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GYROTRANSVERSALS OF ORDER p^3

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Abstract. In this paper, we compute the isomorphism classes of gyrotransversals of order p^3 corresponding to a fixed subgroup of order p in the group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^3}$, where p is an odd prime. This yields a lower bound for the number of right gyrogroups of order p^3 upto isomorphism. In addition, we obtain a lower bound for the non-isomorphic right gyrogroups of order p^3 of nilpotency class 2.

1. Introduction

Let H be a subgroup of a group G and S be a right transversal to H in G with $e \in S$, where e is the identity of the group G. Then, the set S with the induced binary operation $\circ$ defined by $\{x \circ y\} = S \cap Hxy$ becomes a right loop with the identity e. S is also a right transversal to the subgroup $H \cap \langle S \rangle$ in the group $\langle S \rangle$ (see [7]). Using the identification of the members of the set S with the corresponding right cosets of H in G, we get a group homomorphism $\lambda : G \longrightarrow Sym(S)$ defined by $\{\lambda(g)(x)\} = S \cap Hxg$, $g \in G$, $x \in S$. The kernel of λ is $Core_G(H)$, the core of H in G. The group $G_S = \lambda(\langle S \rangle \cap H)$ is called the group torsion of S (see [7, Defination 3.1]). If we identify S with $\lambda(S)$, then $\lambda(\langle S \rangle) = G_S S$. Note that, the group $G_S S$ depends only on S and not on H (see [7]). Also, S is a right transversal to the subgroup G_S in the group $G_S S$ (see [7]). Moreover, S is a group if and only if G_S is trivial. Also, in a group G, if a subgroup H is normal, then the corresponding right transversal S is a group which is isomorphic to the quotient group G/H. Thus for a finite abelian p-group, a gyrotransversal is a group [2].

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The gyrogroup structure is the result of a classic work of A. A. Ungar [13] in the study of Lorentz groups. A gyrogroup is a generalization of a group. In, [3], [13] and [14], Ungar and Foguel studied left gyrogroups and gyrotransversals to a subgroup in a group (see [3, Defination 2.9, p. 31]).

Agore and Militaru [1] observed that all the finite groups of order n can be obtained through the factorization $S_n = S_{n-1}\mathbb{Z}_n$, where S_n is the symmetric group of degree n. Lal and Yadav [9] studied the right gyrogroups, gyrotransversals and their deformations and they showed that there is a unique gyrotransversal to S_{n-1} in S_n. They also proved that right gyrogroups and gyrotransversals are the same in some sense (see [3, Theorem 2.12, p. 33]). Therefore it is reasonable to study the isomorphism classes of gyrotransversals in different groups to get the lower bound of non-isomorphic gyrogroups. The semidirect product of two groups H and K is denoted by $H \ltimes K$, where K is regarded as the normal subgroup in $H \ltimes K$. In this paper, we have used the *Cauchy-Frobenius Formula* to calculate the number of gyrotransversals upto isomorphism in the group $G = \mathbb{Z}_p \ltimes \mathbb{Z}_{p^3}$ to a fixed subgroup H of G of order p, where p is an odd prime. For this, we take the natural action of $Aut_H(G)$ on the set of all the gyrotransversals to the subgroup H in G, where $Aut_H(G) = \{\theta \in Aut(G) \mid \theta(H) = H\}$.

Let S and T be two right transversals to the subgroup H in G such that $\langle S\rangle = G = \langle T\rangle$. Then by [8, Proposition 2.7, p. 652], if $S \simeq T$, then there exists $\theta \in Aut_H(G)$ such that $\theta(S) = T$. Thus, the action of $Aut_H(G)$ on the set of all right transversals isomorphic to S is a transitive action. Since $|H| = p$, if S is a right transversal to H in G, then either $S = \langle S\rangle$ or $\langle S\rangle = G$. Therefore, the number of orbits under the action of $Aut_H(G)$ is equal to the number of isomorphism classes of right loops. Thus we get a lower bound for the number of gyrotransversals of order p^3 upto isomorphism in the group G. As a result, we get a lower bound for the number of non-isomorphic right gyrogroups of order p^3. Also, a lower bound for the number of non-isomorphic right gyrogroups of nilpotency class 2 is obtained. Throughout the paper, $\mathbb{Z}_n$ denotes the cyclic group of order n and $U(p)$ denotes the group of units $(mod\, p)$.

2. Preliminaries

In this section, we give the preliminaries that we will use throughout the paper.

