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ON HIGHER ORDER HOMODERIVATIONS IN SEMI-PRIME RINGS

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Abstract. Considering $\mathfrak{R}$ as an associative ring, a map h which is additive on $\mathfrak{R}$ with the property $h(zw) = h(z)h(w) + h(z)w + zh(w)$ valid for every $z, w \in \mathfrak{R}$ is called a homoderivation on $\mathfrak{R}$. In this paper our purpose is to demonstrate results about this kind of mappings on rings. The link between n-Jordan homoderivations that are mappings satisfying $\mathfrak{h}(u^n) = \sum_{t=1}^{n} \binom{n}{t} \big(\mathfrak{h} + \mathfrak{d}_u\big)^t(u)u^{n-t}$ for all $u \in \mathfrak{R}$ and homoderivations is investigated aa well as a result associating the mappings that for $n > 1$ satisfy

$$\Gamma(u^n) = \Gamma(u)u^{n-1} + \big(\Gamma(u) + u\big)\sum_{t=1}^{n-1} \binom{n-1}{t} (\mathfrak{h} + \mathfrak{d}_u)^t(u)u^{n-1-t} \text{ for every } u \in \mathfrak{R}$$

called n-Jordan generalized homoderivations with generalized homoderivations i.e maps with the property $\Gamma(uv) = \Gamma(u)\mathfrak{h}(v) + \Gamma(u)v + u\mathfrak{h}(v)$ for every $u, v \in \mathfrak{R}$ under suitable conditions is proved.

1. Introduction

All along this article, $\mathfrak{R}$ always considered as a ring with the associativity property and that has a center $Z(\mathfrak{R})$. $\mathfrak{R}$ is claimed to be semi-prime (prime) when for any $\mu \in \mathfrak{R}$, $\mu\mathfrak{R}\mu = 0$ entails that $\mu = 0$ ($\mu\mathfrak{R}\nu = 0$ implies that $\mu = 0$ or $\nu = 0$). $\mathfrak{R}$ will be κ-torsion free provided that $\kappa u = 0$ ($u \in \mathfrak{R}$) signifies $u = 0$. U will be an essential ideal of $\mathfrak{R}$ when for each non zero ideal I of $\mathfrak{R}$ we keep $U \cap I \neq \{0\}$. For any $u, v \in \mathfrak{R}$, the commutator $[u, v]$ will stand for $uv - vu$. A derivation is defined as a mapping $\mathfrak{d}$ on $\mathfrak{R}$ which is additive and validates the identity $\mathfrak{d}(zw) = \mathfrak{d}(z)w + z\mathfrak{d}(w)$

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for all $z, w \in \mathfrak{R}$. A typical example of such mapping is the inner derivation linked to a fixed element $i \in \mathfrak{R}$, $\mathfrak{d}_i(j) = [i, j]$ for all $j \in \mathfrak{R}$. An additive map $\mathfrak{T} : \mathfrak{R} \mapsto \mathfrak{R}$ can designated a generalized derivation assuming that we found a derivation $\mathfrak{d}$ on $\mathfrak{R}$ with $\mathfrak{T}(zw) = \mathfrak{T}(z)w + z\mathfrak{d}(w)$ holds for all $z, w \in \mathfrak{R}$. Derivations are natural examples of generalized derivations, also inner generalized derivations (maps with form $u \mapsto tu + us$ for any $t, s \in \mathfrak{R}$) represent another example. a Jordan homomorphism (resp. n-Jordan homomorphism) is an additive map f between two rings having the character $f(u^2) = \big(f(u)\big)^2$ (resp. $f(u^n) = \big(f(u)\big)^n$) for any $u \in \mathfrak{R}$. Finally, f is called an anti-homomorphism if $f(uv) = f(v)f(u)$ holds for each $u, v \in \mathfrak{R}$.

Numerous authors have inspected the notion of Jordan homomorphisms including Kaplansky, Jacobson and Herstein. The concept defining n-Jordan homomorphism between rings made known in the fifties by the work of Herstein [7]. It is acclaimed that any Jordan homomorphism mapping two rings will always be a n-Jordan homomorphism on the assumption that $n > 2$, However, in general the opposite could be false.

