_{(n)}$.
- The indicator function is denoted by $\mathbb{I}_{\{.\}}$.
- The zero matrix is denoted by $O_{(n,m)}$. For further clarification, we use $O_{(n)}$ to represent $O_{(n,n)}$ and $\underline{O}_{(n)}$ to represent $O_{(n,1)}$.
- $\rho(M)$ represents the spectral radius of a matrix $M_{(n,n)}$.
- The vec operator is denoted by $\widetilde{M} = vec(M)$.
- The Kronecker product of matrices is denoted by $\otimes$.
- The symbol $\rightsquigarrow$ denotes convergence in distribution.

## 2. Strict stationarity

Consider the $r = (2p+1)-$dimensional random vectors defined as follows: $\underline{h}_t := (f_{s_t}(e_t)\,h_t, ..., f_{s_t}(e_{t-p+1})\,h_{t-p+1}, h_t, g_{s_t}(e_t), ..., g_{s_t}(e_{t-p+1}))'$, $\underline{H}' := \left(\underline{O}'_{(p)}, 1, \underline{O}'_{(p)}\right)$ and $\underline{e}'_t = (\beta_0(s_t)\,f_{s_t}(e_t),\ \underline{O}'_{(p-1)}, \beta_0(s_t), g_{s_t}(e_t), \underline{O}'_{(p-1)})'$. Additionally, we have an $r \times$

$r-$matrix denoted as $\Lambda_t$ with

$$\Lambda_{s_t}(e_t) := \begin{pmatrix} \alpha_1(s_t) f_{s_t}(e_t) & ... & \alpha_p(s_t) f_{s_t}(e_t) & 0 & \beta_1(s_t) f_{s_t}(e_t) & ... & \beta_p(s_t) f_{s_t}(e_t) \\ I_{(p-1)} & & \underline{O}_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & & \underline{O}_{(p-1)} \\ \alpha_1(s_t) & ... & \alpha_p(s_t) & 0 & \beta_1(s_t) & ... & \beta_p(s_t) \\ 0 & ... & 0 & 0 & 0 & ... & 0 \\ O_{(p-1)} & & \underline{O}_{(p-1)} & \underline{O}_{(p-1)} & I_{(p-1)} & & \underline{O}_{(p-1)} \end{pmatrix}.$$

The process described by theEq. (1.1) is equivalent to the following process

$$\underline{h}_t = \Lambda_{s_t}(e_t)\, \underline{h}_{t-1} + \underline{e}_t, \tag{2.1}$$

with $h_t = \underline{H}'\underline{h}_t$, Eq. (2.1) is similar to the generalized $PVAR$ process presented lately by Franses and Paap [10]. Furthermore, since Gladychev [14], it has become possible to utilize a non-periodic multivariate stationary process $(\underline{H}_n)_n$, where $\underline{H}_n := \left(\underline{h}'_{ns+1}, ..., \underline{h}'_{ns+s}\right)' \in \mathbb{R}^{rs}$ represents a non-periodic generalized $VAR$ process, i.e.,

$$\underline{H}_n = \Gamma_n \underline{H}_{n-1} + \underline{\eta}_n, \tag{2.3}$$

where

$$\Gamma_n := \begin{pmatrix} O_{(r)} & \cdots & O_{(r)} & \Lambda_1(e_{ns+1}) \\ O_{(r)} & \cdots & O_{(r)} & \Lambda_2(e_{ns+2})\Lambda_1(e_{ns+1}) \\ \vdots & \ddots & \vdots & \vdots \\ O_{(r)} & \cdots & O_{(r)} & \left\{\prod_{v=0}^{s-1} \Lambda_{s-v}(e_{ns+s-v})\right\} \end{pmatrix}_{rs\times rs},$$

$$\underline{\eta}_n := \begin{pmatrix} \underline{e}_{ns+1} \\ \Lambda_2(e_{ns+2})\,\underline{e}_{ns+1} + \underline{e}_{ns+2} \\ \vdots \\ \sum_{k=1}^{s} \left\{\prod_{v=0}^{s-k-1} \Lambda_{s-v}(e_{ns+s-v})\right\} \underline{e}_{ns+k} \end{pmatrix}_{rs\times 1},$$

where $\prod_{v=0}^{s-1} \Lambda_v = I_{(r)}$ if $s < 1$. Moreover, the process solution of (2.3) is strictly stationary (resp. ergodic) equivalent to the solution of (2.1) being strictly periodically stationary (abbreviated as $SPS$) (resp. periodically ergodic, abbreviated as $PE$. See Boyles and Gardner ( [9], 1983 )). The conditions that ensure the existence of $SPS$ solution of $PGARCH$ models were studied by Bibi et al. ([1], [4]). However, in our article, we derive the manageable conditions for general $PGARCH$.

Since $(e_t)_{t\in\mathbb{Z}}$ is an independent and identically distribution process, stationary and ergodic, the process $\left(\Gamma_n, \underline{\eta}_n\right)_n$ is also stationary and ergodic. This is ensured by the conditions $E\left\{\log^+ \|\Gamma_1\|\right\} < \infty$ and $E\left\{\log^+ \left\|\underline{\eta}_1\right\|\right\} < \infty$, where $\log^+(y) = \log y \vee 0$ for any $y > 0$. The results presented in this subsection are based on the theorems proven by Bougerol and Picard [8].

**Theorem 2.1.** *Eq.* (2.3) *has a unique strictly stationary solution and ergodic if and only if the top-Lyapunov exponent* $\gamma_L(\Gamma)$ *of* $(\Gamma_n)_n$ *,*

$$\gamma_L(\Gamma) := \inf_{n>0}\left\{\frac{1}{n}E\left\{\log\left\|\prod_{j=0}^{n-1}\Gamma_{n-j}\right\|\right\}\right\} \overset{a.s.}{=} \lim_{t\longrightarrow\infty}\left\{\frac{1}{n}\log\left\|\prod_{j=0}^{n-1}\Gamma_{n-j}\right\|\right\}, \tag{2.4}$$

*is strictly negative. The unique stationary solution is ergodic, causal and given by*

$$\underline{H}_n = \sum_{k\geq 0}\left\{\prod_{j=0}^{k-1}\Gamma_{n-j}\right\}\underline{\eta}_{n-k}, \tag{2.5}$$

*where the series* (2.5) *converges almost surely (a.s.) and*

$$\sigma^2_{ns+v} = h^{-1}\left(\sum_{k\geq 0}\left(\left\{\prod_{j=0}^{k-1}\Gamma_{n-j}\right\}\underline{\eta}_{n-k}\otimes\underline{H}\right)'\widetilde{F'}\right),$$

*with* $\widetilde{F'} := \left(O_{(r)},\ldots,O_{(r)},\underbrace{I_{(r)}}_{v^{th}-block},O_{(r)},\ldots,O_{(r)}\right)_{r\times rs}.$

*Proof.* The proof of this theorem is identical to the Theorem 1.3 presented by Bougerol and Picard [7]. □

**Proposition 2.1.** *If the top-Lyapunov exponent of* $\left(\left\{\prod_{v=0}^{s-1}\Lambda_{s-v}\left(e_{ns+s-v}\right)\right\}\right)_n$ *is strictly negative, then Eq.* (2.3) *also has a unique, strictly stationary solution, ergodic and can be represented by the series* (2.5).