Definition 2.1. [3, Definition 2.3, p. 29] A groupoid $(S, \circ)$ is said to be a right gyrogroup if,

(*i*) there exists an element $e \in S$ such that $x \circ e = x$ for all $x \in S$,

(*ii*) for each element $a \in S$, there exists an element $a' \in S$ such that $a \circ a' = e$,

(*iii*) there exists a map $f : S \times S \longrightarrow Aut(S, \circ)$ such that for any $x, y, z \in S$,

$$(x \circ y) \circ z = f(y, z)(x) \circ (y \circ z),$$

(*iv*) $f(y, y') = I_S$ for all $y \in S$.

By [9, Corollary 5.7, p. 3566], $(S, \circ)$ is a right loop with the identity e and a' is also the left inverse for each $a \in S$.

Definition 2.2. ([3, Defination 2.9, p. 31]) A gyrotransversal is a right transversal S to a subgroup H in a group G if

(i) $e \in S$, where e is the identity of the group G,

(ii) $S^{-1} \subseteq S$,

(iii) $h^{-1}Sh \subseteq S$ for all $h \in H$.

Proposition 2.1. [9, Corollary 5.11, p. 3569] *Let S be a gyrotransversal to a subgroup H in a group G and $g : S \longrightarrow H$ be a map such that $g(e) = e$. Then the transversal $S_g = \{g(x)x \mid x \in S\}$ is a gyrotransversal if and only if*

$$g(x^{-1}) = g(x)^{-1}$$
$$\textit{and} \quad g(h^{-1}xh) = h^{-1}g(x)h$$

for all $x \in S$ and $h \in H$.

The deformation map is defined as the map $g : S \longrightarrow H$ satisfying the conditions in the Proposition 2.1 and S_g is called the deformed gyrotransversal with respect to the fixed gyrotransversal S to the subgroup H in a group G.

Two right transversals are said to be isomorphic to each other if they are isomorphic as their induced right loop structures. If S and T are two isomorphic right transversals to H in G and S is a gyrotransversal, then T is also a gyrotransversal to H in G.

Theorem 2.1. *(Cauchy-Frobenius Formula)*[11, Theorem 3.1.2, p. 75] *Let a group G acts on a set X. Then the average number of points fixed by the elements of G is equal to the number of orbits of G on X that is,*

$$\textit{number of orbits} = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|,$$

where $Fix(g) = \{x \in X \mid g \cdot x = x\}$.

Let $G = H \ltimes K$ be a group, where H is abelian. Let S be a gyrotransversal to the subgroup H in the group G and $g : S \longrightarrow H$ be a deformation map. Then for all $s \in S$ and $h \in H$, we have

$$g(h^{-1}sh) = h^{-1}g(s)h = g(s). \tag{2.1}$$

The map $(h, s) \mapsto h^{-1}sh$ for all $h \in H$ and $s \in S$ defines an action of H on S. Then for any $s \in S \setminus \{e\}$, the orbit of s is given as $s^\star = \{h^{-1}sh \mid h \in H\}$. Using Equation (2.1), in order to define the map g one can easily observe that it is sufficient to find the images of the representatives of the H-orbits on $S \setminus \{e\}$. Let $\{s_1, s_2, \cdots, s_n\}$ be a set of representatives of the H-orbits on $S \setminus \{e\}$. Note that for all $h, h' \in H$, $h^{-1}(h's_i^\star)h = h'(h^{-1}s_i^\star h) = h's_i^\star$. Thus, $h^{-1}sh \in S$ holds trivially for all $h \in H$ and $s \in S$. Therefore, it is sufficient to check that $S^{-1} = S$ for S to be a gyrotransversal to H in G. Now, we have the total number of gyrotransversals to the subgroup H in the group G.

Theorem 2.2. [5] *Let $G = H \ltimes K$ be a group, where H is abelian. Then, the total number of gyrotransversals to the subgroup H in the group G is*

$$= \begin{cases} |H|^{\frac{n}{2}}, & \textit{if } n \textit{ is even} \\ |H|^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd,} \end{cases}$$

where n is the number of H-orbits on $S \setminus \{e\}$.

3. Gyrotransversals in $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^3}$

Let G denotes the group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^3}$ with the presentation

$$G = \langle a, b \mid a^{p^3} = 1 = b^p, bab^{-1} = a^{1+p^2}\rangle,$$

where p is an odd prime. Let $H = \langle b \rangle$ be a subgroup of G of order p. Throughout this section, G and H respectively denote the group and subgroup as discussed above. Then one can easily observe that if S is a gyrotransversal to H in G which is not a group, then $H \simeq G_S$ and $G \simeq G_S S$. To find all the gyrotransversals in G to the subgroup H, we will determine H-orbits in G.

Let $\overline{a^r} = \{a^{pi+r} \mid 0 \leq i \leq p^2 - 1, 1 \leq r \leq p - 1\}$ and X denotes the collection of all the gyrotransversals to the subgroup H in G.

