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ON MULTI-HYPERRINGS

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Abstract. In this paper, we introduce multi-hyperrings and obtain several related results. Also, we study the concept of sub-multi-hyperring and different operations on multi-hyperrings such that intersection, union, direct product, and homomorphism, and investigate their main properties.

1. Introduction

A multiset (a bag or mset) is an extension of the Cantorian set, where the repetition of elements matters. Yager [15] in 1986 studied the bag structures. Hyperstructures are extensions of classical (algebraic) structures. Marty [7] defined hypergroup as a generalization of a group. This "theory" was developed with the contributions of various authors. For instance, Krasner [6] in 1983, introduced the notion of hyperrings and hyperfields.

In this paper, we study a new algebraic structure obtained by associating a multiset with a Krasner hyperring calling it multi-hyperring. Till now, the combination between multisets and hyper compositional structures has primarily been explored within the context of hypergroups, establishing the concept of a (fuzzy) multi-hypergroup. Consequently, utilizing the same terminologies, this association may arise in future researches. Hence, we extend the previous studies to encompass the combination of multisets and Krasner hyperrings. This theory holds for its applicability in both mathematics and computer sciences. The present paper has the following structure: First, we introduce definitions related to multiset and Krasner hyperring. Then we establish the concept of multi-hyperrings and study some of

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their properties. The last section, focuses on analyzing various operations on multi-hyperrings. In particular, we show that the intersection (direct product, image) of two multi-hyperrings is a multi-hyperring. Moreover, multi-hyperrings drawn from non-commutative Krasner hyperrings could be commutative. Also, we analyze some homomorphic properties of multi-hyperrings.

2. Preliminaries

Definition 2.1. [5] We display a multiset (mset) M, over the set X, with a function $C_M : X \longrightarrow \mathbb{N}$, where $\mathbb{N}$ represents the set of all non-negative integers. For each $x \in X$, $C_M(x)$ is the characteristic value of x in A and indicates the number of occurrences of the element x in A. We denote the set of all multisets over X by $MS(X)$. A multiset A is a set if $C_M(x) = 0$ or 1 for all $x \in X$. Let M and N be two msets over a set X. If $C_M(x) \leq C_N(x)$, for all $x \in X$, then M is a sub-multiset (submset) of N. $\emptyset$ is a sub-multiset of any multiset. We define the intersection of two multisets M and N with $C_{M\cap N}(x) = C_M(x) \wedge C_N(x)$ for all $x \in X$, ($\wedge$ =minimum). Similarly, we can define the sum $M + N$ and the union $M \cup N$.

Example 2.1. [11] *Assume that we have several objects non-distinguishable except for their labels a, b, or c. For example, we have two balls with the label a and one ball with b, three with c, but no ball with the label d. Furthermore, we refrain from attaching extra labels to distinguish the two elements. Thus a natural representation of the situation is that we have a collection* $\{2/a, 1/b, 3/c\}$. *In this case, we say that there are three occurrences of c, two occurrences of a, and so on.*

For a set $\emptyset \neq X$ and the family of all non-empty subsets of X ($P^*(X)$), we define a binary hyperoperation on X with $+ : X \times X \longrightarrow P^*(X)$, and call $(X, +)$ a hypergroupoid. Moreover, if $(X, +)$ is associative, we call it semihypergroup, and if $X = X + a = a + X, \forall a \in X$, we call it quasihypergroup. If $(X, +)$ is both a semi-hypergroup and a quasi-hypergroup, we call it hypergroup. As it is defined in [8], A commutative hypergroup $(N, +)$ is called canonical if: *(1)* there exists an element 0 in N such that for each x in N there exists a unique element x' in N, denoted by $-x$, and $0 \in x + x'$; *(2)* $z \in x + y$ implies $y \in z - x$, for each $x, y, z \in N$. As it is proved in [8], $x + 0 = x$, for all x in N.

Definition 2.2. [6] A Krasner hyperring is a triple $(R, +, .)$ which satisfies the following axioms:

R_1 : $(R, +)$ is a canonical hypergroup with 0 as its neutral element;

R_2 : $(R \setminus \{0\}, .)$ is a semigroup, and $a.0 = 0.a = 0$, for all $a \in R$;

R_3 : The operation . distributes over the hyperoperation +, that is, $a.(b + c) = a.b + a.c$ and $(b + c).a = b.a + c.a$, for all $a, b, c \in R$.

If $(R, .)$ is a unitary commutative semigroup, the Krasner hyperring $(R, +, .)$ is unitary commutative. We say that $\emptyset \neq A \subseteq R$ is a sub-hyperring of R, if $b + d \subseteq A$, $-b \in A$ and $b.d \in A$, $\forall b, d \in A$. Let S be a Krasner hyperring. A map $h : R \longrightarrow S$ is a homomorphism, if $h(b + d) = h(b) + h(d)$, $h(b.d) = h(b).h(d)$ for all $b, d \in R$ and $h(0) = 0$. If $(R \setminus \{0\}, .)$ is a commutative group, then the triple $(R, +, .)$ is called hyperfield.

Example 2.2. [6, 9] *Let F be a field and G a sub-group of F^*. Then F/G forms a "hyperfield" if the multiplication and the addition are defined as follows: $xG.yG = xyG$, $xG + yG = \{(xp + yq)G : p, q \in G\}$, for all $xG, yG \in F/G$. If F is a ring and G is a normal subgroup of F^*, then F/G becomes a hyperring.*

All throughout this paper, we consider $X = (X, +, .)$ as a Krasner hyperring with the additive identity 0.

3. Properties of Multi-Hyperrings

Definition 3.1. Let A be a mset drawn from X. Then A is said to be a (Krasner) multi-hyperring if for all a, d in X,

(1) $\bigwedge_{b \in a+d} C_A(b) \geq C_A(a) \wedge C_A(d)$ and $C_A(a.d) \geq C_A(a) \wedge C_A(d)$;

(2) $C_A(-a) \geq C_A(a)$.

we denote the set of all multi-hyperrings over X by *MHR(X)*.