A characteristic conclusion due to Herstein [8] asserts that any Jordan derivation (i.e. a map $\mathfrak{d}$ which is additive verifying the 2nd power aspect : for any $u \in \mathfrak{R}$, $\mathfrak{d}(u^2) = u\mathfrak{d}(u) + \mathfrak{d}(u)u$) is a derivation in case that $\mathfrak{R}$ is prime with $char(\mathfrak{R}) \neq 2$. Cusack [5] generalized Hersteins result to 2-torsion free semiprime rings. Motivated by these works Vukman and Kosi-Ulbl [10] proved : Each n-Jordan derivation in an $n!$-torsion free semi-prime ring is a derivation. Finally, Wei and Xiao [11] extended these results to generalized n-Jordan derivations in an $n!$-torsion free semi-prime ring with identity element. In [6], El Sofy proposed the concept of homoderivation in rings which merges the notions of homomorphisms and derivations, that is, an additive map $\mathfrak{h}$ on $\mathfrak{R}$ going to be a homoderivation whenever it satisfies : $\mathfrak{h}(uv) = \mathfrak{h}(u)\mathfrak{h}(v) + \mathfrak{h}(u)v + u\mathfrak{h}(v)$ for any $u, v \in \mathfrak{R}$. A case of these mappings is $\mathfrak{h}(u) = f(u) - u$ for every $u \in \mathfrak{R}$ wherever f is an endomorphism on $\mathfrak{R}$. In a prime ring, a homoderivation $\mathfrak{h}$ will be a derivation if and only if $\mathfrak{h} = 0$. Indeed, $\mathfrak{h}$ is a derivation if $\mathfrak{h}(u)\mathfrak{h}(v) = 0$ for each $u, v \in \mathfrak{R}$. At this point, $\mathfrak{h}(u)\mathfrak{R}\mathfrak{h}(v) = \{0\}$ for every $u, v \in \mathfrak{R}$. Thus, On the assumption of the primality of $\mathfrak{R}$, the unique additive mapping at the same time that is a derivation but also a homoderivation is $0_{\mathfrak{E}(\mathfrak{R})}$ where

$\mathfrak{E}(\mathfrak{R})$ denotes the set of all endomorphisms of $\mathfrak{R}$. In the meanwhile of the previous decade, there has been a concern in progress devoted to the link connecting a ring $\mathfrak{R}$ with the behaviour of a derivation or a homoderivation on $\mathfrak{R}$. The study of homoderivations is important because they provide a way to gain insight into the algebraic structure of rings. Our goal is to refine these results to new areas. More specifically, and going along behind the similar path of research, we will first give conditions ensuring that an n-Jordan homoderivation in an $n!$-torsion free semi-prime unital ring is the sum of a homoderivation with an anti-homomorphism Precisely, the alluded result is : Consider $\mathfrak{h} : \mathfrak{R} \to \mathfrak{R}$ as an additive map on a unital semi-prime $n!$-torsion free ring $\mathfrak{R}$ with identity element e such that $\mathfrak{h}(e) = 0$ besides

$$\mathfrak{h}(u^n) = \sum_{t=1}^{n} \binom{n}{t} \big(\mathfrak{h} + \mathfrak{d}_u\big)^t(u)u^{n-t}, \quad \text{for all } u \in \mathfrak{R}, \quad n > 1,$$

this results in the existence of an essential ideal U of $\mathfrak{R}$ making the restriction of $\mathfrak{h}$ to U a direct sum, $\mathfrak{h}_1 \oplus \mathfrak{h}_2$.

$\mathfrak{h}_1$ denotes a homoderivation of U into $\mathfrak{R}$ alongside with $\mathfrak{h}_2$ that denotes an anti-homomorphism of U into $\mathfrak{R}$.

Then we will extend the same result to generalized n-Jordan homoderivations under fitting conditions by showing the following : Assuming that there exist additive mappings Γ and $\mathfrak{h}$ onto a $n!$-torsion free semi-prime ring $\mathfrak{R}$ beside unit element e. along with $\mathfrak{h}(e) = 0$ and

$$\Gamma(u^n) = \Gamma(u)u^{n-1} + \big(\Gamma(u) + u\big) \sum_{t=1}^{n-1} \binom{n-1}{t} (\mathfrak{h} + \mathfrak{d}_u)^t(u)u^{n-1-t} \text{ for every } u \in \mathfrak{R},\ n > 1.$$

Then this results in the existence of an essential ideal U of $\mathfrak{R}$ making the restriction of $\mathfrak{h}$ to U a direct sum, $\mathfrak{h}_1 \oplus \mathfrak{h}_2$.

$\mathfrak{h}_1$ denotes a homoderivation of U into $\mathfrak{R}$ alongside $\mathfrak{h}_2$ denoting an anti-homomorphism of U into $\mathfrak{R}$. Moreover, the restriction of Γ to U is a generalized homoderivation on $\mathfrak{R}$ associated with $\mathfrak{h}_1$.