Lemma 3.1. *Let $b^j a^i$ be any element of $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^3}$, where $0 \leqslant i \leqslant p^3 - 1$, $0 \leqslant j \leqslant p-1$. Then*

(i) $a^i b^j = b^j a^{i(1-jp^2)}$,

(ii) $(b^j a^i)^n = b^{nj} a^{ni - \frac{n(n-1)}{2} p^2 ij}$ *for all n.*

Proof. (i) Using $bab^{-1} = a^{1+p^2}$, we get $b^{-1}ab = a^{1-p^2}$. Therefore, $b^{-1}a^i b = a^{i(1-p^2)}$ for all $0 \leq i \leq p^3 - 1$. Now, $b^{-2}a^i b^2 = b^{-1}a^{i(1-p^2)}b = (b^{-1}ab)^{i(1-p^2)} = a^{i(1-p^2)^2}$. Using the similar argument, we get $b^{-3}a^i b^3 = b^{-1}a^{i(1-p^2)^2}b = (b^{-1}ab)^{i(1-p^2)^2} = a^{i(1-p^2)^3}$. Inductively, we get $b^{-j}a^i b^j = a^{i(1-p^2)^j}$ for all $0 \leq j \leq p-1$. Using $a^{p^3} = 1$, we have $b^{-j}a^i b^j = a^{i(1-jp^2)}$. Hence, $a^i b^j = b^j a^{i(1-jp^2)}$.

(ii) Using the part (i), for all $0 \leq i \leq p^3 - 1$ and $0 \leq j \leq p-1$, we get $(b^j a^i)^2 = b^j(a^i b^j)a^i = b^j b^j a^{i(1-jp^2)} a^i = b^{2j} a^{i(2-jp^2)}$. Now $(b^j a^i)^3 = b^j a^i (b^j a^i)^2 = b^j(a^i b^{2j})a^{i(2-jp^2)} = b^j b^{2j} a^{i(1-2jp^2)} a^{i(2-jp^2)} = b^{3j} a^{i(3-3jp^2)}$. Similarly, we get $(b^j a^i)^4 = b^j(a^i b^{3j})a^{i(3-3jp^2)} = b^j b^{3j} a^{i(1-3jp^2)} a^{i(3-3jp^2)} = b^{4j} a^{4i-6ijp^2}$. Inductively, we get $(b^j a^i)^n = b^{nj} a^{ni - \frac{n(n-1)}{2} ijp^2}$ for all n.

□

Theorem 3.1. *The total number of gyrotransversals to the subgroup H in the group G is equal to $p^{\frac{(p-1)(p+2)}{2}}$.*

Proof. The right cosets of the subgroup H in the group G are given by Ha^i $(0 \leq i \leq p^3 - 1)$. Thus, a right transversal of H in G is given by

$$S = \bigcup_{i=1}^{p^3-1} (\{1\} \cup \{b^{j_i} a^i\}),$$

where $0 \leqslant j_i \leqslant p-1$. Now, for $b^j a^{pi+r} \in S$, we have $(b^j a^{pi+r})^{-1} = b^{-j} a^{p(-prj-i)-r}$ $= b^{-j} a^{p(-prj-i-1)+(p-r)}$. Then $S^{-1} = S$ if and only if $b^j a^{pi+r} \in S$ implies $(b^j a^{pi+r})^{-1}$ $\in S$. Therefore, a gyrotransversal S in G is given as

(3.1)

$$S = \{1\} \cup \bigcup_{i=1}^{\frac{p-1}{2}} (b^{j_i}\overline{a^{r_i}} \cup b^{-j_i}\overline{a^{-r_i}} \cup \{b^{j_i}a^{p^2 i}\} \cup \{b^{-j_i}a^{-p^2 i}\}) \cup \bigcup_{l=1}^{\frac{p^2-1}{2}} (\{b^{j_l}a^{pl}\} \cup \{b^{-j_l}a^{-pl}\}),$$

where $0 \leqslant j_i, j_l \leqslant p-1$, $r_i \in \{1, 2, \cdots, p-1\}$ and $gcd(l, p) = 1$. Therefore, the set of representatives of H-orbits is

$$\{a, a^2, \cdots, a^{p-1}\} \cup \{a^p, a^{2p}, \cdots, a^{p(p^2-1)}\}.$$

Now, the total number of orbits is equal to $n = (p-1) + (p^2-1) = (p-1)(p+2)$. Hence, using the Theorem 2.2, the total number of gyrotransversals is $p^{\frac{(p-1)(p+2)}{2}}$.