*Proof.* A simple computation shows that

$$\prod_{j=0}^{n}\Gamma_{n-j} = \Gamma_n\begin{pmatrix} O_{(r)} & \cdots & O_{(r)} & O_{(r)} \\ O_{(r)} & \cdots & O_{(r)} & O_{(r)} \\ \vdots & \ddots & \vdots & \vdots \\ O_{(r)} & \cdots & O_{(r)} & \prod_{j=1}^{n-1}\left\{\prod_{v=0}^{s-1}\Lambda_{s-v}\left(e_{ns+s-v-j}\right)\right\} \end{pmatrix},$$

therefore, due to the independence of the top-Lyapunov exponent from the norm, we can conclude that $\gamma_L(\Gamma) \leq \gamma_L(\Lambda)$. □

**Corollary 2.1.** *If $p = 1$, a sufficient condition that ensures $\gamma_L(\Lambda) < 0$ is that $E\left\{\prod_{v=1}^{s} \alpha_1(v) |f_{v-1}(e_0)|\right\} < 1$. This condition is the same as the one presented in* [13] *for the scalar case.*

*Proof.* If $p = 1$, we have $\gamma_L(\Lambda) = E\left\{\log\left\{\prod_{v=1}^{s} \alpha_1(v) |f_{v-1}(e_0)|\right\}\right\}$. □

**Example 2.1.** *In this example, Table 2 provides a summary of the available results indicating that the condition $\gamma_L(\Lambda) < 0$ holds for certain specifications.*

| *Model* | *The condition* $\gamma_L(\Lambda) < 0$ |
|---|---|
| *Periodic GARCH* $(1,1)$ | $\left\{\prod_{v=1}^{s} (\alpha_1(v) + \gamma_1(v))\right\} < 1$ |
| *Periodic LGARCH* $(1,1)$ | $\prod_{v=1}^{s} E\{\lvert\alpha_1(v) e_0 + \gamma_1(v)\rvert\} < 1$ |
| *Periodic exponential GARCH* $(1,1)$ | $\left\{\prod_{v=1}^{s} \alpha_1(v)\right\} < 1$ |
| *Periodic multiplicative GARCH* $(1,1)$ | $\left\{\prod_{v=1}^{s} \alpha_1(v)\right\} < 1$ |
| *Periodic* $\log GARCH\,(1,1)$ | $\left\{\prod_{v=1}^{s} (\alpha_1(v) + \gamma_1(v))\right\} < 1$ |
| *Periodic asym-power GARCH* $(1,1)$ | $\prod_{v=1}^{s} E\left\{\left\lvert\alpha_1(v) + (\lvert e_0\rvert + \gamma(v-1) e_0)^{\delta} \beta_1(v)\right\rvert\right\} < 1$ |
| *Periodic threshold GARCH* $(1,1)$ | $E\left\{\prod_{v=1}^{s} \left\lvert\alpha_1(v) + \gamma_1(v) e_0^{+} + \beta_1(v) e_0^{-}\right\rvert\right\} < 1$ |
| *Periodic GJR power GARCH* $(1,1)$ | $E\left\{\prod_{v=1}^{s} \lvert\alpha_1(v) + \left(\beta_1(v) e_0^{\delta} + \gamma_1(v) e_0^{\delta} \mathbb{I}_{\{e_0>0\}}\right)\rvert\right\} < 1$ |
| *Periodic absolute value GARCH* $(1,1)$ | $E\left\{\prod_{v=1}^{s} (\alpha_1(v) + \lvert e_0\rvert \beta_1(v))\right\} < 1$ |
| *Periodic nonlinear GARCH* $(1,1)$ | $E\left\{\prod_{v=1}^{s} \left(\alpha_1(v) + \beta_1(v) \lvert e_0\rvert^{\delta}\right)\right\} < 1$ |
| *Periodic VGARCH* $(1,1)$ | $\left\{\prod_{v=1}^{s} \alpha_1(v)\right\} < 1$ |

*Table 2 : The condition $\gamma_L(\Lambda) < 0$ holds for some specifications.*

When it is difficult to obtain the top Lyapunov exponent criterion, we propose the following result

**Theorem 2.2.** $(\underline{H}_n)_n$ *is a stationary process solution of model* (2.3) *under the assumption $\tau_l := E\left\{e_t^{2l}\right\} < \infty$ with $l > 1$,*

***1.*** *if $\rho\left(\prod_{v=0}^{s-1} E\left\{\Lambda_{s-v}^{\otimes l}(e_0)\right\}\right) < 1$, then $\underline{H}_n \in \mathbb{L}_l$.*

***2.*** *if $\rho\left(\prod_{v=0}^{s-1} E\left\{\Lambda_{s-v}^{\otimes l}(e_0)\right\}\right) \geq 1$, then there is no strictly stationary solution $(\underline{H}_n)_n$ to the process* (2.3) *such that $\underline{H}_n \in \mathbb{L}_l$.*

*Proof.* The proof is identical to that of Theorem 4.1 in the work by Bibi and Aknouche [4]. Firstly, we define $\mathbb{R}^{rs}-$random vectors

$$\underline{P}_n(t) = \underline{O}_{(rs)}\mathbb{I}_{\{n<0\}} + \Gamma_t \underline{P}_{n-1}(t-1)\mathbb{I}_{\{n\geq 0\}},$$

and $\underline{Q}_n(t) = \underline{P}_n(t) - \underline{P}_{n-1}(t)$ for all $n \in \mathbb{Z}$. It is easy to prove that for all $n \geq 0$, $\underline{P}_n(t)$ and $\underline{Q}_n(t)$ are measurable functions of $\underline{e}_t, ..., \underline{e}_{t-n}$. We have for all $n \in \mathbb{Z}$

$$\underline{Q}_n(t) = \underline{O}_{(rs)}\mathbb{I}_{\{n<0\}} + \underline{\eta}_t\mathbb{I}_{\{n=0\}} + \Gamma_t \underline{P}_{n-1}(t-1)\mathbb{I}_{\{n>0\}},$$

which implies that $E\left\{\underline{Q}_n^{\otimes l}(t)\right\} = \left(E\left\{\Gamma_t^{\otimes l}\right\}\right)^n E\left\{\underline{\eta}_{t-n}^{\otimes l}\right\}$ for all $n > 0$. Since $\rho\left(\prod_{v=0}^{s-1} E\left\{\Lambda_{s-v}^{\otimes l}(e_0)\right\}\right) = \rho\left(E\left\{\Gamma_t^{\otimes l}\right\}\right) < 1$, we can conclude that $\underline{P}_n(t) \overset{L_l \text{ and } a.s.}{\longrightarrow} \underline{H}_t \in \mathbb{L}_l$ satisfies Eq. (2.3). Secondly, from (2.3), we obtain for any $n > 0$

$$\underline{H}_t = \left\{\prod_{k=0}^{n} \Gamma_{t-k}\right\} \underline{H}_{t-n-1} + \sum_{j=0}^{n} \left\{\prod_{k=0}^{j-1} \Gamma_{t-k}\right\} \underline{\eta}_{t-j},$$

and $E\left\{\underline{H}_t^{\otimes l}\right\} \geq \sum_{j=0}^{n} \left(E\left\{\Gamma_t^{\otimes l}\right\}\right)^k E\left\{\underline{\eta}_{t-j}^{\otimes l}\right\}$. This completes the proof. □

## 3. Asymptotic properties of the $QMLE$

In this section, we display the SC and AN of the $QMLE$ for general periodic $GARCH$ models. The study conducted by Aknouche and Bibi [1] investigates the asymptotic properties of $PGARCH(p, q)$ models. Additionally, Straumann and Mikosch [15] provide an analysis of the asymptotic properties of augmented $GARCH$ for the scalar case.