2. The Main Results

2.1. **On n-Jordan homoderivations.**

Definition 2.1. An additive map $\mathfrak{h}$ from $\mathfrak{R}$ to $\mathfrak{R}$ is known as a Jordan homoderivation with the condition

$$\mathfrak{h}(u^2) = \mathfrak{h}^2(u) + \mathfrak{h}(u)u + u\mathfrak{h}(u) \quad \text{holds for all } u \in \mathfrak{R}. \tag{2.1}$$

Any additive map on $\mathfrak{R}$ of the structure $u \mapsto g(u) - u$, wherever g is a Jordan homomorphism, will be a Jordan homoderivation.

Definition 2.2. Let $n \geq 1$ considered an integer, An additive map $\mathfrak{d} : \mathfrak{R} \to \mathfrak{R}$, that establishes the relation

$$\mathfrak{d}(u^n) = \sum_{t=0}^{n} u^{n-t}\mathfrak{d}(u)u^{t-1} \quad \text{for all } u \in \mathfrak{R} \tag{2.2}$$

will be known as an n-Jordan derivation.

The following definition provides an analogous formula for n-Jordan homoderivations in rings.

Definition 2.3. Let $n \geq 1$ considered an integer, An additive map $\mathfrak{h} : \mathfrak{R} \to \mathfrak{R}$, that assures the relation

$$\mathfrak{h}(u^n) = \sum_{t=1}^{n} \binom{n}{t} \left(\mathfrak{h} + \mathfrak{d}_u\right)^t (u)u^{n-t} \quad \text{for all } u \in \mathfrak{R} \tag{2.3}$$

is called an n-Jordan homoderivation.

The next lemma proves that there is no problem for an n-Jordan homoderivation to be a homoderivation.

Lemma 2.1. *Each homoderivation on $\mathfrak{R}$ is an n-Jordan homoderivation on $\mathfrak{R}$.*

Proof. Taking $n = 1$, nothing to demonstrate.

Let n be a positive integer, and suppose we have (2.3)

$$\begin{aligned}
\mathfrak{h}(u^{n+1}) &= \mathfrak{h}(u.u^n) \\
&= \mathfrak{h}(u)\mathfrak{h}(u^n) + \mathfrak{h}(u)u^n + u\mathfrak{h}(u^n) \\
&= \mathfrak{h}(u)\sum_{t=1}^{n}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n-t} + \mathfrak{h}(u)u^n + u\sum_{t=1}^{n}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n-t} \\
&= \sum_{t=1}^{n}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^{t+1}(u)u^{n-t} + \mathfrak{h}(u)u^n + \sum_{t=1}^{n}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n+1-t} \\
&= \sum_{t=0}^{n}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^{t+1}(u)u^{n-t} + \sum_{t=1}^{n}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n+1-t} \\
&= \sum_{t=1}^{n+1}\binom{n}{t-1}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n+1-t} + \sum_{t=1}^{n+1}\binom{n}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n+1-t} \\
&= \sum_{t=1}^{n+1}\left[\binom{n}{t-1}+\binom{n}{t}\right]\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n+1-t} \\
&= \sum_{t=1}^{n+1}\binom{n+1}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n+1-t}
\end{aligned}$$

hence the result is proved. □

Example 2.1.

$$\begin{aligned}
\mathfrak{h}\left(u^2\right) &= \sum_{t=1}^{2}\binom{2}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{2-t} \\
&= \binom{2}{1}\big(\mathfrak{h}+\mathfrak{d}_u\big)(u)u + \binom{2}{2}\big(h+\mathfrak{d}_u\big)^2(u) \\
&= 2\mathfrak{h}(u)u + \big(\mathfrak{h}+\mathfrak{d}_u\big)\big(\big(\mathfrak{h}+\mathfrak{d}_u\big)(u)\big) \\
&= 2\mathfrak{h}(u)u + \big(\mathfrak{h}+\mathfrak{d}_u\big)\big(\mathfrak{h}(u)\big) \\
&= 2\mathfrak{h}(u)u + \mathfrak{h}^2(u) + \mathfrak{d}_u\big(\mathfrak{h}(u)\big) \\
&= 2\mathfrak{h}(u)u + \mathfrak{h}^2(u) + u\mathfrak{h}(u) - \mathfrak{h}(u)u \\
&= \mathfrak{h}^2(u) + \mathfrak{h}(u)u + u\mathfrak{h}(u).
\end{aligned}$$