Example 3.1. *Let $X = \{0, 1, 2, 3\}$ be a set with,*

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{0, 1}	{3}	{2,3}
2	{2}	{3}	{0}	{1}
3	{3}	{2,3}	{1}	{0, 1}

and $a.d = \begin{cases} 2 & if\ a, d \in \{2, 3\} \\ 0 & otherwise. \end{cases}$

Then $(X, +, .)$ *is a Krasner hyperring* [14]. *Clearly,* $A = \{4/0, 2/1, 3/2, 2/3\}$ *is a multi-hyperring over X.*

Example 3.2. *Consider* $R = \{\bar{r}G \mid \bar{r} \in \mathbb{Z}_{12}\} = \{\bar{\bar{r}} \mid \bar{r} \in \mathbb{Z}_{12}\}$, *where* $G = \{\bar{1}, \bar{5}, \bar{7}, \bar{11}\}$ *is multiplicative sub-group of units of* $\mathbb{Z}_{12}$. *Now define on R, the hyperoperation* $\oplus$ *and multiplication . by* $\bar{\bar{r}} \oplus \bar{\bar{s}} = \{\bar{\bar{t}} : (\bar{\bar{r}} + \bar{\bar{s}}) \cap \bar{\bar{t}} \neq \emptyset\}$ *and* $\bar{\bar{r}}.\bar{\bar{s}} = \bar{\bar{x.y}}$. *Then,* $(R, \oplus, .)$ *is a Krasner hyperring* [1] *and* $A = \{5/\bar{\bar{0}}, 2/\bar{\bar{1}}, 3/\bar{\bar{2}}, 2/\bar{\bar{3}}, 4/\bar{\bar{4}}, 3/\bar{\bar{6}}\}$ *is a multi-hyperring over R.*

Example 3.3. *Let m be a fixed element in X. Consider multiset A with* $C_A(a) = C_A(m)$, $\forall a \in X$. *A is a multi-hyperring over X, and is called the constant multi-hyperring over X. For any Krasner hyperring X with* $\mid X \mid = n$, *we can define at least "n" multi-hyperrings over X, based on what happens in this example.*

Proposition 3.1. *For all multi-hyperring A over X and* $a \in X$,

(1) $C_A(0) \geq C_A(a)$;
(2) $\bigwedge_{d \in n.a} C_A(d) \geq C_A(a), \ \forall n \in \mathbb{N}$;
(3) $C_A(-a) = C_A(a)$.

Proof. Let $a \in X$.

(1) Since A is a multi-hyperring over X, then

$$C_A(0) \geq \wedge_{d \in -a+a} C_A(d) \geq C_A(-a) \wedge C_A(a) \geq C_A(a).$$

(2) We have

$$\begin{aligned} \wedge_{d \in n.a} C_A(d) &= \wedge_{d \in a+(n-1)a} C_A(d) \geq C_A(a) \wedge (\wedge_{d \in (n-1)a} C_A(d)) \\ &\geq C_A(a) \wedge C_A(a) \wedge (\wedge_{d \in (n-2)a} C_A(d)) \geq ... \\ &\geq C_A(a) \wedge C_A(a) \wedge ... \wedge C_A(a) = C_A(a). \end{aligned}$$

(3) It is sufficient to prove that,

$$C_A(a) \geq \wedge_{d\in 0+a} C_A(d) = \wedge_{d\in 0+(-(-a))} C_A(d) \geq C_A(0) \wedge C_A(-(-a))$$
$$= C_A(-(-a)) \geq C_A(-a).$$

□

Proposition 3.2. *Let A be a multi-hyperring over X. $\bigwedge_{b\in a-d} C_A(b) = C_A(0)$ implies $C_A(a) = C_A(d)$, $\forall a, d \in X$.*

Proof. Let $a, d \in X$. Since A is a multi-hyperring over X,

$$C_A(a) \geq \wedge_{b\in(a-d)+d} C_A(b) \geq (\wedge_{b\in a-d} C_A(b)) \wedge C_A(d) = C_A(0) \wedge C_A(d) = C_A(d).$$

Also

$$C_A(d) = C_A(-d) \geq \wedge_{b\in -a+(a-d)} C_A(b) \geq C_A(-a) \wedge (\wedge_{b\in a-d} C_A(b))$$
$$\geq C_A(a) \wedge C_A(0) = C_A(a).$$

□

Theorem 3.1. *For $A \in MS(X)$, the following assertions are equivalent:*

(1) $\wedge_{b\in a+d} C_A(b) \geq C_A(a) \wedge C_A(d)$ *and* $C_A(-a) \geq C_A(a)$*;*

(2) $\wedge_{b\in a-d} C_A(b) \geq C_A(a) \wedge C_A(d)$*,*

for all $a, d \in X$.

Proof. Let $a, d \in X$. If the condition (1) holds, then

$$\wedge_{b\in a-d} C_A(b) = \wedge_{b\in a+(-d)} C_A(b) \geq C_A(a) \wedge C_A(-d) \geq C_A(a) \wedge C_A(d).$$

Now, if the condition (2) holds, then we have

$$C_A(0) \geq \wedge_{b\in a-a} C_A(b) \geq C_A(a) \wedge C_A(a) = C_A(a).$$

Thus $C_A(-a) \geq \wedge_{b\in 0-a} C(b) \geq C_A(0) \wedge C_A(a) = C_A(a)$. Hence

$$\wedge_{b\in a+d} C_A(b) = \wedge_{b\in a-(-d)} C_A(b) \geq C_A(a) \wedge C_A(-d) \geq C_A(a) \wedge C_A(d).$$

□

Proposition 3.3. *Let $A \in MHR(X)$. Then $\forall a_1, ..., a_r \in X$, $r \in \mathbb{N}$ and $r \geq 2$;*

(1) $\bigwedge_{b\in a_1-...-a_r} C_A(b) \geq C_A(a_1) \wedge ... \wedge C_A(a_r)$;

(2) $C_A(a_1.....a_r) \geq C_A(a_1) \wedge ... \wedge C_A(a_r)$.