□

Now, we calculate the isomorphism classes of gyrotransversals to the subgroup H in the group G. We will use the *Cauchy-Frobenius Formula* to find the isomorphism class of gyrotransversals. As given in [6], any automorphism $\theta \in Aut_H(G)$ is given by

$$\theta(a) = b^j a^i \text{ and } \theta(b) = b,$$

where $0 \leqslant j \leqslant p-1$ and $i \in \mathbb{Z}_{p^3}$ such that $gcd(p, i) = 1$. Here, we see that the map $\theta \in Aut_H(G)$ fixes all the elements of H. Therefore any $\theta \in Aut_H(G)$ will give us an isomorphism between S and $\theta(S)$. In this way, $Aut_H(G)$ acts naturally on the set X. Note that, the image of any map $\theta \in Aut_H(G)$ depends only on the images of the elements of the subgroup $\langle a \rangle$ of G. Now, we find the set $Fix(\theta) = \{S \in X \mid \theta(S) = S\}$, for all $\theta \in Aut_H(G)$.

Lemma 3.2. *Let $\theta_i \in Aut_H(G)$ be defined by $\theta_i(a) = a^{p^2 i+1}$, where $0 \leqslant i \leqslant p-1$. Then $|Fix(\theta_i)| = |X|$.*

Proof. Let $\theta_i(a) = a^{p^2 i+1}$, where $0 \leqslant i \leqslant p-1$. Let S be a gyrotransversal to H in G such that $\theta_i(S) = S$. By (3.1), S consists of the elements from the sets $b^j\overline{a^u}$ and the elements of the form $b^j a^{p^2 u}$ and $b^j a^{pv}$, where $0 \leqslant j \leqslant p-1$, $u \in \{1, 2, \cdots, p-1\}$

and $v \in \{1, 2, \cdots, p^2-1\}$ with $gcd(v,p) = 1$. Note that, $\theta_i(b^j\overline{a^u}) = b^j\overline{a^u}$, $\theta_i(b^ja^{p^2u}) = b^ja^{p^2u}$ and $\theta_i(b^ja^{pv}) = b^ja^{pv}$. Hence, the map θ_i fixes all gyrotransversals. □

Lemma 3.3. *Let $\theta_i \in Aut_H(G)$ be defined by $\theta_i(a) = a^{pi+1}$, where $0 < i \leqslant p^2-1$ and* $\gcd(p,i) = 1$. *Then $|Fix(\theta_i)| = p^{\frac{3(p-1)}{2}}$.*

Proof. Let $\theta_i(a) = a^{pi+1}$, where $0 \leqslant i \leqslant p^2-1$. Let S be a gyrotransversal to H in G such that $\theta_i(S) = S$. Using Equation (3.1), S consists of the elements from the sets $b^j\overline{a^u}$ and the elements of the form $b^ja^{p^2u}$ and b^ja^{pv}, where $0 \leqslant j \leqslant p-1$, $u \in \{1, 2, \cdots, p-1\}$ and $v \in \{1, 2, \cdots, p^2-1\}$ with $gcd(v,p) = 1$. Note that, $\theta_i(b^j\overline{a^u}) = b^j\overline{a^u}$and $\theta_i(b^ja^{p^2u}) = b^ja^{p^2u}$.

Observe that, $\theta_i(b^ja^{pv}) = b^ja^{p^2vi+pv}$, $\theta_i(b^ja^{p^2vi+pv}) = b^ja^{2p^2vi+pv}$, $\theta_i(b^ja^{2p^2vi+pv}) = b^ja^{3p^2vi+pv}$ and $\theta_i(b^ja^{(p-1).p^2vi+pv}) = b^ja^{p.p^2vi+pv} = b^ja^{pv}$. Here we see that this map makes a cycle of p elements and $p(p-1)$ total elements of these types. Hence, b^ja^{pv} have $p^{\frac{p-1}{2}}$ choices. So the map θ_i fixes $p^{\frac{p-1}{2}} \times p^{\frac{p-1}{2}} \times p^{\frac{p-1}{2}} = p^{\frac{3(p-1)}{2}}$ gyrotransversals. □

Lemma 3.4. *Let $\theta_{i,k} \in Aut_H(G)$ be defined by $\theta_{i,k}(a) = b^ka^{pi+1}$, where $0 \leqslant i \leqslant p^2-1$ and $1 \leqslant k \leqslant p-1$. Then $|Fix(\theta_{i,k})| = 0$.*

Proof. Let $\theta_{i,k}(a) = b^ka^{pi+1}$, where $0 \leqslant i \leqslant p^2-1$, $1 \leqslant k \leqslant p-1$. Let S be a gyrotransversal to H in G such that $\theta_{i,k}(S) = S$. Using Equation (3.1), S consists of the elements from the sets $b^j\overline{a^u}$ and the elements of the form $b^ja^{p^2u}$ and b^ja^{pv}, where $0 \leqslant j \leqslant p-1$, $u \in \{1, 2, \cdots, p-1\}$ and $v \in \{1, 2, \cdots, p^2-1\}$ with $gcd(v,p) = 1$. Note that, $\theta_{i,k}(b^j\overline{a^u}) = b^{uk+j}\overline{a^u} \notin S$ because if $b^{uk+j}\overline{a^u} \in S$ then $j \equiv uk + j \,(mod\, p)$ which implies that $r_\gamma k \equiv 0 \,(mod\, p)$ which is not true. Hence, the map $\theta_{i,k}$ does not fix any gyrotransversal. □