We call equation (2.3) the nth-power property. For $n = 2,\ 3$, the nth-power property generates 2-Jordan homoderivations and 3-Jordan homoderivations respectively. In particular, we called 2-Jordan homoderivation a Jordan homoderivation for simplicity. Basic examples are homoderivations. However, the converse of lemma 2.4, need not be true as shown by the following example :

Example 2.2. *If $\mathfrak{R}$ has a non-trivial central idempotent e with $u^2 = 0$ for all $u \in \mathfrak{R}$, but $uv \neq 0$ for some non-zero elements u and v in $\mathfrak{R}$, then the mappings of the form $u \mapsto eu$ for all $u \in \mathfrak{R}$ are Jordan homoderivations which are not homoderivations.*

Now it appears legitimate requesting what supplementary presumptions the contrary will be correct. We should first mention that Herstein proved :

Lemma 2.2 ([7], Theorem H)**.** *Jordan homomorphisms mapping a ring onto a prime ring with characteristic neither 2 nor 3 are one of two : homomorphisms or anti-homomorphisms.*

Smiley [9], sharpened this result by taking out the necessity of the characteristic be different from 3. After that, Baxter and Martindale, studied a more general case and proved the following result :

Lemma 2.3 ([1], Theorem 2.6)**.** *Suppose ϑ is a Jordan homomorphism mapping a ring $\mathfrak{S}$ onto a semi-prime 2-torsion free ring $\mathfrak{R}$. This results in the existence of an essential ideal I of $\mathfrak{S}$ making the restriction of ϑ to I a direct sum : $\alpha_1 \oplus \alpha_2$.*
α_1 denoted a homomorphism of $\mathfrak{S}$ into $\mathfrak{R}$ besides α_2 that denoted an anti-homomorphism of $\mathfrak{S}$ into $\mathfrak{R}$.

At this moment we are able to declare and show our leading main conclusion in this paper.

Theorem 2.1. *Consider $\mathfrak{h} : \mathfrak{R} \to \mathfrak{R}$ as an additive map on a unital semi-prime $n!$-torsion free ring $\mathfrak{R}$ with identity element e such that $\mathfrak{h}(e) = 0$ besides*

$$\mathfrak{h}(u^n) = \sum_{t=1}^{n} \binom{n}{t} \left(\mathfrak{h} + \mathfrak{d}_u\right)^t(u)u^{n-t}, \quad \text{for all } u \in \mathfrak{R}, \quad n > 1,$$

this results in the existence of an essential ideal U of $\mathfrak{R}$ making the restriction of $\mathfrak{h}$ to U a direct sum, $\mathfrak{h}_1 \oplus \mathfrak{h}_2$.
$\mathfrak{h}_1$ denotes a homoderivation of U into $\mathfrak{R}$ alongside with $\mathfrak{h}_2$ that denotes an anti-homomorphism of U into $\mathfrak{R}$.

Proof. Let $v \in Z(\mathfrak{R})$ with $\mathfrak{h}(v) = 0$. Replace u by $u + v$ in (2.3), we attain

$$\begin{aligned}
\mathfrak{h}\big((u+v)^n\big) &= \mathfrak{h}\Big(\sum_{s=0}^{n}\binom{n}{s}u^{n-s}v^s\Big)\\
&= \sum_{s=0}^{n}\binom{n}{s}\mathfrak{h}(u^{n-s})v^s\\
&= \sum_{s=0}^{n}\binom{n}{s}\sum_{t=1}^{n-s}\binom{n-s}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n-s-t}v^s.
\end{aligned} \tag{2.4}$$

Alternatively, we get

$$\begin{aligned}
\mathfrak{h}\big((u+v)^n\big) &= \sum_{s=1}^{n}\binom{n}{s}\big(\mathfrak{h}+\mathfrak{d}_u\big)^s(u)(u+v)^{n-s}\\
&= \sum_{s=1}^{n}\binom{n}{s}\big(\mathfrak{h}+\mathfrak{d}_u\big)^s(u)\sum_{t=0}^{n-s}\binom{n-s}{t}u^{n-s-t}v^t.
\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) we get

$$\begin{aligned}
&\sum_{s=0}^{n}\binom{n}{s}\sum_{t=1}^{n-s}\binom{n-s}{t}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n-s-t}v^s\\
&= \sum_{s=1}^{n}\binom{n}{s}\big(\mathfrak{h}+\mathfrak{d}_u\big)^s(u)\sum_{t=0}^{n-s}\binom{n-s}{t}u^{n-s-t}v^t\\
&= \sum_{t=1}^{n}\binom{n}{t}\sum_{s=0}^{n-t}\binom{n-t}{s}\big(\mathfrak{h}+\mathfrak{d}_u\big)^t(u)u^{n-t-s}v^s.
\end{aligned} \tag{2.6}$$