### 3.1. **Strong consistency.**

The vector parameters is denoted as $\underline{\theta} := \left(\underline{\theta}'(1), ..., \underline{\theta}'(s)\right)' \in \Theta \subset \left(\mathbb{R}_+^* \times \mathbb{R}_+^{2p}\right)^s$. Let $\underline{\theta}_0 := \left(\underline{\theta}_0'(1), ..., \underline{\theta}_0'(s)\right)'$ the true parameter value is unknown. Consider a time series $(\varepsilon_1, ..., \varepsilon_{ns})'$ defined by (1.1) with the parameter $\underline{\theta}_0$, given as follows

$$(3.1)\qquad \begin{cases} \varepsilon_{st+v} = \sigma_{st+v}(\underline{\theta})\, e_{st+v} \\ h_{st+v}(\underline{\theta}) = \beta_{0,0}(v) + \sum_{i=1}^{p} \beta_{i,0}(v)\, g_{v,\underline{\theta}}(e_{st+v-i}) \\ \qquad\qquad + \sum_{j=1}^{p} \alpha_{j,0}(v)\, f_{v,\underline{\theta}}(e_{st+v-j})\, h_{st+v-j}(\underline{\theta}) \end{cases},$$

with $\sigma_t^2(\underline{\theta}) = h_t^{-1}(\underline{\theta})$, the Gaussian log −quasi-likelihood, conditional on initial values $e_0, ..., e_{1-p}, \tilde{h}_0, ..., \tilde{h}_{1-p}$ is expressed as $\tilde{L}_{ns}(\underline{\theta}) = \frac{-1}{ns}\sum_{t=0}^{n-1}\sum_{v=1}^{s}\tilde{l}_{st+v}(\underline{\theta})$. It is accompanied by the contribution function $\tilde{l}_t(\underline{\theta}) = \frac{\varepsilon_t^2}{\tilde{\sigma}_t^2} + \log\tilde{\sigma}_t^2$, $t \geq 1$, where $\tilde{\sigma}_t^2 = \tilde{\sigma}_t^2(\underline{\theta})$ and it can be computed for $t \geq 1$,

$$\begin{cases} \varepsilon_{st+v} = \tilde{\sigma}_{st+v}(\underline{\theta})\, e_{st+v} \\ \tilde{h}_{st+v}(\underline{\theta}) = \beta_0(v) + \sum_{i=1}^{p}\beta_i(v)\, g_{v,\underline{\theta}}(e_{st+v-i}) + \sum_{j=1}^{p}\alpha_j(v)\, f_{v,\underline{\theta}}(e_{st+v-j})\, \tilde{h}_{st+v-j}(\underline{\theta}) \end{cases},$$

with $\tilde{\sigma}_t^2(\underline{\theta}) = \tilde{h}_t^{-1}(\underline{\theta})$ by giving initial values $e_0, ..., e_{1-p}, \tilde{h}_0, ..., \tilde{h}_{1-p}$. The $QMLE$ of the true parameters $\underline{\theta}_0$, denoted by $\hat{\underline{\theta}}_{ns}$, is defined as $\hat{\underline{\theta}}_{ns} = \underset{\theta\in\Theta}{arg\,min}\tilde{J}_{ns}(\underline{\theta}) = \underset{\theta\in\Theta}{arg\,max}\tilde{L}_{ns}(\underline{\theta})$, where $\tilde{J}_{ns}(\underline{\theta}) = \frac{1}{ns}\sum_{t=0}^{n-1}\sum_{v=1}^{s}\tilde{l}_{st+v}(\underline{\theta})$. Now, we will use the following hypotheses to prove the strong consistency of the $QMLE$.

**A1**. $\underline{\theta}_0 \in \Theta$ and $\Theta$ is compact.

**A2**. The top Lyapunov exponent $\gamma_L(\Lambda^0)$ of the sequence $\left(\Lambda_{s_t}^0(e_t)\right)_t$ is negative. Here, $\left(\Lambda_{s_t}^0(e_t)\right)_t$ denotes the sequence obtained by replacing $\underline{\theta}_0$ with $\underline{\theta}$ in $(\Lambda_{s_t}(e_t))_t$.

**A3**. If, for all $v \in \mathbb{P}$, $\sigma_v^2(\underline{\theta}) = \sigma_v^2(\underline{\theta}_0)$ $a.s.$ then $\underline{\theta} = \underline{\theta}_0$.

**A4**. For any $\underline{\theta} \in \Theta$, $g_{v,\underline{\theta}}(e_0) \geq 0$, $f_{v,\underline{\theta}}(e_0) \geq 0$ and $\sigma_v^2(\underline{\theta}) \geq \sigma > 0$ for all $v \in \mathbb{P}$.

**A5**. The function $h$ is three times continuously differentiable and there exist constants $K > 0$, $\delta$ such that the following conditions hold

**a.** $\left|\left(\frac{\partial h}{\partial\theta(i)}\left(h^{-1}\left(\sigma_v^2(\underline{\theta})\right)\right)\right)^{-1}\right| \leq K\sigma^\delta$,

**b.** $\left|\frac{\partial^2 h}{\partial\theta(i)\partial\theta(j)}\left(h^{-1}\left(\sigma_v^2(\underline{\theta})\right)\right)\right| \leq K\sigma^\delta$,

**c.** $\left|\frac{\partial^3 h}{\partial\theta(i)\partial\theta(j)\partial\theta(k)}\left(h^{-1}\left(\sigma_v^2(\underline{\theta})\right)\right)\right| \leq K\sigma^\delta$,

for all $\sigma_v^2(\underline{\theta}) \geq \sigma > 0$, $v \in \mathbb{P}$ and $i, j, k \in \{1, ..., s(d+1)\}$.

Assumption **A1** is a commonly used assumption in various real analysis results. Assumption **A2** guarantees strict stationarity, ergodic (in the periodic sense), and the existence of finite moments of (3.1). Assumption **A3** is made to ensure the identifiability of the parameter. Assumptions **A4** and **A5** are similar to those used in the work of Aue et al.[2].

Let $(l_t(\underline{\theta}))_t$ be defined as $l_t(\underline{\theta}) = \frac{\varepsilon_t^2}{\sigma_t^2} + \log \sigma_t^2$, where $\sigma_t^2 = \sigma_t^2(\underline{\theta})$ and is obtained by

$$\begin{cases} \varepsilon_{st+v} = \sigma_{st+v}(\underline{\theta})\, e_{st+v} \\ h_{st+v}(\underline{\theta}) = \beta_0(v) + \sum_{i=1}^{p} \beta_i(v)\, g_{v,\underline{\theta}}(e_{st+v-i}) + \sum_{j=1}^{p} \alpha_j(v)\, f_{v,\underline{\theta}}(e_{st+v-j})\, h_{st+v-j}(\underline{\theta}) \end{cases},$$

with $\sigma_t^2(\underline{\theta}) = h_t^{-1}(\underline{\theta})$, it is worth noting that $\sigma_t^2 = \sigma_t^2(\underline{\theta}_0)$ and

$$L_{ns}(\underline{\theta}) = \frac{-1}{ns} \sum_{t=0}^{n-1} \sum_{v=1}^{s} l_{st+v}(\underline{\theta}).$$

Now, we will present some of the results used to establish the $SC$

**Lemma 3.1.** *Assume that* ***A1-A5*** *are satisfied, the following items hold*
*1.* $\lim_{n \longrightarrow \infty} \sup_{\underline{\theta} \in \Theta} \left| \tilde{L}_{ns}(\underline{\theta}) - L_{ns}(\underline{\theta}) \right| = 0$ *a.s.* *2.* $\sum_{v=1}^{s} E_{\underline{\theta}_0} \{ l_v(\underline{\theta}_0) \} < \infty$.
*3. If* $\underline{\theta} \neq \underline{\theta}_0$, *then* $\sum_{v=1}^{s} E_{\underline{\theta}_0} \{ l_v(\underline{\theta}) - l_v(\underline{\theta}_0) \} \geq 0$.
*4. For any* $\underline{\theta} \neq \underline{\theta}_0$, *consider a neighborhood* $\mathcal{V}(\underline{\theta})$ *such that*

$$\lim_{n \longrightarrow \infty} \inf \inf_{\underline{\tilde{\theta}} \in \Theta} \left( -\tilde{L}_{ns}\left(\underline{\tilde{\theta}}\right) \right) > \sum_{v=1}^{s} E_{\underline{\theta}_0} \{ l_v(\underline{\theta}_0) \}.$$