Lemma 3.5. *Let $\theta_{i,r,k} \in Aut_H(G)$ be defined by $\theta_{i,r,k}(a) = b^ka^{pi+r}$, where $0 \leqslant i \leqslant p^2-1$, $0 \leq k \leq p-1$ and $1 < r \leq p-1$ with α being the order of r $(mod\, p)$ is even. Then $|Fix(\theta_{i,r,k})| = 1$.*

Proof. Let $\theta_{i,r,k}(a) = b^ka^{pi+r}$, where $0 \leqslant i \leqslant p^2-1$, $0 \leq k \leq p-1$ and $1 \leq r \leq p-1$ with α be the order of r $(mod\, p)$ which is even. Let S be a gyrotransversal to H in

G such that $\theta_{i,r,k}(S) = S$. Using Equation (3.1), S consists of the elements from the sets $b^j\overline{a^u}$ and the elements of the form $b^ja^{p^2u}$ and b^ja^{pv}, where $0 \leqslant j \leqslant p-1$, $u \in \{1, 2, \cdots, p-1\}$ and $v \in \{1, 2, \cdots, p^2-1\}$ with $gcd(v, p) = 1$. Note that $\theta_{i,r,k}(b^j\overline{a^u}) = b^{j+ku}\overline{a^{ur}}$, $\theta_{i,r,k}(b^{j+ku}\overline{a^{ur}}) = b^{j+ku+kur}\overline{a^{ur^2}}$, $\theta_{i,r,k}(b^{j+ku+kur}\overline{a^{ur^2}}) = b^{j+ku+kur+kur^2}\overline{a^{ur^3}}$ and

$$\begin{aligned}&\theta_{i,r,k}(b^{j+ku+kur+\cdots+kur^{\frac{\alpha}{2}-2}}\overline{a^{ur^{\frac{\alpha}{2}-1}}})\\&= b^{j+ku+kur+\cdots+kur^{\frac{\alpha}{2}-1}}\overline{a^{ur^{\frac{\alpha}{2}}}}\\&= b^{j+ku+kur+\cdots+kur^{\frac{\alpha}{2}-1}}\overline{a^{-u}}\end{aligned}$$

thus $b^{j+ku+kur+\cdots+kur^{\frac{\alpha}{2}-1}} \in S$ if and only if $b^{j+ku+kur+kur^2+\ldots+kur^{\frac{\alpha}{2}-1}} = b^{-j}$ which implies that $b^{ku(\frac{r^{\frac{\alpha}{2}}-1}{r-1})} = b^{-2j}$. This shows that $b^{ku(\frac{-2}{r-1})} = b^{-2j}$. Thus,

$$ku = j(r-1)\,(mod\,p) \tag{3.2}$$

Since $gcd(r-1, p) = 1$ for each u, there is a unique j that satisfies Equation (3.2). Observe that, $\theta_{i,r,k}(b^ja^{p^2u}) = b^ja^{p^2ur}$, $\theta_{i,r,k}(b^ja^{p^2ur}) = b^ja^{p^2ur^2}$ and $\theta_{i,r,k}(b^ja^{p^2ur^{\frac{\alpha}{2}-1}}) = b^ja^{p^2ur^{\frac{\alpha}{2}}} = b^ja^{-p^2u}$. Since S is a gyrotransversal and $(b^ja^{p^2u})^{-1} = b^{-j}a^{-p^2u}$ we have, $\theta_{i,r,k}(b^ja^{p^2ur^{\frac{\alpha}{2}-1}}) \in S$ if and only if $j \equiv -j\ (mod\,p)$. This implies that $j = 0$. Also observe that, $\theta_{i,r,k}(b^ja^{pv}) = b^ja^{pv(pi+r)}$, $\theta_{i,r,k}(b^ja^{pv(pi+r)}) = b^ja^{pv(pi+r)^2}$ and $\theta_{i,r,k}(b^ja^{pv(pi+r)^{\frac{\beta}{2}-1}}) = b^ja^{pv(pi+r)^{\frac{\beta}{2}}} = b^ja^{-pv}$, where β denotes the order of $(pi+r)$ $(mod\,p^2)$. Since S is a gyrotransversal and $(b^ja^{pv})^{-1} = b^{-j}a^{-pv}$, for $\theta(S) = S$ we have, $\theta_{i,r,k}(b^ja^{pv(pi+r)^{\frac{\beta}{2}-1}}) \in S$ if and only if $j \equiv -j\ (mod\,p)$ this implies that $j = 0$. Hence, the map $\theta_{i,r,k}$ fixes only one gyrotransversal.