Now let $\delta_s(u, v)$ be the expression of terms containing s factors of v, hence by reorganizing the terms holding equal number of factors of v in (2.6), we obtain

$$\sum_{s=1}^{n}\delta_s(u,v) = 0, \quad u \in \mathfrak{R} \tag{2.7}$$

Replacing v in (2.7) by the terms from e to $(n-1)e$ yields a homogeneous system with $(n-1)$ equations giving a matrix formed in the following way

$$\mathbb{V} = \begin{pmatrix}
1 & 1 & \cdots & 1\\
2 & 2^2 & \cdots & 2^{n-1}\\
. & . & \cdots & .\\
. & . & \cdots & .\\
. & . & \cdots & .\\
(n-1) & (n-1)^2 & \cdots & (n-1)^{n-1}
\end{pmatrix}.$$

The related system to this Van Der Monde matrix could exclusively have a trivial solution. Particularly, one has

$$\begin{aligned}\delta_{n-2}(u,v) = \binom{n}{n-2}\sum_{t=1}^{2}\binom{2}{t}\left(\mathfrak{h}+\mathfrak{d}_u\right)^t(u)u^{2-t} \\ -\binom{n}{n-2}\sum_{t=0}^{2}\binom{2}{t}\left(\mathfrak{h}+\mathfrak{d}_u\right)^{n-2}(u)u^{2-t} = 0\end{aligned} \tag{2.8}$$

We can rewrite (2.8) as

$$\frac{n(n-1)}{2}\mathfrak{h}(u^2) = \binom{n}{1}\binom{n-1}{n-2}\mathfrak{h}(u)u + \binom{n}{2}\binom{n-2}{n-2}\Big(\mathfrak{h}^2(u) + u\mathfrak{h}(u) - \mathfrak{h}(u)u\Big) \tag{2.9}$$

or more explicitly

$$\frac{n(n-1)}{2}\mathfrak{h}(u^2) = n(n-1)\mathfrak{h}(u)u + \frac{n(n-1)}{2}\Big(\mathfrak{h}^2(u) + u\mathfrak{h}(u) - \mathfrak{h}(u)u\Big) \tag{2.10}$$

which in turn gives

$$\frac{n(n-1)}{2}\mathfrak{h}(u^2) = \frac{n(n-1)}{2}\Big(\mathfrak{h}^2(u) + u\mathfrak{h}(u) + \mathfrak{h}(u)u\Big). \tag{2.11}$$

Considering the $n!$-torsion freeness of $\mathfrak{R}$, the precedent equation gives

$$\mathfrak{h}(u^2) = \mathfrak{h}^2(u) + u\mathfrak{h}(u) + \mathfrak{h}(u)u \quad \textit{for all } u \in \mathfrak{R}. \tag{2.12}$$

Thus, $\mathfrak{h}$ is a Jordan homoderivation. By putting $\mathfrak{g}(u) = \mathfrak{h}(u) + u$

$$\begin{aligned}\mathfrak{g}(u^2) &= \mathfrak{h}(u^2) + u^2 \\ &= \mathfrak{h}^2(u) + u\mathfrak{h}(u) + \mathfrak{h}(u)u + u^2 \\ &= \mathfrak{h}(u)\Big(\mathfrak{h}(u) + u\Big) + u\Big(\mathfrak{h}(u) + u\Big) \\ &= \Big(\mathfrak{h}(u) + u\Big)\Big(\mathfrak{h}(u) + u\Big) \\ &= \mathfrak{g}(u)\mathfrak{g}(u) \quad \text{for all } u \in \mathfrak{R}.\end{aligned}$$

Thus $\mathfrak{g}$ is a Jordan homomorphism and the theorem follows from Lemma 2.3. This proves the theorem completely. □

2.2. **On generalized n-Jordan homoderivations.** Analogously to generalized derivations definition in rings, we define a generalized homoderivation.

Definition 2.4. Let $\mathfrak{h}$ be a homoderivation on $\mathfrak{R}$. A generalized homoderivation Γ associated to $\mathfrak{h}$ is an additive mapping on $\mathfrak{R}$ that for any $u, v \in \mathfrak{R}$, verifies $\Gamma(uv) = \Gamma(u)\mathfrak{h}(v) + \Gamma(u)v + u\mathfrak{h}(v)$.