*Proof.* First, we have

$$\sup_{\underline{\theta} \in \Theta} \left| \tilde{L}_{ns}(\underline{\theta}) - L_{ns}(\underline{\theta}) \right| \leq$$

$$\frac{1}{n} \sup_{\underline{\theta} \in \Theta} \sum_{t=0}^{n-1} \sum_{v=1}^{s} \left( \frac{\varepsilon_{st+v}^2}{\sigma^2} \left| \tilde{\sigma}_{st+v}^2(\underline{\theta}) - \sigma_{st+v}^2(\underline{\theta}) \right| + \left| \log \tilde{\sigma}_{st+v}^2(\underline{\theta}) - \log \sigma_{st+v}^2(\underline{\theta}) \right| \right),$$

and

$$\left| \tilde{h}_t - h_t \right| \leq \underline{H}' \left\{ \prod_{j=0}^{t} \Lambda_{s_{t-j}}(e_{t-j}) \right\} \left| \tilde{h}_0 - h_0 \right|,$$

under assumption **A2**, we have $\left| \tilde{h}_t - h_t \right| \xrightarrow{a.s.} 0$. Applying the mean value theorem, we get

$$\begin{aligned} \left| \tilde{\sigma}_t^2(\underline{\theta}) - \sigma_t^2(\underline{\theta}) \right| &= \left| h^{-1}\left( \tilde{h}_t(\underline{\theta}) \right) - h^{-1}(h_t(\underline{\theta})) \right| \leq K\sigma^{\delta} \left| \tilde{h}_t(\underline{\theta}) - h_t(\underline{\theta}) \right| \xrightarrow{a.s.} 0, \\ \left| \log \tilde{\sigma}_t^2(\underline{\theta}) - \log \sigma_t^2(\underline{\theta}) \right| &\leq \frac{1}{\sigma} \left| \tilde{\sigma}_t^2(\underline{\theta}) - \sigma_t^2(\underline{\theta}) \right| \xrightarrow{a.s.} 0. \end{aligned}$$

By the last inequality, we can establish that the supremum of the absolute difference between $\tilde{L}_{ns}(\underline{\theta})$ and $L_{ns}(\underline{\theta})$ is bounded as follows

$$\underset{\underline{\theta}\in\Theta}{sup}\left|\tilde{L}_{ns}(\underline{\theta})-L_{ns}(\underline{\theta})\right|\leq\frac{1}{n\sigma}\underset{\underline{\theta}\in\Theta}{sup}\sum_{t=0}^{n-1}\sum_{v=1}^{s}\left(1+\frac{\varepsilon_{st+v}^{2}}{\sigma}\right)\left|\tilde{\sigma}_{st+v}^{2}(\underline{\theta})-\sigma_{st+v}^{2}(\underline{\theta})\right|.$$

By applying Lemma 1 proposed by Straumann and Mikosch [15], we find $\sum\limits_{t\geq0}\varepsilon_t^2\left|\tilde{\sigma}_t^2(\underline{\theta})-\sigma_t^2(\underline{\theta})\right|$ converges a.s., if $E\left\{\log^{+}\varepsilon_t^2\right\}<\infty$. This result holds under assumption **A2**.

Second, we can note that $l_v(\underline{\theta}_0)=(l_v^{+}-l_v^{-})(\underline{\theta}_0)$, where $l_v^{+}(\underline{\theta}_0)=\max(l_v(\underline{\theta}_0),0)$ and $l_v^{-}(\underline{\theta}_0)=\max(0,-l_v(\underline{\theta}_0))$. If we assume that $\sum\limits_{v=1}^{s}E_{\underline{\theta}_0}\{l_v^{+}(\underline{\theta}_0)\}<\infty$ and $\sum\limits_{v=1}^{s}E_{\underline{\theta}_0}\{l_v^{-}(\underline{\theta}_0)\}<\infty$. Under the condition **A4**, we get

$$\begin{aligned}\sum_{v=1}^{s}E_{\underline{\theta}_0}\{l_v^{-}(\underline{\theta}_0)\}&\leq\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\log\left(\max\left(1,\frac{1}{\sigma_v^2(\underline{\theta}_0)}\right)\right)\right\}\\&\leq\sum_{v=1}^{s}\log\left(E_{\underline{\theta}_0}\left\{\max\left(1,\frac{1}{\sigma_v^2(\underline{\theta}_0)}\right)\right\}\right)\leq\max(0,-s\log\sigma)<\infty,\end{aligned}$$

on the other side

$$\sum_{v=1}^{s}E_{\underline{\theta}_0}\{l_v^{+}(\underline{\theta}_0)\}\leq s+\frac{1}{\delta}\sum_{v=1}^{s}\log\left(\max\left(\frac{1}{e^{\delta}},E_{\underline{\theta}_0}\left\{\left(\sigma_v^2(\underline{\theta}_0)\right)^{\delta}\right\}\right)\right),$$

where $e$ is the base of the natural logarithm function and $\delta>0$. Under the conditions **A2** and **A4**, we have

$$E_{\underline{\theta}_0}\left\{\left(\sigma_v^2(\underline{\theta}_0)\right)^{\delta}\right\}=E_{\underline{\theta}_0}\left\{\left(h^{-1}\left(\sum_{k\geq0}\left(\left\{\prod_{j=0}^{k-1}\Gamma_{-j}\right\}\underline{\eta}_{-k}\otimes\underline{H}\right)'\widetilde{F'}\right)\right)^{\delta}\right\}<\infty.$$

Third, we have

$$\begin{aligned}\sum_{v=1}^{s}E_{\underline{\theta}_0}\{l_v(\underline{\theta})-l_v(\underline{\theta}_0)\}&=\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\log\left(\frac{\sigma_v^2(\underline{\theta})}{\sigma_v^2(\underline{\theta}_0)}\right)+\frac{\sigma_v^2(\underline{\theta}_0)}{\sigma_v^2(\underline{\theta})}-1\right\}\\&\geq\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\log\left(\frac{\sigma_v^2(\underline{\theta})}{\sigma_v^2(\underline{\theta}_0)}\right)+\log\left(\frac{\sigma_v^2(\underline{\theta}_0)}{\sigma_v^2(\underline{\theta})}\right)\right\}\geq0,\end{aligned}$$

because, for all $x>0$, $x-1\geq\log x$. Fourth, is similar the proof of Lemma$B.4$ in Bibi and Aknouche [1]. □

Based on the previous results, we can establish the following result regarding the $SC$ of $\hat{\underline{\theta}}_{ns}$.

**Theorem 3.1.** *According to Assumptions* ***A1-A5****,* $\hat{\underline{\theta}}_{ns} \longrightarrow \underline{\theta}_0$ *a.s. as* $n \longrightarrow \infty$.

*Proof.* The proof of this theorem follows a similar approach to the proof of Theorem 3 presented in the work of Bibi and Aknouche [1]. □

3.2. **Asymptotic normality.** In this subsection, we present the AN of $\hat{\underline{\theta}}_{ns}$. We begin by considering the Taylor series expansion of $\frac{\partial}{\partial \underline{\theta}} \tilde{L}_{ns}(\underline{\theta})$ around $\underline{\theta}_0$, we have

$$0 = \frac{1}{\sqrt{ns}} \sum_{t=1}^{ns} \frac{\partial \tilde{l}_t}{\partial \underline{\theta}} \left(\hat{\underline{\theta}}_{ns}\right) = \frac{1}{\sqrt{ns}} \sum_{t=1}^{ns} \frac{\partial \tilde{l}_t}{\partial \underline{\theta}} (\underline{\theta}_0) + \left(\frac{1}{ns} \sum_{t=1}^{ns} \frac{\partial^2 \tilde{l}_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}^*)\right) \sqrt{ns} \left(\hat{\underline{\theta}}_{ns} - \underline{\theta}_0\right),$$

where $\underline{\theta}^*$ lies between $\hat{\underline{\theta}}_{ns}$ and $\underline{\theta}_0$. Then

$$\sqrt{ns} \left(\hat{\underline{\theta}}_{ns} - \underline{\theta}_0\right) = \left(\frac{1}{ns} \sum_{t=1}^{ns} \frac{\partial^2 \tilde{l}_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}^*)\right)^{-1} \left(-\frac{1}{\sqrt{ns}} \sum_{t=1}^{ns} \frac{\partial \tilde{l}_t}{\partial \underline{\theta}} (\underline{\theta}_0)\right).$$