Proof. 1. By induction on the value of n, the assertion is true for $n = 2$. Assume that

$$\wedge_{z\in x_1-x_2-...-x_n} C_A(z) \geq C_A(x_1) \wedge C_A(x_2) \wedge ... \wedge C_A(x_n),$$

and let $z' \in x_1 - x_2 - ... - x_n - x_{n+1}$. Then there exists $x \in x_1 - x_2 - ... - x_n$ such that $z' \in x - x_{n+1}$. Since A is a multi-hyperring over X, then

$$\wedge_{z'\in x-x_{n+1}} C_A(z') \geq C_A(x) \wedge C_A(x_{n+1}).$$

Using our assumption, it is implied that our statement is true for $n + 1$.
2. The proof is similar to *1.* □

Proposition 3.4. *If A is a multi-hyperring over X, then $-A$ is a multi-hyperring over X, where $C_{-A}(x) = C_A(-x)$, $\forall x \in X$.*

Definition 3.2. Let $a \in X$. For $A, B \in MHR(X)$, we define $A \odot B$ and $A \circ B$ by $C_{A\odot B}(a) = \vee\{C_A(b) \wedge C_B(d) : b, d \in X, a \in b + d\}$, and $C_{A\circ B}(a) = \vee\{C_A(b) \wedge C_B(d) : b, d \in X, a = b.d\}$, respectively. If a cannot be expressed as above, then $C_{A\circ B}(a) = 0$.

Example 3.4. *Let $(X, +, .)$ be that given in Example 3.1. Consider two multi-hyperrings $A = \{3/0, 1/1, 2/2, 1/3\}$ and $B = \{4/0, 2/1, 3/2, 2/3\}$ over X. It is clear that $A \odot B = \{3/0, 2/1, 3/2, 2/3\}$ and $A \circ B = \{3/0, 2/2\}$.*

Proposition 3.5. *For $A \subseteq B \in MHR(X)$, then for any multi-hyperring K of X, $K \odot A \subseteq K \odot B$.*

Proof. We have, for $a \in X$

$$\begin{aligned} C_{K\odot B}(a) &= \vee\{C_K(b) \wedge C_B(d) : b, d \in X, a \in b + d\} \\ &\geq \vee\{C_K(b) \wedge C_A(d) : b, d \in X, a \in b + d\} = C_{K\odot A}. \end{aligned}$$

□

Proposition 3.6. *For $A \in MHR(X)$,*

(1) *$A \odot A = A$;*

(2) $-A \subseteq A$ *and* $A \subseteq -A$ *and* $-A = A$.

Proof. Let $a \in X$.

(1) Since A is a multi-hyperring over X, then $C_A(a) \geq C_A(b) \wedge C_A(d)$, $\forall a \in b + d$. Hence $C_A(a) \geq \vee\{C_A(b) \wedge C_A(d) : b, d \in X, a \in b + d\} = C_{A \odot A}(x)$. Again

$$C_{A \odot A}(a) = \vee\{C_A(b) \wedge C_A(d) : b, d \in X, a \in b + d\} \geq C_A(a) \wedge C_A(0) = C_A(a).$$

(2) Since $C_{-A}(a) = C_A(-a) = C_A(a)$, the given assertion is true. □

Theorem 3.2. *Let A and B be two multi-hyperrings over X.*

(1) *If $C_A(0) = C_B(0)$, then A and B are submsets of $A \odot B$;*
(2) $-(A \circ B) = A \circ B$.

Proof. (1) Let $a \in X$. Since $C_A(0) = C_B(0)$, then

$$C_{A \odot B}(a) = \vee\{C_A(b) \wedge C_B(d) : b, d \in X, a \in b + d\} \geq C_A(0) \wedge C_B(a) = C_B(a).$$

Similarly, $A \subseteq A \odot B$.

(2) Let $x \in X$. Then by Proposition 3.1*(3)*, we have

$$C_{-(A \circ B)}(a) = C_{A \circ B}(-a) = \vee\{C_A(b) \wedge C_B(d) : b, d \in X, -a = b.d\}$$
$$= \vee\{C_A(-b) \wedge C_B(d) : b, d \in X, a = (-b).d\} = C_{A \circ B}(a).$$

□

Definition 3.3. Let X be a Krasner hyperring with unity 1. If $C_A(1) > 0$, then the multi-hyperring A over X is said to be a unitary multi-hyperring. Also, if for all $a, d \in X$, $C_A(a.d) = C_A(d.a)$, we say that A is commutative.

Example 3.5. *Let $X = \{0, 1, 2\}$, and*

+	0	1	2
0	{0}	{1}	{2}
1	{1}	{1}	{0,1,2}
2	{2}	{0,1,2}	{2}

and

.	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

Then, $(X, +, .)$ is a Krasner hyperring [2]. Also, X is non-cmmutative, and $A = \{4/0, 1/1, 1/2\}$ is a commutative multi-hyperring.

Theorem 3.3. *Let $A \in MS(X)$.*

(1) *If $A_n = \{a \in X : C_A(a) \geq n\}$ is a sub-hyperring of X, for all $n \in \mathbb{N}$ and define $C_A(a) = \sum_{n\in\mathbb{N}} \chi_{A_n}(a)$, then A is a multi-hyperring over X;*

(2) *If $A \in MHR(X)$, then for all $n \in \mathbb{N}$, A_n is a sub-hyperring of X.*

Proof. Let $A \in MS(X)$.

(1) Let $a, d \in X$ and $C_A(a) = r, C_A(d) = s$. Then $a \in A_r$ and $d \in A_s$, so that $a \notin A_{r+n}$ and $d \notin A_{s+n}$, $\forall n \in \mathbb{N}$. Let $r \wedge s = r$ and so $y \in A_r$. Since $b \in A_r$, for all $b \in a + d$, $\bigwedge_{b\in a+d} C_A(b) \geq r = r \wedge s = C_A(a) \wedge C_A(d)$, and $C_A(a.d) \geq r = r \wedge s = C_A(a) \wedge C_A(d)$, and $C_A(-a) \geq r = C_A(a)$. Thus A is a multi-hyperring over X.