Example 2.3. *Let $a \in \mathfrak{R}$ be an invertible element. Since for every $u \in \mathfrak{R}$ the additive mapping $u \mapsto aua^{-1}$ is a homomorphism of $\mathfrak{R}$, then the mapping $\mathfrak{h}_a$ defined by $\mathfrak{h}_a(u) = aua^{-1} - u$ for all $u \in \mathfrak{R}$ determines a homoderivation on $\mathfrak{R}$. We call $\mathfrak{h}_a$ the inner homoderivation on $\mathfrak{R}$. Now, let $b \in \mathfrak{R}$ be an invertible element. The additive mapping $\Gamma_{a,b}$ defined by $\Gamma_{a,b}(u) = aub^{-1} - u$ for any $u \in \mathfrak{R}$ represents a generalized homoderivation on $\mathfrak{R}$ associated to h_b. Indeed, let $u, v \in \mathfrak{R}$, we obtain*

$$\begin{aligned}
&\Gamma_{a,b}(u)h_b(v) + \Gamma_{a,b}(u)v + u\mathfrak{h}_b(v) \\
&= (aub^{-1} - u)(bvb^{-1} - v) + (aub^{-1} - u)v + u(bvb^{-1} - v) \\
&= auvb^{-1} - aub^{-1}v - ubvb^{-1} + uv + aub^{-1}v - uv + ubvb^{-1} - uv \\
&= auvb^{-1} - uv \\
&= \Gamma_{a,b}(uv).
\end{aligned}$$

Moreover, if $a \neq b$, then $\Gamma_{a,b}$ is not a homoderivation.

Definition 2.5. Let $\mathfrak{h}$ be a Jordan homoderivation on $\mathfrak{R}$. A generalized Jordan homoderivation η associated to $\mathfrak{h}$ is the additive mapping on $\mathfrak{R}$ having for each $u \in \mathfrak{R}$, the identity $\eta(u^2) = \eta(u)\mathfrak{h}(u) + \eta(u)u + u\mathfrak{h}(u)$.

Example 2.4. *Let $a \in \mathfrak{R}$ be an invertible element along with an involution σ on $\mathfrak{R}$. Since for any $u \in \mathfrak{R}$ the additive mapping $u \mapsto a\sigma(u)a^{-1}$ is a homomorphism of $\mathfrak{R}$ and $\sigma(u^2) = \sigma^2(u)$ then the mapping $\mathfrak{h}_a$ defined by $\mathfrak{h}_a(u) = a\sigma(u)a^{-1} - u$ for each $u \in \mathfrak{R}$ determines a Jordan homoderivation on $\mathfrak{R}$.*
Now, let $b \in \mathfrak{R}$ be an invertible element. The additive mapping $\eta_{a,b}$ defined by $\eta_{a,b}(u) = a\sigma(u)b^{-1} - u$ for each $u \in \mathfrak{R}$ is a generalized Jordan homoderivation on $\mathfrak{R}$ associated to $\mathfrak{h}_b$ which is not a Jordan homoderivation if $a \neq b$.

Furthermore, $\eta_{a,b}$ is not a generalized homoderivation as shown below.
Let $u, v \in \mathfrak{R}$, one has

$$\begin{aligned}
&\eta_{a,b}(u)\mathfrak{h}_b(v) + \eta_{a,b}(u)v + u\mathfrak{h}_b(v) \\
&= (a\sigma(u)b^{-1} - u)(b\sigma(v)b^{-1} - v) + (a\sigma(u)b^{-1} - u)v + u(b\sigma(v)b^{-1} - v) \\
&= a\sigma(uv)b^{-1} - a\sigma(u)b^{-1}v - ub\sigma(v)b^{-1} + uv + a\sigma(u)b^{-1}v - uv + ub\sigma(v)b^{-1} - uv \\
&= a\sigma(u)\sigma(v)b^{-1} - uv \\
&\neq a\sigma(v)\sigma(u)b^{-1} - uv = \eta_{a,b}(uv).
\end{aligned}$$

It is obvious that every generalized homoderivation Γ associated to a homoderivation $\mathfrak{h}$ on $\mathfrak{R}$ is a generalized n-Jordan homoderivation i.e. it has the following nth power property

$$\Gamma(u^n) = \Gamma(u)u^{n-1} + \big(\Gamma(u) + u\big)\sum_{t=1}^{n-1}\binom{n-1}{t}(\mathfrak{h} + \mathfrak{d}_u)^t(u)u^{n-1-t} \quad \text{for all } u \in \mathfrak{R}.$$

Next, we will show, under mild conditions, that the converse is also true by providing an extended version of theorem 2.1.