□

Lemma 3.6. *Let $\theta_{i,r,k} \in Aut_H(G)$ be defined by $\theta_{i,r,k}(a) = b^ka^{pi+r}$, where $0 \leqslant i \leqslant p^2-1$, $0 \leqslant k \leq p-1$ and $1 < r \leq p-1$ with the order of r $(mod\,p)$ is odd which denoted by α. Then $|Fix(\theta_{i,r,k})| = p^{2t+m}$, where $t = \frac{p-1}{2\alpha}$ and $m = \frac{p(p-1)}{2\beta}$ with β denotes the order of $(pi+r)$ $(mod\,p^2)$.*

Proof. Let $\theta_{i,r,k}(a) = b^ka^{pi+r}$, where $0 \leqslant i \leqslant p^2-1$, $0 \leqslant k \leq p-1$ and $1 < r \leqslant p-1$ with α be the order of r $(mod\,p)$, which is odd. Let S be a gyrotransversal to H in G such that $\theta_{i,r,k}(S) = S$. Using Equation (3.1), S consists of the elements from the sets $b^j\overline{a^u}$ and the elements of the form $b^ja^{p^2u}$ and b^ja^{pv}, where $0 \leqslant j \leqslant p-1$,

$u \in \{1, 2, \cdots, p-1\}$ and $v \in \{1, 2, \cdots, p^2-1\}$ with $gcd(v,p) = 1$. Note, $\theta_{i,r,k}(b^j\overline{a^u}) = b^{j+ku}\overline{a^{ur}}$, $\theta_{i,r,k}(b^{j+ku}\overline{a^{ur}}) = b^{j+ku+kur}\overline{a^{ur^2}}$, $\theta_{i,r,k}(b^{j+ku+kur}\overline{a^{ur^2}}) = b^{j+ku+kur+kur^2}\overline{a^{ur^3}}$ and

$$\begin{aligned}&\theta_{i,r,k}(b^{j+ku+kur+kur^2+\cdots+kur^{\alpha-2}}\overline{a^{ur^{\alpha-1}}})\\&= b^{j+ku+kur+kur^2+\cdots+kur^{\alpha-1}}\overline{a^{ur^{\alpha}}}\\&= b^j\overline{a^u}.\end{aligned}$$

Hence this map make a cycle of α elements. Therefor we have, only $p^{\frac{p-1}{2\alpha}} = p^t$ choices for these type of elements.

Observe that, $\theta_{i,r,k}(b^j a^{p^2u}) = b^j a^{p^2ur}$, $\theta_{i,r,k}(b^j a^{p^2ur}) = b^j a^{p^2ur^2}$ and $\theta_{i,r,k}(b^j a^{p^2ur^{\alpha-1}}) = b^j a^{p^2ur^{\alpha}} = b^j a^{p^2u}$. As above we have, only $p^{\frac{p-1}{2\alpha}} = p^t$ choices for these type of elements. Also observe that, $\theta_{i,r,k}(b^j a^{pv}) = b^j a^{pv(pi+r)}$, $\theta_{i,r,k}(b^j a^{pv(pi+r)}) = b^j a^{pv(pi+r)^2}$ and $\theta_{i,r,k}(b^j a^{pv(pi+r)^{\beta-1}}) = b^j a^{pv(pi+r)^{\beta}} = b^j a^{pv}$. Here we say that this map makes a cycle of β elements. So we have, only $p^{\frac{p(p-1)}{2\beta}} = p^m$ choices for these type of elements. Hence the map $\theta_{i,r,k}$ fixes $p^t \times p^t \times p^m = p^{2t+m}$ gyrotransversals. □

Theorem 3.2. *The number of isomorphism classes of gyrotransversals to the subgroup H in the group G is equal to*

$$\frac{1}{p(p-1)}\left(\alpha_e p + p^{\frac{p^2+p-4}{2}} + (p-1)p^{\frac{3p-5}{2}} + \sum_{\eta=1}^{\alpha_o}\sum_{i=0}^{p-1} p^{2t_\eta+m_{\eta,i}}\right), \tag{3.3}$$

where α_e is the total number of even order elements in $U(p)$, α_o is the total number of odd order elements except the identity in $U(p)$, t_η is the t value corresponding to η^{th} odd order element except the identity and $m_{\eta,i}$ is the m value of η^{th} odd order element except the identity corresponding to different i's.