(2) Let $a, d \in A_n$. Since A is a multi-hyperring over X, then $\bigwedge_{b\in a+d} C_A(b) \geq C_A(a) \wedge C_A(d) \geq n$. Thus $a+d \subseteq A_n$; because for all $b \in a+d$, we get $C_A(b) \geq n$ and so $b \in A_n$. Also, we have $C_A(a.d) \geq C_A(a) \wedge C_A(d) \geq n$. Hence $a.d \in A_n$. Moreover, $C_A(-a) \geq C_A(a) \geq n$. Thus $-a \in A_n$. Therefore, $\forall n \in \mathbb{N}$, A_n is a sub-hyperrings of X.

□

Example 3.6. *Let $X = \{0, 1\}$. Consider hyperoperation "+" as:*

$$0 + 1 = 1 + 0 = \{1\},\ 1 + 1 = \{0, 1\} \text{ and } 0 + 0 = \{0\}.$$

and operation "." the usual multiplication. Then (X, +, .) is a Krasner hyperring [13]. Consider the multi-hyperring $A = \{2/0, 1/1\}$ over X. Then $A_1 = \{0, 1\}$, $A_2 = \{0\}$, and $A_n = \emptyset$, $n \geq 3$. Moreover, A_1, A_2 and A_n, $n \geq 3$ are sub-hyperrings of X.

Proposition 3.7. *Let $A, B \in MHR(X)$. Then $-(A \circ B)_n = (A \circ B)_n$, $\forall n \in \mathbb{N}$.*

Proof. Let $a \in X$. Then $a \in -(A \circ B)_n$ iff $C_{-(A\circ B)}(a) \geq n$ iff $C_{A\circ B}(-a) \geq n$ iff $\vee\{C_A(b) \wedge C_B(d) : b, d \in X, -a = b.d\} \geq n$ iff $\vee\{C_A(-b) \wedge C_B(d) : b, d \in X, a = (-b).d\} \geq n$ iff $C_{A\circ B}(a) \geq n$ iff $a \in (A \circ B)_n$. Thus the result holds. □

Proposition 3.8. *Let $A \in MHR(X)$. Consider*

$$A_* = \{a \in X : C_A(a) > 0\} \text{ and } A^* = \{a \in X : C_A(a) = C_A(0)\}.$$

Then A^* and A_* are sub-hyperrings of X.

Proof. It is similar to Theorem 3.3(1). □

Proposition 3.9. *Let $A \in MHR(X)$. Then $(-A)_* = A_*$ and $(-A)^* = A^*$.*

Proof. For all $a \in X$, we have $a \in (-A)_*$ iff $C_{-A}(a) > 0$ iff $C_A(-a) > 0$ iff $C_A(a) > 0$ iff $a \in A_*$. Therefore $(-A)_* = A_*$. Similarly, we can prove another result. □

Definition 3.4. Let $A \in MHR(X)$. The nonempty submset B of A is said to be a sub-multi-hyperring of A, if $B \in MHR(X)$. The sub-multi-hyperring B of A is proper, if $A \neq B$. Moreover, B is said to be complete, if $B_* = A_*$.

Example 3.7. *Let X be the Krasner hyperring in Example 3.1. Consider $A = \{5/0, 3/1, 4/2, 3/3\}$ and $B = \{3/0, 1/1, 2/2, 1/3\}$. Then B is a proper sub-multi-hyperring of A. Moreover, we have $B_* = X = A_*$ and so B is complete.*

4. Operations on Multi-Hyperrings

Proposition 4.1. *Let $A, B \in MHR(X)$. The intersection of A and B is a multi-hyperring over X.*

Proof. It is straightforward. □

Example 4.1. *Consider $X = \{0, 1, 2, 3\}$, and*

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{0, 2}	{1, 3}	{2}
2	{2}	{1, 3}	{0, 2}	{1}
3	{3}	{2}	{1}	{0}

and

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

Then $(X, +, .)$ is a hyperring [2]. Two multi-hyperrings $A = \{3/0, 1/2\}$ and $B = \{3/0, 1/3\}$, and $A \cap B = \{3/0\}$ are multi-hyperrings over X, but since

$$\bigwedge_{z \in 3+2} C_{A \cup B}(z) = 0 \ngeq 1 = C_{A \cup B}(3) \wedge C_{A \cup B}(2),$$

then $A \cup B = \{3/0, 1/2, 1/3\}$ is not a multi-hyperring over X.

Theorem 4.1. *Let $A, B \in MHR(X)$.*

(1) *If* $C_A(0) = C_B(0)$, $A^* \cap B^* = (A \cap B)^*$, *;*

(2) $A_n \cap B_n = (A \cap B)_n$, *for all* $n \in \mathbb{N}$.

Proof. (1) Since $A, B \in MHR(X)$, their intersection is a multi-hyperring over X. Additionally, A^* and B^* are sub-hyperrings of X, this implies that their intersection is a sub-hyperring of X (See Propositions 3.8 and 4.1). Therefore $A^* \cap B^*$, and $(A \cap B)^*$ are well-defined. If $a \in A^* \cap B^*$, then $a \in A^*$, and $a \in B^*$. Thus $C_A(a) \wedge C_B(a) = C_A(0) \wedge C_B(0)$ and $C_{A\cap B}(a) = C_{A\cap B}(0)$. Therefore, $a \in (A \cap B)^*$.

Now $a \in (A \cap B)^*$, implies $C_A(a) \geq C_{A\cap B}(a) = C_{A\cap B}(0) = C_A(0) \wedge C_B(0) = C_A(0)$. So, by Proposition 3.1*(1)*, we get $C_A(a) = C_A(0)$, that is $a \in A^*$. Similarly, we can prove that $a \in B^*$, so $a \in A^* \cap B^*$, which completes the proof.