Theorem 2.2. *Assuming that there exist additive mappings Γ and h on a $n!$-torsion free prime ring $\mathcal{R}$ beside unit element e along with $h(e) = 0$, $\Gamma(e) \neq e$, $h + id_{\mathcal{R}}$ is an onto mapping that is not an anti-homomorphism and*

(2.13)
$$\Gamma(u^n) = \Gamma(u)u^{n-1} + \big(\Gamma(u) + u\big)\sum_{t=1}^{n-1}\binom{n-1}{t}(\mathfrak{h} + \mathfrak{d}_u)^t(u)u^{n-1-t} \text{ for every } u \in \mathfrak{R},\ n > 1.$$

Then Γ is a generalised homoderivation associated with the homoderivation h on $\mathcal{R}$.

Proof. Our statement indicates that

(2.14)
$$\Gamma(u^n) - \Gamma(u)u^{n-1} - \big(\Gamma(u) + u\big)\sum_{t=1}^{n-1}\binom{n-1}{t}(\mathfrak{h} + \mathfrak{d}_u)^t(u)u^{n-1-t} = 0 \quad \text{for all } u \in \mathfrak{R}.$$

Let α be an integer. Changing u by $u + \alpha v$ in (2.14) we get

$$\sum_{t=1}^{n}\alpha^t T_t(u, v) = 0 \quad \text{for all } u, v \in \mathfrak{R}. \tag{2.15}$$

Where $T_t(u,v)$ express the number of quantities containing t elements of v in the expression

$$\begin{aligned}&\Gamma\big((u+\alpha v)^n\big)-\Gamma(u+\alpha v)(u+\alpha v)^{n-1}-\big[\Gamma(u+\alpha v)+(u+\alpha v)\big]\\&\sum_{t=1}^{n-1}\binom{n-1}{t}(\mathfrak{h}+\mathfrak{d}_{u+\alpha v})^t(u+\alpha v)(u+\alpha v)^{n-1-t}=0\end{aligned}\tag{2.16}$$

for all $u,v\in\mathfrak{R}$. By [[4], Lemma 1] it follows that

(2.17)

$$\begin{aligned}T_1(u,v)&=\Gamma(u^{n-1}v+u^{n-2}vu+\cdots+vu^{n-1})-\Gamma(v)u^{n-1}\\&-\Gamma(u)(u^{n-2}v+u^{n-3}vu+\cdots+vu^{n-2})\\&-\big(\Gamma(u)+u\big)\Big[\big(\mathfrak{h}(u)+u\big)^{n-2}\big(\mathfrak{h}(v)+v\big)+\big(\mathfrak{h}(u)+u\big)^{n-3}\big(\mathfrak{h}(v)+v\big)\big(\mathfrak{h}(u)+u\big)+\dots\\&\cdots+\big(\mathfrak{h}(v)+v\big)\big(\mathfrak{h}(u)+u\big)^{n-2}\Big]+\big(\Gamma(u)+u\big)(u^{n-2}v+u^{n-3}vu+\cdots+vu^{n-2})=0\end{aligned}$$

for all $u,v\in\mathfrak{R}$. Taking $u=e$ into (2.17) leads to

$$n\Gamma(v)-\Gamma(v)-(n-1)\Gamma(e)v-(n-1)\big(\Gamma(e)+e\big)\big(\mathfrak{h}(v)+v\big)+(n-1)\big(\Gamma(e)+e\big)v=0$$

for each $v\in\mathfrak{R}$. Considering the $n!$-torsion freeness of $\mathfrak{R}$ one gets

$$\Gamma(v)=\Gamma(e)h(v)+\Gamma(e)v+\mathfrak{h}(v)\quad\text{for any } v\in\mathfrak{R}.\tag{2.18}$$

Converting v into u^2 in (2.18) yields

$$\Gamma(u^2)=\Gamma(e)\mathfrak{h}(u^2)+\Gamma(e)u^2+\mathfrak{h}(u^2)\quad\text{for all } u\in\mathfrak{R}.\tag{2.19}$$

Also replacing v by u in (2.18) then right multiplying by u implies that

$$\Gamma(u)u=\Gamma(e)\mathfrak{h}(u)u+\Gamma(e)u^2+\mathfrak{h}(u)u\quad\text{for all } u\in\mathfrak{R}.\tag{2.20}$$