Consequently, we show that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{v=1}^{s} \frac{\partial \tilde{l}_{st+v}}{\partial \underline{\theta}} (\underline{\theta}_0) \rightsquigarrow \mathcal{N}(\underline{O}, \Omega), \tag{3.2}$$

$$\frac{1}{n} \sum_{t=1}^{n} \sum_{v=1}^{s} \frac{\partial^2 \tilde{l}_{st+v}}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}^*) \to \Xi \text{ a.s.},$$

where the matrix $\Omega$ is defined by

$$\begin{aligned} \Omega &:= (\tau_2 - 1) \left(\sum_{v=1}^{s} E_{\underline{\theta}_0} \left\{\frac{\partial^2 l_v}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}_0)\right\}\right)^{-1} \\ &= (\tau_2 - 1) \Xi^{-1} \text{ with } \Xi := \sum_{v=1}^{s} E_{\underline{\theta}_0} \left\{\frac{1}{\sigma_v^4(\underline{\theta}_0)} \frac{\partial \sigma_v^2}{\partial \underline{\theta}} (\underline{\theta}_0) \frac{\partial \sigma_v^2}{\partial \underline{\theta}'} (\underline{\theta}_0)\right\} \end{aligned}$$

. The partial derivatives of $l_t(\underline{\theta})$ are obtained by

$$\frac{\partial l_t}{\partial \underline{\theta}} (\underline{\theta}) = \left(1 - \frac{\varepsilon_t^2}{\sigma_t^2(\underline{\theta})}\right) \frac{1}{\sigma_t^2(\underline{\theta})} \frac{\partial \sigma_t^2}{\partial \underline{\theta}} (\underline{\theta})$$

$$\begin{aligned} \frac{\partial^2 l_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}) &= \left(1 - \frac{\varepsilon_t^2}{\sigma_t^2(\underline{\theta})}\right) \frac{1}{\sigma_t^2(\underline{\theta})} \frac{\partial^2 \sigma_t^2}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}) \\ &\quad + \left(2 \frac{\varepsilon_t^2}{\sigma_t^2(\underline{\theta})} - 1\right) \frac{1}{\sigma_t^4(\underline{\theta})} \frac{\partial \sigma_t^2}{\partial \underline{\theta}} (\underline{\theta}) \frac{\partial \sigma_t^2}{\partial \underline{\theta}'} (\underline{\theta}) \end{aligned} \tag{3.3}$$

**Remark 1.** Since $\frac{\partial l_t}{\partial \underline{\theta}} (\underline{\theta})$ and $\frac{\partial^2 l_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta})$ are measurable functions of the $SPS$ and $PE$ process $(\varepsilon_{st+v})$, then $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \sum_{v=1}^{s} \frac{\partial l_{st+v}}{\partial \underline{\theta}} (\underline{\theta})$ is a $SPS$ and $PE$ zero$-$mean martingale difference.

But consider the following additional conditions

**A6**. $\underline{\theta}_0 \in \mathring{\Theta}$, where $\mathring{\Theta}$ is the interior of $\Theta$.

**A7**. $\tau_2 < \infty$.

**A8**. There exists a neighborhood $\mathcal{V}(\underline{\theta}_0)$ of $\underline{\theta}_0$ and $\sigma_t^2(\underline{\theta})$ is $3-$times continuously differentiable in $\underline{\theta}$ with measurable derivatives such that

$$\begin{aligned}
&\mathbf{a}. \quad \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\left\|\frac{1}{\sigma_v^2(\underline{\theta}_0)}\frac{\partial\sigma_v^2}{\partial\underline{\theta}}(\underline{\theta}_0)\right\|\right\} < \infty,\\
&\mathbf{b}. \quad \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\left\|\frac{1}{\sigma_v^2(\underline{\theta}_0)}\frac{\partial^2\sigma_v^2}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\} < \infty,\\
&\mathbf{c}. \quad \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\left\|\frac{1}{\sigma_v^4(\underline{\theta}_0)}\frac{\partial\sigma_v^2}{\partial\underline{\theta}}(\underline{\theta}_0)\frac{\partial\sigma_v^2}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\} < \infty,\\
&\mathbf{d}. \quad \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in V(\underline{\theta}_0)}\left|\frac{\partial^3 l_v}{\partial\theta(i)\partial\theta(j)\partial\theta(k)}(\underline{\theta})\right|\right\} < \infty,
\end{aligned}$$

for all $i,\ j,\ k \in \{1, ..., s(d+1)\}$.

**A9**. The components of $\frac{\partial\sigma_t^2(\underline{\theta})}{\partial\underline{\theta}}$ are considered as linearly independent random variables.

The assumptions **A6-A9** are similar to the assumptions in Strumann and Mikosch [15]. We use some of our results to prove the $AN$.

**Lemma 3.2.** *Assume that **A1-A9**, the following items are satisfied*

***1***. *$\Xi$ is invertible matrix.*

***2***. $\frac{1}{\sqrt{n}}\left\|\sum_{t=1}^{n}\sum_{v=1}^{s}\left(\frac{\partial\tilde{l}_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0) - \frac{\partial l_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\right)\right\| \to 0,$

*and* $\frac{1}{n}\sup_{\underline{\theta}\in V(\underline{\theta}_0)}\left\|\sum_{t=1}^{n}\sum_{v=1}^{s}\left(\frac{\partial^2\tilde{l}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0) - \frac{\partial^2 l_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right)\right\| \to 0$ *in probability when* $n \to \infty$.

***3***. $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{v=1}^{s}\frac{\partial\tilde{l}_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0) \rightsquigarrow \mathcal{N}(\underline{O},\Omega)$ *and* $\frac{1}{n}\sum_{t=1}^{n}\sum_{v=1}^{s}\frac{\partial^2\tilde{l}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}^*) \to \Xi$ *a.s.*

*Proof.* First, it is show that the matrix $\Xi$ is positive definite. Assuming $\underline{X}_0'\Xi\underline{X}_0 = 0$ for some $\underline{X}_0 \in \mathbb{R}^d$, this is equivalent to $\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\frac{1}{\sigma_v^4(\underline{\theta}_0)}\left(\frac{\partial\sigma_v^2}{\partial\underline{\theta}'}(\underline{\theta}_0)\underline{X}_0\right)^2\right\} = 0$, which implies $\frac{\partial\sigma_v^2}{\partial\underline{\theta}'}(\underline{\theta}_0)\underline{X}_0 = 0$ a.s. for all $v \in \mathbb{S}$. Under **A9** implies $\underline{X}_0 = \underline{O}_d$. This concludes the first item.

Second, we have for all $i \in \{1, ..., s(d+1)\}$

$$\left|\frac{\partial \tilde{l}_t}{\partial \theta(i)}(\underline{\theta}_0) - \frac{\partial l_t}{\partial \theta(i)}(\underline{\theta}_0)\right| = \left| \begin{array}{l} \left(\frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} - \frac{1}{\sigma_t^2(\underline{\theta}_0)}\right) \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) \left(1 - \frac{\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)} - \frac{\varepsilon_t^2}{\sigma_t^2(\underline{\theta}_0)}\right) \\ + \left(1 - \frac{\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)}\right) \frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} \left(\frac{\partial \tilde{\sigma}_t^2}{\partial \theta(i)}(\underline{\theta}_0) - \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0)\right) \end{array} \right|$$

$$\leq K\left(1 + e_t^2\right)\left(\frac{1}{\sigma_t^2(\underline{\theta}_0)} \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) \left|\sigma_t^2(\underline{\theta}_0) - \tilde{\sigma}_t^2(\underline{\theta}_0)\right| + \left|\frac{\partial \tilde{\sigma}_t^2}{\partial \theta(i)}(\underline{\theta}_0) - \frac{\partial \sigma_t^2}{\partial \theta(j)}(\underline{\theta}_0)\right|\right),$$

applying the mean value theorem, we have

$$\left|\frac{\partial \tilde{\sigma}_t^2}{\partial \theta(i)}(\underline{\theta}_0) - \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0)\right| \leq K\sigma^\delta \left|\tilde{h}_t(\underline{\theta}_0) - h_t(\underline{\theta}_0)\right|,$$