(2) It is straightforward. □

Proposition 4.2. *Let A and B be multi-hyperrings over X and $C_A(0) = C_B(0)$. Then $A \cup B$ and $A \cap B$ are submsets of $A \odot B$.*

Proof. A and B are two multi-hyperrings over X, so $\forall a \in X$

$$C_{A\odot B}(a) = \vee\{C_A(b) \wedge C_B(d) : b, d \in X, a \in b + d\} \geq C_A(a) \wedge C_B(0) = C_A(a).$$

In the same way, we have $C_{A\odot B}(a) \geq C_B(a)$. So $C_{A\odot B}(a) \geq C_A(a) \vee C_B(a) = C_{A\cup B}(a)$. Similarly, we can prove this result for the intersection. □

Theorem 4.2. *Let $\{A_i : i = 1, 2, ...\}$ be an arbitrary family of multi-hyperrings over X. Then*

(1) $\cap A_i$ *is a multi-hyperring of X, where $C_{\cap_i A_i}(a) = \wedge_i C_{A_i}(a)$, $\forall a \in X$;*

(2) $\cup_i A_i$, *is a multi-hyperring over X; if for every chain of multi-hyperrings $A_1 \subseteq A_2 \subseteq ...$ (resp. $A_1 \supseteq A_2 \supseteq ...$), there exists $n \in \mathbb{N}$ such that $A_n = A_m$, $\forall m \geq n$.*

Proof. Let $a, b \in X$ and A_i, $i = 1, 2, ...$, be the multi-hyperrings over the same Krasner hyperring X.

(1) By Definition 3.1, we get

$$\wedge_{b\in a+d} C_{\cap_i A_i}(b) = \wedge_{b\in a+d}(\wedge_i C_{A_i}(b)) = \wedge_i(\wedge_{b\in a+d} C_{A_i}(b)) \geq \wedge_i(C_{A_i}(a) \wedge C_{A_i}(d))$$
$$= (\wedge_i C_{A_i}(a)) \wedge (\wedge_i C_{A_i}(d)) = C_{\cap_i A_i}(a) \wedge C_{\cap_i A_i}(d).$$

Moreover,

$$C_{\cap_i A_i}(a.d) = \wedge_i C_{A_i}(a.d) \geq \wedge_i (C_{A_i}(a) \wedge C_{A_i}(d))$$
$$= (\wedge_i C_{A_i}(a)) \wedge (\wedge_i C_{A_i}(d)) = C_{\cap_i A_i}(a) \wedge C_{\cap_i A_i}(d).$$

Also, $C_{\cap_i A_i}(-a) = \wedge_i C_{A_i}(-a) \geq \wedge_i C_{A_i}(a) = C_{\cap_i A_i}(a)$.

(2) From hypothesis $\exists i_n = 1, 2, ...$ s.t. $C_{A_{i_n}}(a) = \vee_{i \in I} C_{A_i}(a) = C_{\cup_i A_i}(a)$. Therefore,

$$\wedge_{b \in a+d} C_{\cup_i A_i}(b) = \wedge_{b \in a+d} C_{A_{i_n}}(b) \geq C_{A_{i_n}}(a) \wedge C_{A_{i_n}}(d) = C_{\cup_i A_i}(a) \wedge C_{\cup_i A_i}(d).$$

Furthermore, $C_{\cup_i A_i}(a.d) = C_{A_{i_n}}(a.d) \geq C_{A_{i_n}}(a) \wedge C_{A_{i_n}}(d) = C_{\cup_i A_i}(a) \wedge C_{\cup_i A_i}(d)$. Additionally, $C_{\cup_i A_i}(-a) = C_{A_{i_n}}(-a) \geq C_{A_{i_n}}(a) = C_{\cup_i A_i}(a)$, and this completes the proof. If $A_1 \supseteq A_2 \supseteq ...$, then $C_{A_1}(a) = \bigvee_i C_{A_i}(a) = C_{\cup_i A_i}(k)$ and, by the same procedure, we can arrive at the result. □

Definition 4.1. For $A \in MS(X)$, we define $< A >= \bigcap \{B : B \in MHR(X), A \subseteq B\}$, which implies $< A >$ is the smallest multi-hyperring containing A. Also, we say that A is generated by two multi-hyperrings if A is the smallest multi-hyperring over X containing the same two multi-hyperrings, and so on.

Theorem 4.3. *Let A and B be multi-hyperrings over X with $C_A(0) = C_B(0)$, and $A \odot B \in MHR(X)$. Then, $A \odot B$ is generated by A and B.*

Proof. Let $a \in X$. Since $C_A(0) = C_B(0)$, then

$$C_{A \odot B}(a) = \vee \{C_A(b) \wedge C_B(d) : b, d \in X, a \in b + d\} \geq C_A(a) \wedge C_B(0) = C_A(a).$$

Thus $A \subseteq A \odot B$. Similarly, $B \subseteq A \odot B$. Now, if K is a multi-hyperring over X s.t. $A, B \subseteq K$, then

$$C_{K \odot K}(a) = \vee \{C_K(b) \wedge C_K(d) : b, d \in X, a \in b + d\}$$
$$\geq \vee \{C_A(b) \wedge C_B(d) : b, d \in X, a \in b + d\} = C_{A \odot B}(a).$$

Therefore, we have

$$A \odot B \subseteq K \odot K. \tag{4.1}$$

Now, since K is a multi-hyperring over X, then for all $a \in b+d$, $C_K(a) \geq \wedge_{a \in b+d} C_K(a) \geq C_K(b) \wedge C_K(d)$, So $C_K(a) \geq \vee \{C_K(b) \wedge C_K(d) : b, d \in X, a \in b + d\} = C_{K \odot K}(a)$. Consequently, $K \supseteq K \odot K$, so (4.1) completes the proof. □

Recall that if X and Y be Krasner hyperrings, then $X \times Y$ is a Krasner hyperring such that $+$ and $.$ on it are defined component-wise.