Proof. Using Lemma 3.2 each θ_i fixes $p^{\frac{(p-1)(p+2)}{2}}$ number of gyrotransversals. Therefore total number of gyrotransversals fixed by all the maps θ_i is $pp^{\frac{(p-1)(p+2)}{2}}$. Similarly from Lemma 3.3 fixes $(p^2-p)p^{\frac{3(p-1)}{2}}$ gyrotransversals, Lemma 3.4 does not fix any gyrotransversal, Lemma 3.5 fixes $\alpha_e p^2 p$ gyrotransversals and from Lemma 3.6 fixes $p\sum_{\eta=0}^{\alpha_o}\sum_{i=0}^{p^2-1} p^{2t_\eta+m_{\eta,i}}$ gyrotransversals.

Now by the *Cauchy-Frobenius Formula*, we have, the number of isomorphism classes of gyrotransversals is equal to the number of orbits, that is,

$$
\begin{aligned}
=&\frac{1}{|Aut_H(G)|}\sum_{\theta\in Aut_H(G)}|Fix(\theta)|\\
=&\frac{1}{p^3(p-1)}\left(\alpha_e p^3+p^{\frac{p(p+1)}{2}}+(p-1)p^{\frac{3p-1}{2}}+p\sum_{\eta=1}^{\alpha_o}\sum_{i=0}^{p^2-1}p^{2t_\eta+m_{\eta,i}}\right)\\
=&\frac{1}{p(p-1)}\left(\alpha_e p+p^{\frac{p^2+p-4}{2}}+(p-1)p^{\frac{3p-5}{2}}+\sum_{\eta=1}^{\alpha_o}\sum_{i=0}^{p-1}p^{2t_\eta+m_{\eta,i}}\right).
\end{aligned}
$$

□

Corollary 3.1. *The lower bound of the number of non isomorphic right gyrogroups of order p^3 is given in Equation (3.3).*

Next, we find the lower bound of number of non-isomorphic gyrotransversals such that the corresponding right gyrogroup is of nilpotency class 2. Note that, the center of the group G, $Z(G) = \langle a^p\rangle$. Now, we will find the deformations of the gyrotransversals such that $g(Z(G)) = 1$, that is, a gyrotransversal of the form given below,

$$T = \overline{a^p} \cup \bigcup_{r=1}^{p-1} b^j\overline{a^r}.$$

Theorem 3.3. [5] *If T be any gyrotransversal to the subgroup H in the group G such that $g(Z(G)) = 1$, then T is of nilpotency class 2.*

Now, we find the total number of such gyrotransversals to the subgroup H in the group G.

Theorem 3.4. *The total number of gyrotransversals in the group G to the subgroup H such that $g(Z(G)) = 1$ is equal to $p^{\frac{p-1}{2}}$.*

Proof. The proof is similar to the proof of the Theorem 3.1. □

Theorem 3.5. *The number of isomorphism classes of gyrotransversals to the subgroup H in the group G such that $g(Z(G)) = 1$ is equal to*

$$\frac{1}{(p-1)}\left(\alpha_e+p^{\frac{p-3}{2}}+\sum_{\eta=1}^{\alpha_o}p^{t_\eta}\right),$$

where α_e is the total number of even order elements in $U(p)$, α_o is the total number of odd order elements except the identity in $U(p)$ and t_η is the t value of η^{th} odd order element except the identity.

Proof. The proof is similar to the proof of the Theorem 3.2. □

Corollary 3.2. *The lower bound of the number of non isomorphic right gyrogroups of order p^3 having the nilpotency class 2 is given by the number in Theorem 3.5.*

As an illustration, we find the isomorphism classes of gyrotransversals of order 3^3 in the group $\mathbb{Z}_3 \ltimes \mathbb{Z}_{27}$.

Example 3.1. *Consider the group $G = \mathbb{Z}_3 \ltimes \mathbb{Z}_{27} = \langle a, b \mid a^{27} = 1 = b^3, bab^{-1} = a^{10}\rangle$ and the subgroup $H = \langle b\rangle$ of order 3. Then there are $3^{\frac{(3-1)(3+2)}{2}} = 3^5 = 243$ gyrotransversals to the subgroup H in the group G. These are given as*

$$S_j = \{1\} \cup (b^j\overline{a^1} \cup b^{-j}\overline{a^2}) \cup (\{b^j a^9\} \cup \{b^{-j}a^{18}\}) \cup \bigcup_{l=1}^{4} (\{b^{j_l}a^{3l}\} \cup \{b^{-j_l}a^{-3l}\}),$$