which implies

$$\left|\frac{\partial \tilde{l}_t}{\partial \theta(i)}(\underline{\theta}_0) - \frac{\partial l_t}{\partial \theta(i)}(\underline{\theta}_0)\right| \leq K\left(1 + e_t^2\right)\left|\tilde{h}_t(\underline{\theta}_0) - h_t(\underline{\theta}_0)\right| \left(\frac{1}{\sigma_t^2(\underline{\theta}_0)} \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) + \sigma^\delta\right),$$

using the Markov inequality, we have for all $\sigma > 0$

$$P\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(1 + e_t^2\right)\left|\tilde{h}_t(\underline{\theta}_0) - h_t(\underline{\theta}_0)\right| \left(\frac{1}{\sigma_t^2(\underline{\theta}_0)} \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) + \sigma^\delta\right) > \sigma\right)$$

$$\leq \frac{2}{\sigma\sqrt{n}}\left(E_{\underline{\theta}_0}\left\{\frac{1}{\sigma_t^2(\underline{\theta}_0)} \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0)\right\} + \sigma^\delta\right) \sum_{t=1}^{n} E\left\{\left|\tilde{h}_t(\underline{\theta}_0) - h_t(\underline{\theta}_0)\right|\right\},$$

by Eq. (3.3), we get

$$\left|\frac{\partial^2 \tilde{l}_t}{\partial \theta(i)\partial \theta(j)}(\underline{\theta}_0) - \frac{\partial^2 l_t}{\partial \theta(i)\partial \theta(j)}(\underline{\theta}_0)\right| \leq \sum_{i=1}^{3} |I_i|,$$

with

$$I_1 = \left(\frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} - \frac{1}{\sigma_t^2(\underline{\theta}_0)}\right) \left\{ \begin{array}{l} \left(1 - \frac{\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)} - \frac{\varepsilon_t^2}{\sigma_t^2(\underline{\theta}_0)}\right) \frac{\partial^2 \sigma_t^2}{\partial \theta(i)\partial \theta(j)}(\underline{\theta}_0) \\ + \left(\frac{2\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)} + \frac{2\varepsilon_t^2}{\sigma_t^2(\underline{\theta}_0)} - 1\right) \frac{1}{\sigma_t^2(\underline{\theta}_0)} \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) \frac{\partial \sigma_t^2}{\partial \theta(j)}(\underline{\theta}_0) \\ + \left(\frac{2\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)} - 1\right) \frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} \frac{\partial \tilde{\sigma}_t^2}{\partial \theta(j)}(\underline{\theta}_0) \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) \end{array} \right\},$$

$$I_2 = \left(1 - \frac{\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)}\right) \frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} \left(\frac{\partial^2 \tilde{\sigma}_t^2}{\partial \theta(i)\partial \theta(j)}(\underline{\theta}_0) - \frac{\partial^2 \sigma_t^2}{\partial \theta(i)\partial \theta(j)}(\underline{\theta}_0)\right),$$

$$I_3 = \left(\frac{2\varepsilon_t^2}{\tilde{\sigma}_t^2(\underline{\theta}_0)} - 1\right) \frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} \left\{ \begin{array}{l} \left(\frac{\partial \tilde{\sigma}_t^2}{\partial \theta(i)}(\underline{\theta}_0) - \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0)\right) \frac{1}{\tilde{\sigma}_t^2(\underline{\theta}_0)} \frac{\partial \tilde{\sigma}_t^2}{\partial \theta(j)}(\underline{\theta}_0) \\ + \left(\frac{\partial \tilde{\sigma}_t^2}{\partial \theta(j)}(\underline{\theta}_0) - \frac{\partial \sigma_t^2}{\partial \theta(j)}(\underline{\theta}_0)\right) \frac{1}{\sigma_t^2(\underline{\theta}_0)} \frac{\partial \sigma_t^2}{\partial \theta(i)}(\underline{\theta}_0) \end{array} \right\},$$

applying the mean value theorem and under assumption **A5**, we have

$$\left|\frac{\partial^2 \tilde{\sigma}_t^2}{\partial\theta(i)\,\partial\theta(j)}\left(\underline{\theta}_0\right)-\frac{\partial^2 \sigma_t^2}{\partial\theta(i)\,\partial\theta(j)}\left(\underline{\theta}_0\right)\right| \leq K\sigma^{\delta}\left|\tilde{h}_t\left(\underline{\theta}_0\right)-h_t\left(\underline{\theta}_0\right)\right|,$$

and using the Markov inequality, we have $\frac{1}{n}\underset{\underline{\theta}\in V(\underline{\theta}_0)}{sup}\left\|\sum_{t=1}^{n}\sum_{v=1}^{s}\left(\frac{\partial^2 \tilde{l}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}_0\right)-\frac{\partial^2 l_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}_0\right)\right)\right\|$ $\to 0$ in probability when $n\to\infty$.

Third, let $\mathcal{G}_t := \sigma\left(e_{t-u}, u\geq 0\right)$. Since $E_{\underline{\theta}_0}\left\{\frac{\partial l_t}{\partial\underline{\theta}}\left(\underline{\theta}_0\right)\Big|\,\mathcal{G}_{t-1}\right\}=0$, we can utilize the Central Limit Theorem (C.L.T.) of Bilingsley [5] and the Wold-Cramér theorem to obtain (3.2). By employing a second Taylor series expansion of $\frac{\partial^2\tilde{L}_{ns}}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}\right)$ at $\underline{\theta}_0$, we have for all $i$, $j$, $k$.

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{ns}\frac{\partial^2 l_t}{\partial\theta(i)\,\partial\theta(j)}\left(\underline{\theta}^*\right) &= \frac{1}{n}\sum_{t=1}^{ns}\frac{\partial^2 l_t}{\partial\theta(i)\,\partial\theta(j)}\left(\underline{\theta}_0\right)\\ &\quad+\left(\frac{1}{n}\sum_{t=1}^{ns}\frac{\partial^3 l_t}{\partial\theta(i)\,\partial\theta(j)\,\partial\theta(k)}\left(\tilde{\underline{\theta}}\right)\right)\left(\underline{\theta}^*-\underline{\theta}_0\right),\end{aligned}$$

with $\tilde{\underline{\theta}}$ being between $\underline{\theta}^*$ and $\underline{\theta}_0$. Applying the ergodic theorem, we have, for all $i$, $j$, $k$,

$$\begin{aligned}&\underset{n\to\infty}{lim\,sup}\left|\frac{1}{n}\sum_{t=1}^{ns}\frac{\partial^3 l_t}{\partial\theta(i)\,\partial\theta(j)\,\partial\theta(k)}\left(\tilde{\underline{\theta}}\right)\right|\\ \leq\;& E_{\underline{\theta}_0}\left\{\underset{\underline{\theta}\in V(\underline{\theta}_0)}{sup}\left|\frac{\partial^3 l_t}{\partial\theta(i)\,\partial\theta(j)\,\partial\theta(k)}\left(\underline{\theta}\right)\right|\right\}<\infty \;a.s.,\end{aligned}$$

and

$$\underset{n\to\infty}{lim\,sup}\left|\frac{1}{n}\sum_{t=1}^{ns}\frac{\partial^2 l_t}{\partial\theta(i)\,\partial\theta(j)}\left(\underline{\theta}_0\right)\right|\to\Xi\text{ a.s.}$$

□

The next result checks the AN of $\hat{\underline{\theta}}_{ns}$.

**Theorem 3.2.** *Under conditions* ***A1-A9****, we get*

$$\sqrt{ns}\left(\hat{\underline{\theta}}_{ns}-\underline{\theta}_0\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Omega\right)\;\; as\; n\to\infty$$

*Proof.* The proof is very analogous to that of Francq and Zakoïan [11]. We utilize Lemma 3.2 and apply the Slutsky Lemma to establish this result. □

## 4. Examples

To provide a clearer illustration of the outcomes, we will now examine specific instances that fall within the scope of our overarching theory. Our objective is to demonstrate the superior characteristics and advantages of the $QMLE$ in various econometric models such as $PEGARCH$, $PMGARCH$, $PAGARCH$ and $PTGARCH$. Throughout these examples, we aim to show the $SC$ and $AN$ of the $QMLE$.