Theorem 4.4. *Let X and Y be Krasner hyperrings, and $A \in MS(X)$, $B \in MS(Y)$, such that $C_A(0) = C_B(0')$, where $0 \in X$ and $0' \in Y$ are the identity elements. Consider $C_{A\times B}(a,d) = C_A(a) \wedge C_B(d)$, for all $a \in X$, $d \in Y$. Then A and B are (commutative) multi- hyperrings over X and Y, respectively, iff $A \times B$ is a (commutative) multi-hyperring over $X \times Y$.*

Proof. For all $(m,t),(k,p)$ in $X \times Y$,

$$\begin{aligned}
\wedge_{(a,d)\in(m,t)+(k,p)} C_{A\times B}(a,d) &= \wedge_{(a,d)\in(m+k,t+p)} C_{A\times B}(a,d) \\
&= (\wedge_{a\in m+k} C_A(a)) \wedge (\wedge_{d\in t+p} C_B(d)) \\
&\geq (C_A(m) \wedge C_A(k)) \wedge (C_B(t) \wedge C_B(p)) \\
&= (C_A(m) \wedge C_B(t)) \wedge (C_A(k) \wedge C_B(p)) \\
&= C_{A\times B}(m,t) \wedge C_{A\times B}(k,p).
\end{aligned}$$

Besides,

$$\begin{aligned}
C_{A\times B}((m,t).(k,p)) &= C_{A\times B}(m.k, t.p) = C_A(m.k) \wedge C_B(t.p) \\
&\geq (C_A(m) \wedge C_A(k)) \wedge (C_B(t) \wedge C_B(p)) \\
&= (C_A(m) \wedge C_B(t)) \wedge (C_A(k) \wedge C_B(p)) \\
&= C_{A\times B}(m,t) \wedge C_{A\times B}(k,p).
\end{aligned}$$

Moreover,

$$\begin{aligned}
C_{A\times B}(-(m,t)) &= C_{A\times B}(-m,-t) = C_A(-m) \wedge C_B(-t) \\
&\geq C_A(m) \wedge C_B(t) = C_{A\times B}(m,t).
\end{aligned}$$

Therefore $A \times B \in MHR(X \times Y)$. Now, if A, and B are commutative, then

$$\begin{aligned}
C_{A\times B}((m,t).(k,p)) &= C_{A\times B}(m.k, t.p) = C_A(m.k) \wedge C_B(t.p) \\
&= C_A(k.m) \wedge C_B(p.t) = C_{A\times B}(k.m, p.t) \\
&= C_{A\times B}((k,p).(m,t)).
\end{aligned}$$

Thus $A \times B$ is commutative.

Conversely, for $a \in A$, $d \in B$,

$$\begin{aligned}
\wedge_{b\in a+d} C_A(b) &= \wedge_{b\in a+d}(C_A(b) \wedge C_A(0)) = \wedge_{b\in a+d}(C_A(b) \wedge C_B(0')) \\
&= \bigwedge_{(b,0')\in(a+d,0'+0')} C_{A\times B}(b,0') = \wedge_{(b,0')\in(a,0')+(d,0')} C_{A\times B}(b,0') \\
&\geq C_{A\times B}(a,0') \wedge C_{A\times B}(d,0') \\
&= (C_A(a) \wedge C_B(0')) \wedge (C_A(d) \wedge C_B(0')) \\
&= C_A(a) \wedge C_A(d) \wedge C_B(0') = C_A(a) \wedge C_A(d).
\end{aligned}$$

Also,

$$\begin{aligned}
C_A(a.d) &= C_A(a.d) \wedge C_A(0.0) = C_A(a.d) \wedge C_B(0'.0') = C_{A\times B}(a.d, 0'.0') \\
&= C_{A\times B}((a,0').(d,0')) \geq C_{A\times B}(a,0') \wedge C_{A\times B}(d,0') \\
&\geq (C_A(a) \wedge C_B(0')) \wedge (C_A(d) \wedge C_B(0')) \\
&= C_A(a) \wedge C_A(d) \wedge C_B(0') = C_A(a) \wedge C_A(d).
\end{aligned}$$

Moreover,

$$\begin{aligned}
C_A(-a) &= C_A(-a) \wedge C_A(0) = C_A(-a) \wedge C_B(-0') = C_{A\times B}(-a, -0') \\
&= C_{A\times B}(-(a,0')) \geq C_{A\times B}(a,0') = C_A(a) \wedge C_B(0') = C_A(a).
\end{aligned}$$

Therefore, A is a multi-hyperring over X. Similarly, B is a multi-hyperring over Y. Also, if $A \times B$ is commutative, then

$$\begin{aligned}
C_A(a.d) &= C_A(a.d) \wedge C_A(0.0) = C_A(a.d) \wedge C_B(0'.0') = C_{A\times B}(a.d, 0'.0') \\
&= C_{A\times B}((a,0').(d,0')) = C_{A\times B}((d,0').(a,0')) = C_{A\times B}(d.a, 0'.0') \\
&= C_A(d.a) \wedge C_B(0'.0') = C_A(d.a).
\end{aligned}$$

Therefore A is commutative on X. Similarly, B is commutative on Y. □

Example 4.2. *Let $X = \mathbb{Z}_{12}/H$, $Y = \{0,1,2,3\}$ be Krasner hyperrings with hyper operations and binary operations defined in Example 3.2 and Example 3.1, respectively. Consider multi-hyperring $A = \{4/(\bar{0}H), 1/(\bar{1}H), 2/(\bar{2}H), 1/(\bar{3}H), 3/(\bar{4}H), 2/(\bar{6}H)\}$ over X and $B = \{6/0, 1/1, 5/2, 1/3\}$ over Y. We have $A \times B = \{4/(\bar{0}H, 0), 1/(\bar{0}H, 1),$*

$4/(\bar{0}H,2),1/(\bar{0}H,3),1/(\bar{1}H,0),1/(\bar{1}H,1),1/(\bar{1}H,2),1/(\bar{1}H,3),2/(\bar{2}H,0),1/(2\bar{H},1),$
$2/(\bar{2}H,2),1/(\bar{2}H,3),1/(\bar{3}H,0),1/(\bar{3}H,1),1/(\bar{3}H,2),1/(\bar{3}H,3),3/(\bar{4}H,0),1/(\bar{4}H,1),$
$3/(\bar{4}H,2),1/(\bar{4}H,3),2/(\bar{6}H,0),1/(\bar{6}H,1),2/(\bar{6}H,2),1/(\bar{6}H,3)\}.$