where $l \neq 3$. Any map $\theta \in Aut_H(G)$ is given by

$$\theta(a) = b^j a^i \text{ and } \theta(b) = b,$$

where $j \in \{0, 1, 2\}$, $0 \leq i \leq 26$ and $gcd(i, 3) = 1$. So, $|Aut_H(G)| = 54$. Now, one can easily check that the map $\theta_i(a) = a^i$, where $i \in \{1, 10, 19\}$ fixes all gyrotransversals. Therefore, $|Fix(\theta_i)| = 243$ for all $i \in \{1, 10, 19\}$. The map $\theta_i(a) = a^i$, where $i \in \{4, 7, 13, 16, 22, 25\}$ fixes $3^1 \cdot 3^{2(1)}$ gyrotransversals given by

$$\begin{aligned} S_j =\{&1, b^{j_1}a, b^{-j_1}a^2, b^{j_2}a^3, b^{j_1}a^4, b^{-j_1}a^5, b^{-j_2}a^6, b^{j_1}a^7, b^{-j_1}a^8, b^{j_3}a^9, b^{j_1}a^{10}, b^{-j_1}a^{11}, b^{j_2}a^{12},\\ &b^{j_1}a^{13}, b^{-j_1}a^{14}, b^{-j_2}a^{15}, b^{j_1}a^{16}, b^{-j_1}a^{17}, b^{-j_3}a^{18}, b^{j_1}a^{19}, b^{-j_1}a^{20}, b^{j_2}a^{21}, b^{j_1}a^{22}, b^{-j_1}a^{23},\\ &b^{-j_2}a^{24}, b^{j_1}a^{25}, b^{-j_1}a^{26}\}, \end{aligned}$$

where $0 \leq j_1, j_2, j_3 \leq 8$. So, $|Fix(\theta_j)| = 27$ for all $j \in \{4, 7, 13, 16, 22, 25\}$. The map $\theta_l(a) = a^l$, where $l \in \{2, 5, 8, \cdots, 26\}$ fixes the gyrotransversal

$$\begin{aligned} S =\{&1, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, a^{16}, a^{17}, a^{18}, a^{19}, a^{20}, a^{21},\\ &a^{22}, a^{23}, a^{24}, a^{25}, a^{26}\}. \end{aligned}$$

Note that for any b^ja^i, $\theta_l(b^ja^i) = b^ja^{li}$. *Therefore, if the map* θ_l *fixes any other gyrotransversal, then it is given by*

$$S_j = \{1, b^ja, b^ja^2, b^ja^3, b^ja^4, b^ja^5, b^ja^6, b^ja^7, b^ja^8, b^ja^9, b^ja^{10}, b^ja^{11}, b^ja^{12}, b^ja^{13}, b^ja^{14}, b^ja^{15}, b^ja^{16}, b^ja^{17}, b^ja^{18}, b^ja^{19}, b^ja^{20}, b^ja^{21}, b^ja^{22}, b^ja^{23}, b^ja^{24}, b^ja^{25}, b^ja^{26}\}.$$

Then $b^ja \in S_j$, *but* $(b^ja)^{-1} = b^{-j}a^{-i-p^2j} \notin S_j$. *This is a contradiction. Thus the map* θ_l *fails to fix any other gyrotransversal. So,* $|Fix(\theta_l)| = 1$ *for all* $l \in \{2, 5, 8, \cdots, 26\}$. *Using the similar arguments, the map* $\theta_{l,k}(a) = b^ka^l$, *where* $l \in \{2, 5, 8, \cdots, 26\}$ *and* $j \in \{1, 2\}$ *fixes only one gyrotransversal.*

Now, for the map $\theta_j(a) = b^ja$, *where* $j \in \{1, 2\}$, *we have if* S *is a gyrotransversal to the subgroup* H *in* G *such that* $\theta_j(S) = S$ *and* $b^ka \in S$, *then* $\theta_j(b^ka) = b^{k+j}a \in S$ *for all* $k \in \{0, 1, 2\}$. *This is a contradiction as* S *is a right transversal to* H *in* G. *Therefore, the map* θ_j *do not fix any gyrotransversal. Similarly, the map* $\theta_j(a^i) = b^ja^i$, *where* $i \in \{1, 4, 7, \cdots, 25\}$ *and* $j \in \{1, 2\}$ *do not fix any gyrotransversal. Therefore,* $|Fix(\theta_{i,j})| = 0$. *Thus,*

$$\sum_{\theta \in Aut_H(G)} |Fix(\theta)| = (3 \times 243) + (6 \times 27) + (9 \times 1) + (2 \times 9 \times 0) + (2 \times 9 \times 1) = 918.$$

Hence, using the Cauchy-Frobenius Formula, we get, the number of H *orbits is equal to* $\frac{918}{54} = 17$. *Hence, there are at least* 17 *non isomorphic right gyrogroups of order* 27

Also, the number of gyrotransversals of H *in the group* G *such that* $g(Z(G)) = 1$ *is equal to* 3. *Hence, the number of isomorphism class of gyrotransversals of order* 27 *having nilpotency class* 2 *is equal to* 1.

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