**Example 4.1.** ***The $QMLE$ in the $PEGARCH$ model***

The $PEGARCH$ model is obtained by setting in model (1.2)

$$\begin{aligned} h\left(\sigma_{sn+v}^2\right) &= \log \sigma_{sn+v}^2, f_v\left(e_{sn+v}\right) = 1, \\ g_v\left(e_{sn+v}\right) &= a\left(v\right) e_{sn+v} + b\left(v\right)\left(\left|e_{sn+v}\right| - E\left\{\left|e_{sn+v}\right|\right\}\right), \end{aligned}$$

where $|a(v)| < b(v)$ for all $v \in \mathbb{P}$ and we denote $a_i(v) = a(v)\beta_i(v)$ and $b_i(v) = b(v)\beta_i(v)$. Let $\underline{\theta}'(v) := (\beta_0(v), a_1(v), ..., a_p(v), b_1(v), ..., b_p(v))$. The next corollary presents the $SC$ of the $QMLE$.

**Corollary 4.1.** *If for all $v \in \mathbb{P}$, $\sigma_v^2(\underline{\theta}) \overset{a.s.}{=} \sigma_v^2(\underline{\theta}_0)$, then $\hat{\underline{\theta}}_{ns} \xrightarrow{a.s.} \underline{\theta}_0$ as $n \longrightarrow \infty$.*

*Proof.* It suffices to prove the validity of Assumptions **A1**-**A5**. Assumptions **A1**-**A2** are immediate. Note that $\sigma_v^2(\underline{\theta}) = \sigma_v^2(\underline{\theta}_0)$ $a.s.$is aquivalent to $\log \sigma_v^2(\underline{\theta}) = \log \sigma_v^2(\underline{\theta}_0)$ $a.s.$ for all $v \in \mathbb{P}$. Now, just prove that $\underline{\theta} = \underline{\theta}_0$, let the polynomials $\mathcal{A}_v(z) = \sum_{i=1}^{p} \beta_i(v) z^i$ and $\mathcal{B}_v(z) = 1 - \sum_{j=1}^{p} \alpha_j(v) z^j$ have no common root with $\mathcal{A}_v(1) \neq 0$ and $\alpha_p(v) + \beta_p(v) \neq 0$ for all $v \in \mathbb{P}$. By convention, if $p = 0$, $\mathcal{A}_v(z) = 0$ and $\mathcal{B}_v(z) = 1$ for all $v \in \mathbb{P}$. It follows from the proof of Bibi and Aknouche [1], Lemma $B.2$, that $\underline{\theta} = \underline{\theta}_0$, thus satisfying **A3**. Additionally, since $f_v(e_{sn+v}) = 1 \geq 0$ and $g_v(e_{sn+v}) = a(v) e_{sn+v} + b(v)(|e_{sn+v}| - E\{|e_{sn+v}|\}) \geq 0$, because if $e_n > 0$, then $0 < (a(v) + b(v)) e_{sn+v} < 2b(v) e_{sn+v}$ and if $e_n < 0$, then $0 < (a(v) - b(v)) e_{sn+v} < -2b(v) e_{sn+v}$. This implies that **(A4)** holds, and (**A5)** is realized. This completes the proof. □

Another result shows the $AN$ of the $QMLE$.

**Corollary 4.2.** *We have*

$$\sqrt{ns}\left(\underline{\hat{\theta}}_{ns}-\underline{\theta}_{0}\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Omega\right),$$

*as* $n\to\infty$ *where* $\Omega:=\left(\tau_{2}-1\right)\left(\sum\limits_{v=1}^{s}E_{\underline{\theta}_{0}}\left\{\frac{1}{\sigma_{v}^{4}\left(\underline{\theta}_{0}\right)}\frac{\partial\sigma_{v}^{2}}{\partial\underline{\theta}}\left(\underline{\theta}_{0}\right)\frac{\partial\sigma_{v}^{2}}{\partial\underline{\theta}^{\prime}}\left(\underline{\theta}_{0}\right)\right\}\right)^{-1}.$

*Proof.* Assumptions (**A6-A7)** are immediate. The proofs for (**A8)** and (**A9)** are provided in Francq and Zakoïan [11] (Theorem 2.2) and Berkes et al. [3] (Lemma 5.7). □

**Example 4.2.** ***The*** $QMLE$ ***in the*** $PMGARCH$ ***model***

The $PMGARCH$ model is obtained by setting in model (1.2)

$$h\left(\sigma_{sn+v}^{2}\right)=\log\sigma_{sn+v}^{2},f_{v}\left(e_{sn+v}\right)=1\text{ and }g_{v}\left(e_{sn+v}\right)=\log e_{sn+v}^{2}\text{ for all }v\in\mathbb{P}.$$

Let $\underline{\theta}\left(v\right):=\left(\beta_{0}\left(v\right),\beta_{1}\left(v\right),...,\beta_{p}\left(v\right),\alpha_{1}\left(v\right),...,\alpha_{p}\left(v\right)\right)^{\prime}.$ The next corollary presents the $SC$ of the $QMLE$.

**Corollary 4.3.** *If for all* $v\in\mathbb{P}$, $\sigma_{v}^{2}\left(\underline{\theta}\right)\overset{a.s.}{=}\sigma_{v}^{2}\left(\underline{\theta}_{0}\right)$, *then* $\underline{\hat{\theta}}_{ns}\overset{a.s.}{\longrightarrow}\underline{\theta}_{0}$ *as* $n\longrightarrow\infty$.

*Proof.* It suffices to prove the validity of Assumptions **A1-A5**. Assumptions (**A1-A2)** are immediate. Note that $\sigma_{v}^{2}\left(\underline{\theta}\right)=\sigma_{v}^{2}\left(\underline{\theta}_{0}\right)$ $a.s.$is aquivalent to $\log\sigma_{v}^{2}\left(\underline{\theta}\right)=\log\sigma_{v}^{2}\left(\underline{\theta}_{0}\right)$ $a.s.$ for all $v\in\mathbb{P}$. Now, just prove that $\underline{\theta}=\underline{\theta}_{0}$, let the polynomials $\mathcal{A}_{v}\left(z\right)=\sum\limits_{i=1}^{p}\beta_{i}\left(v\right)z^{i}$ and $\mathcal{B}_{v}\left(z\right)=1-\sum\limits_{j=1}^{p}\alpha_{j}\left(v\right)z^{j}$ have no common root with $\mathcal{A}_{v}\left(1\right)\neq0$ and $\alpha_{p}\left(v\right)+\beta_{p}\left(v\right)\neq0$ for all $v\in\mathbb{P}$. By convention, if $p=0$, $\mathcal{A}_{v}\left(z\right)=0$ and $\mathcal{B}_{v}\left(z\right)=1$ for all $v\in\mathbb{P}$. It follows from the proof of Bibi and Aknouche [1], Lemma $B.2$, that $\underline{\theta}=\underline{\theta}_{0}$, thus satisfying **A3**. Assumptions (**A4-A5)** are holds. This completes the proof. □

Another result shows the $AN$ of the $QMLE$.

**Corollary 4.4.** *We have*

$$\sqrt{ns}\left(\underline{\hat{\theta}}_{ns}-\underline{\theta}_{0}\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Omega\right),$$

*as* $n\to\infty$ *where* $\Omega:=\left(\tau_{2}-1\right)\left(\sum\limits_{v=1}^{s}E_{\underline{\theta}_{0}}\left\{\frac{1}{\sigma_{v}^{4}\left(\underline{\theta}_{0}\right)}\frac{\partial\sigma_{v}^{2}}{\partial\underline{\theta}}\left(\underline{\theta}_{0}\right)\frac{\partial\sigma_{v}^{2}}{\partial\underline{\theta}^{\prime}}\left(\underline{\theta}_{0}\right)\right\}\right)^{-1}.$

*Proof.* Assumptions (**A6-A7**) are immediate. The proofs for (**A8**) and (**A9**) are provided in Francq and Zakoïan [11] (Theorem 2.2) and Berkes et al. [3] (Lemma 5.7). $\square$

**Example 4.3.** ***The*** $QMLE$ ***in the*** $PAGARCH$ ***model***

The $PAGARCH$ model is obtained by setting in model (1.2)

$$\begin{aligned} h\left(\sigma_{sn+v}^2\right) &= \sigma_{sn+v}^2, f_v\left(e_{sn+v-j}\right) = \alpha_j\left(v\right) + \beta_j\left(v\right)\left(\left|e_{sn+v-j}\right| + \gamma\left(v\right) e_{sn+v-j}\right)^2, \\ g_v\left(e_{sn+v}\right) &= 0, \end{aligned}$$

for all $v \in \mathbb{P}$. Let $\underline{\theta}\left(v\right) := \left(\beta_0\left(v\right), \beta_1\left(v\right), ..., \beta_p\left(v\right), \alpha_1\left(v\right), ..., \alpha_p\left(v\right), \gamma\left(v\right)\right)'$. The next corollary presents the $SC$ of the $QMLE$.