Definition 4.2. Let h be a mapping from Krasner hyperring X to Krasner hyperring Y, $A \in MHR(X)$, and $B \in MHR(Y)$. We define

$$C_{h(A)}(d) = \begin{cases} \vee_{a\in h^{-1}(d)} C_A(a), & h^{-1}(d) \neq \emptyset; \\ 0, & otherwise, \end{cases}$$

and $C_{h^{-1}(B)}(a) = C_B(h(a))$, for all $a \in X$, $d \in Y$. If $h(A) = B$, A is called homomorphic to B ($A \approx B$).

Theorem 4.5. *Let h be a homomorphism from Krasner hyperring X to Krasner hyperring Y.*

(1) *If $A \in MHR(X)$, then $h(A) \in MHR(Y)$;*
(2) *If $B \in MHR(Y)$, then $h^{-1}(B) \in MHR(X)$.*

Proof. (1) Let $a, d \in Y$, and $b \in a + d$. If $h^{-1}(a)$ or $h^{-1}(d)$ is empty, the result holds. Otherwise, $\exists r, s \in X$ such that

$$C_A(r) = \vee_{h(u)=a} C_A(u) = C_{h(A)}(a) \text{ and } C_A(s) = \vee_{h(l)=d} C_A(l) = C_{h(A)}(d).$$

Since h is a homomorphism, then $b \in h(r) + h(s)$ implies $b \in h(r + s)$. Hence, $\exists k \in r + s$ such that $b = h(k)$. Since A is a multi-hyperring over X, then

$$\begin{aligned} C_{h(A)}(b) &= \vee_{h(t)=b} C_A(t) \geq C_A(k) \geq \wedge_{b\in r+s} C_A(b) \\ &\geq C_A(r) \wedge C_A(s) = C_{h(A)}(a) \wedge C_{h(A)}(d). \end{aligned}$$

Moreover,

$$C_{h(A)}(a.d) = \vee_{v\in h^{-1}(a.d)} C_A(v) \geq C_A(r.s) \geq C_A(r) \wedge C_A(s) = C_{h(A)}(a) \wedge C_{h(A)}(d).$$

Also,

$$C_{h(A)}(-a) = \vee_{w\in h^{-1}(-a)} C_A(w) = \vee_{k\in h^{-1}(a)} C_A(-k) \geq \vee_{k\in h^{-1}(a)} C_A(k) = C_{h(A)}(a).$$

(2) Let B be a multi-hyperring over Y. $\forall q,k \in X$,

$$\wedge_{m\in q+k} C_{h^{-1}(B)}(m) = \wedge_{m\in q+k} C_B(h(m)) = \wedge_{s\in h(q)+h(k)} C_B(s)$$
$$\geq C_B(h(q)) \wedge C_B(h(k)) = C_{h^{-1}(B)}(q) \wedge C_{h^{-1}(B)}(k).$$

Moreover,

$$C_{h^{-1}(B)}(q.k) = C_B(h(q.k)) = C_B(h(q).h(k)) \geq C_B(h(q)) \wedge C_B(h(k))$$
$$= C_{h^{-1}(B)}(q) \wedge C_{h^{-1}(B)}(k).$$

Also, $C_{h^{-1}(B)}(-q) = C_B(h(-q)) = C_B(-h(q)) \geq C_B(h(q)) = C_{h^{-1}(B)}(q)$.

□

Proposition 4.3. *Let $\varphi : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ be homomorphisms of Krasner hyperrings and $A \in MHR(X)$, $B \in MHR(Y)$. Then*

(1) *$(h\varphi)(A) = h(\varphi(A))$;*
(2) *$(h\varphi)^{-1}(B) = \varphi^{-1}(h^{-1}(B))$.*

Proof. (1) Let $d \in Z$. If $h^{-1}(d) = \emptyset$, then clearly the result holds. Otherwise,

$$C_{h(\varphi(A))}(d) = \vee_{b\in h^{-1}(d)} C_{\varphi(A)}(b) = \vee_{b\in h^{-1}(d)}(\vee_{a\in\varphi^{-1}(b)} C_A(a))$$
$$= \vee_{a\in(h\varphi)^{-1}(d)} C_A(a) = C_{(h\varphi)(A)}(d).$$

(2) Let $q \in X$. Then

$$C_{(h\varphi)^{-1}(A)}(q) = C_A((h\varphi)(q)) = C_A(h(\varphi(q))) = C_{h^{-1}(A)}(\varphi(q)) = C_{\varphi^{-1}(h^{-1}(A))}(q).$$

□

Theorem 4.6. *Let h be a mapping from Krasner hyperring X to Krasner hyperring Y, $A_i \in MHR(X)$, and $B_j \in MHR(Y)$, $i,j = 1,2,...$ be two arbitrary families of multi-hyperrings over X, and Y. Then*

(1) *$h(-A_i) = -h(A_i)$, and $h^{-1}(-B_j) = -h^{-1}(B_j)$;*
(2) *$h(\cap_i A_i) = \cap_i h(A_i)$, $h(\cup_i A_i) = \cup_i h(A_i)$ and $h^{-1}(\cap_j B_j) = \cap_j h^{-1}(B_j)$, $h^{-1}(\cup_j B_j) = \cup_j h^{-1}(B_j)$;*
(3) *$h(A_n) = (h(A))_n$ and $h^{-1}(B_n) = (h^{-1}(B))_n$, for all $n \in \mathbb{N}$.*

Proof. (1) It is straightforward.

(2) Let $a \in X$, and $d \in Y$. We have

$$C_{h(\cup_i A_i)}(d) = \vee_{a\in h^{-1}(d)} C_{\cup_i A_i}(a) = \vee_{a\in h^{-1}(d)}(\vee_i C_{A_i}(a))$$
$$= \vee_i(\vee_{a\in h^{-1}(d)} C_{A_i}(a)) = \vee_i C_{h(A_i)}(d) = C_{\cup_i h(A_i)}(d).$$

Also, $C_{h^{-1}(\cup_j B_j)}(a) = C_{\cup_j B_j}(h(a)) = \vee_j C_{B_j}(h(a)) = \vee_j C_{h^{-1}(B_j)}(a) = C_{\cup_j h^{-1}(B_j)}(a)$.

Similarly, we can get other results.

(3) Let $n \in \mathbb{N}$. If $a \in h(A_n)$, then $a = h(t)$, for some $t \in A_n$. Therefore, $C_{h(A)}(x) = \vee_{b\in h^{-1}(a)} C_A(b) \geq C_A(t) \geq n$. Thus $a \in (h(A))_n$ and so $h(A_n) \subseteq (h(A))_n$. Now, if $d \in (h(A))_n$, then $C_{h(A)}(d) = \vee_{r\in h^{-1}(d)} C_A(r) \geq n$ and so $C_A(k) \geq n$, for some $k \in h^{-1}(d)$. Therefore, $k \in A_n$ and so $d = h(k) \in h(A_n)$. Thus $(h(A))_n \subseteq h(A_n)$. So equality holds. Now, we have $a \in (h^{-1}(B))_n$ iff $C_{h^{-1}(B)}(a) \geq n$ iff $C_B(h(a)) \geq n$ iff $h(a) \in B_n$ iff $a \in h^{-1}(B_n)$. Thus the result holds. □

Theorem 4.7. *Let $h : X \longrightarrow Z$, and $\rho : Y \longrightarrow W$ be homomorphisms of Krasner hyperrings with $A \in MHR(X)$, $B \in MHR(Y)$, $C \in MHR(Z)$, and $D \in MHR(W)$. Consider $h\times\rho : X\times Y \longrightarrow Z\times W$ by setting $(h\times\rho)(a, d) = (h(a), \rho(d))$, $\forall a \in X, d \in Y$. Then*

1. *$h \times \rho$ is a homomorphism;*
2. *$(h \times \rho)(A \times B)$ is a multi-hyperring over $Z \times W$, where $(h \times \rho)(A \times B) = h(A) \times \rho(B)$;*
3. *$(h\times\rho)^{-1}(C\times D)$ is a multi-hyperring over $X\times Y$, such that $(h\times\rho)^{-1}(C\times D) = h^{-1}(C) \times \rho^{-1}(D)$.*

Proof. (1) Let $(m, t), (k, q) \in X \times Y$.

$$(h \times \rho)((m, t) + (k, q)) = (h \times \rho)(\{(s, v) : (s, v) \in (m + k, t + q)\})$$
$$= \{(h(s), \rho(v)) : s \in m + k, v \in t + q\}$$
$$= (h(m + k), \rho(t + q)) = (h(m) + h(k), \rho(t) + \rho(q))$$
$$= (h(m), \rho(t)) + (h(k), \rho(q)) = (h \times \rho)(m, t) + (h \times \rho)(k, q).$$

Moreover,

$$(h \times \rho)((m,t).(k,q)) = (h \times \rho)(m.k, t.q) = (h(m.k), \rho(t.q)) = (h(m).h(k), \rho(t).\rho(q))$$
$$= (h(m), \rho(t)).(h(k), \rho(q)) = (h \times \rho)(m,t).(h \times \rho)(k,q).$$

Also, we have $(h \times \rho)(0,0) = (h(0), \rho(0)) = (0,0)$.

(2) The first result is obtained by Theorem 4.4 and Theorem 4.5. Let (c,b) be in $Z \times W$. If $(h \times \rho)^{-1}(c,b)$ is empty, we can easily get the result. Otherwise, we have $(h \times \rho)^{-1}(c,b) = (h \times \rho)(c^{-1}, b^{-1}) = (h^{-1}(c), \rho^{-1}(b))$. Therefore

$$\begin{aligned} C_{(h\times\rho)(A\times B)}(c,b) &= \vee_{(a,d)\in(h\times\rho)^{-1}(c,b)} C_{A\times B}(a,d) \\ &= \vee_{(a,d)\in(h^{-1}(c),\rho^{-1}(b))}(C_A(a) \wedge C_B(d)) \\ &= (\vee_{a\in h^{-1}(c)} C_A(a)) \wedge (\vee_{d\in\rho^{-1}(b)} C_B(d)) \\ &= C_{h(A)}(c) \wedge C_{\rho(B)}(b) = C_{h(A)\times\rho(B)}(c,b). \end{aligned}$$

(3) Similarly, by 4.11 and 4.18, we have the first part. Let (a,d) be in $X \times Y$. If $h^{-1}(c)$ or $g^{-1}(w)$ is empty, we can easily get the result. Otherwise, we have

$$C_{(h\times\rho)^{-1}(C\times D)}(a,d) = C_{C\times D}((h \times \rho)(a,d)) = C_{C\times D}(h(a), \rho(d)) = C_C(h(a)) \wedge C_D(\rho(d))$$
$$= C_{h^{-1}(C)}(a) \wedge C_{\rho^{-1}(D)}(d) = C_{h^{-1}(C)\times\rho^{-1}(D)}(a,d).$$

□

5. Conclusion

A hyperring is a system $(R,+,.)$ which satisfies the ring-like axioms. A well-known type of hyperrings is the Krasner hyperring. Krasner hyperrings are essentially rings in which addition is hyperoperation. Also, the theory of multisets is a generalization of the classical sets theory. In this paper, we combined multisets and Krasner hyperrings and introduced the notion of multi-hyperrings. We analyzed the main properties and different operations on multi-hyperrings and obtained several results supported by examples; Finally, we studied some homomorphic properties of multi-hyperrings. Multi-hyperrings are helpful in mathematics and computer sciences; In future we will investigate more properties of multi-hyperrings.

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