**Corollary 4.5.** *If for all* $v \in \mathbb{P}$, $\sigma_v^2\left(\underline{\theta}\right) \overset{a.s.}{=} \sigma_v^2\left(\underline{\theta}_0\right)$, *then* $\underline{\hat{\theta}}_{ns} \overset{a.s.}{\longrightarrow} \underline{\theta}_0$ *as* $n \longrightarrow \infty$.

*Proof.* It suffices to prove the validity of Assumptions **A1**-**A5**. Assumptions (**A1-A2**) are immediate. Note that $\sigma_v^2\left(\underline{\theta}\right) = \sigma_v^2\left(\underline{\theta}_0\right)$ $a.s.$ for all $v \in \mathbb{P}$. Now, just prove that $\underline{\theta} = \underline{\theta}_0$, let the polynomials $\mathcal{A}_v\left(z\right) = \sum\limits_{i=1}^{p} \beta_i\left(v\right) z^i$ and $\mathcal{B}_v\left(z\right) = 1 - \sum\limits_{j=1}^{p} \alpha_j\left(v\right) z^j$ have no common root with $\mathcal{A}_v\left(1\right) \neq 0$ and $\alpha_p\left(v\right) + \beta_p\left(v\right) \neq 0$ for all $v \in \mathbb{P}$. By convention, if $p = 0$, $\mathcal{A}_v\left(z\right) = 0$ and $\mathcal{B}_v\left(z\right) = 1$ for all $v \in \mathbb{P}$. It follows from the proof of Bibi and Aknouche [1], Lemma $B.2$, that $\underline{\theta} = \underline{\theta}_0$, thus satisfying **A3**. Since $f_v\left(e_{sn+v}\right) \geq 0$ and $g_v\left(e_{sn+v}\right) \geq 0$, which implies that (**A4**) holds and (**A5**) is realized. This completes the proof. $\square$

Another result shows the $AN$ of the $QMLE$.

**Corollary 4.6.** *We have*

$$\sqrt{ns}\left(\underline{\hat{\theta}}_{ns} - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, \Omega\right),$$

*as* $n \to \infty$ *where* $\Omega := \left(\tau_2 - 1\right)\left(\sum\limits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\frac{1}{\sigma_v^4\left(\underline{\theta}_0\right)} \frac{\partial \sigma_v^2}{\partial \underline{\theta}}\left(\underline{\theta}_0\right) \frac{\partial \sigma_v^2}{\partial \underline{\theta}'}\left(\underline{\theta}_0\right)\right\}\right)^{-1}$.

*Proof.* Assumptions (**A6-A7**) are immediate. The proofs for (**A8**) and (**A9**) are provided in Hamadeh and Zakoïan [12] (Theorem 2.1) and Berkes et al. [3] (Lemma 5.7). $\square$

**Example 4.4.** ***The*** $QMLE$ ***in the*** $PTGARCH$ ***model***

The $PTGARCH$ model is obtained by setting in model (1.2)

$h\left(\sigma_{sn+v}^{2}\right)=\sigma_{sn+v}, f_{v}\left(e_{sn+v-j}\right)=\alpha_{j}\left(v\right)+\beta_{j}\left(v\right)e_{sn+v-j}^{-}+\gamma_{j}\left(v\right)e_{sn+v-j}^{+}, g_{v}\left(e_{sn+v}\right)=0,$

for all $v\in\mathbb{P}$ with $x^{+}=x\vee 0$ and $x^{-}=(-x)\vee 0$.

Let $\underline{\theta}'\left(v\right):=\left(\beta_{0}\left(v\right),\beta_{1}\left(v\right),...,\beta_{p}\left(v\right),\alpha_{1}\left(v\right),...,\alpha_{p}\left(v\right),\gamma_{1}\left(v\right),...,\gamma_{p}\left(v\right)\right)$. The next corollary presents the $SC$ of the $QMLE$.

**Corollary 4.7.** *If for all* $v\in\mathbb{P}$, $\sigma_{v}^{2}\left(\underline{\theta}\right)\overset{a.s.}{=}\sigma_{v}^{2}\left(\underline{\theta}_{0}\right)$, *then* $\underline{\hat{\theta}}_{ns}\overset{a.s.}{\longrightarrow}\underline{\theta}_{0}$ *as* $n\longrightarrow\infty$.

*Proof.* It suffices to prove the validity of Assumptions **A1**-**A5**. Assumptions (**A1-A2)** are immediate. Note that $\sigma_{v}^{2}\left(\underline{\theta}\right)=\sigma_{v}^{2}\left(\underline{\theta}_{0}\right)$ $a.s.$ for all $v\in\mathbb{P}$. Now, just prove that $\underline{\theta}=\underline{\theta}_{0}$, let the polynomials $\mathcal{A}_{v}\left(z\right)=\sum\limits_{i=1}^{p}\beta_{i}\left(v\right)z^{i}$, $\mathcal{A}_{v}^{+}\left(z\right)=\sum\limits_{i=1}^{p}\gamma_{i}\left(v\right)z^{i}$ and $\mathcal{B}_{v}\left(z\right)=1-\sum\limits_{j=1}^{p}\alpha_{j}\left(v\right)z^{j}$ have no common root with $\mathcal{A}_{v}\left(1\right)+\mathcal{A}_{v}^{+}\left(1\right)\neq 0$ and $\alpha_{p}\left(v\right)+\beta_{p}\left(v\right)+\gamma_{p}\left(v\right)\neq 0$ for all $v\in\mathbb{P}$. By convention, if $p=0$, $\mathcal{A}_{v}\left(z\right)=\mathcal{A}_{v}^{+}\left(z\right)=0$ and $\mathcal{B}_{v}\left(z\right)=1$ for all $v\in\mathbb{P}$. It follows from the proof of Hamadeh and Zakoïan [12], Theorem 2.1, that $\underline{\theta}=\underline{\theta}_{0}$, thus satisfying **A3**. Since $f\left(e_{n}\right)\geq 0$ and $g\left(e_{n}\right)\geq 0$, which implies that (**A4)** holds and (**A5)** is realized. This completes the proof. □

Another result shows the $AN$ of the $QMLE$.

**Corollary 4.8.** *We have*

$$\sqrt{ns}\left(\underline{\hat{\theta}}_{ns}-\underline{\theta}_{0}\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Omega\right),$$

*as* $n\rightarrow\infty$ *with* $\Omega:=4\left(\tau_{2}-1\right)\left(\sum\limits_{v=1}^{s}E_{\underline{\theta}_{0}}\left\{\frac{1}{\sigma_{v}^{2}\left(\underline{\theta}_{0}\right)}\frac{\partial\sigma_{v}}{\partial\underline{\theta}}\left(\underline{\theta}_{0}\right)\frac{\partial\sigma_{v}}{\partial\underline{\theta}'}\left(\underline{\theta}_{0}\right)\right\}\right)^{-1}$.

*Proof.* Assumptions (**A6-A7)** are immediate. The proofs for (**A8)** and (**A9)** are provided in Hamadeh and Zakoïan [12] (Theorem 2.1) and Berkes et al. [3] (Lemma 5.7). □

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