Substituting v by u in (2.18) then right multiplying by $h(u)$ yields

$$\Gamma(u)h(u)=\Gamma(e)\mathfrak{h}^2(u)+\Gamma(e)uh(u)+\mathfrak{h}^2(u)\quad\text{for all } u\in\mathfrak{R}.\tag{2.21}$$

Alternatively, taking $v=e$ in the expression (2.16) we get

$$\begin{aligned}&\Gamma\big((u+\alpha e)^n\big)-\Gamma(u+\alpha e)(u+\alpha e)^{n-1}-\Big[\Gamma\big((u+\alpha e)^{n-1}\big)+(u+\alpha e)^{n-1}\Big]\\&\sum_{t=1}^{n-1}\binom{n-1}{t}(\mathfrak{h}+\mathfrak{d}_{u+\alpha e})^t(u+\alpha e)(u+\alpha e)^{n-1-t}=0\text{ for all } u\in\mathfrak{R}.\end{aligned}\tag{2.22}$$

Extending (2.22) and using (2.13) one obtains

$$\Gamma\left(\sum_{t=1}^{n-1}\binom{n}{t}\alpha^t u^{n-t}\right) = \Gamma(u)\left(\sum_{t=1}^{n-2}\binom{n-1}{t}\alpha^t u^{n-1-t} + \alpha^{n-1}e\right)$$
$$+\alpha\Gamma(e)\left(\sum_{t=1}^{n-2}\binom{n-1}{t}\alpha^t u^{n-t}\right)$$
$$+\binom{n-1}{1}\big[(\Gamma(u)+u)+\alpha(\Gamma(e)+e)\big]h(u)\left(\sum_{i=1}^{n-3}\binom{n-2}{i}\alpha^i x^{n-2-i}+\alpha^{n-2}e\right)$$

(2.23)

$$+\binom{n-1}{2}\big[(\Gamma(u)+u)+\alpha(\Gamma(e)+e)\big](\mathfrak{h}+\mathfrak{d}_{u+\alpha e})^2(u+\alpha e)$$
$$\left(\sum_{i=1}^{n-4}\binom{n-3}{i}\alpha^i u^{n-3-i}+\alpha^{n-3}e\right)$$
$$\vdots$$
$$+\binom{n-1}{n-1}\big[(\Gamma(u)+u)+\alpha(\Gamma(e)+e)\big](\mathfrak{h}+\mathfrak{d}_{u+\alpha e})^{n-1}(u+\alpha e)=0 \text{ for all } u\in\mathfrak{R}.$$

Collecting the terms Q_i containing i factors of α in (2.2), we obtain

$$\alpha Q_1(u,e)+\alpha^2 Q_2(u,e)+\cdots+\alpha^n Q_n(u,e)=0. \tag{2.24}$$

Making use again of [[4], Lemma 1] we have in particular

$$Q_{n-2}(u,e)=0 \quad \text{for all } u\in\mathcal{R}.$$

Which returns

$$n(n-1)\Gamma(u^2) = 2(n-1)\Gamma(u)u$$
$$+(n-1)(n-2)\Gamma(e)u^2 + 2(n-1)\Big(\Gamma(u)+u\Big)h(u)$$
$$+2(n-1)(n-2)\Big(\Gamma(e)+e\Big)h(u)u$$
$$+(n-1)(n-2)\Big(\Gamma(e)+e\Big)\Big[h^2(u)+uh(u)-h(u)u\Big] \tag{2.25}$$
$$=2(n-1)\Gamma(u)u$$
$$+(n-1)(n-2)\Gamma(e)u^2+2(n-1)\Big(\Gamma(u)+u\Big)h(u)$$
$$+(n-1)(n-2)\Big(\Gamma(e)+e\Big)\Big[h^2(u)+uh(u)+h(u)u\Big] \text{ for all } u\in\mathfrak{R}.$$

Now replacing (2.20) in (2.25) we achieve

$$\begin{aligned}n(n-1)\Gamma(u^2) &= n(n-1)\Gamma(e)u^2 + n(n-1)h(u)u + n(n-1)uh(u)\\ &+ n(n-1)\Gamma(e)h(u)u + 2(n-1)\Gamma(u)h(u)\\ &+ (n-1)(n-2)\Big(\Gamma(e)+e\Big)\Big[h^2(u) + uh(u) + h(u)u\Big] \text{ for all } u \in \mathfrak{R}.\end{aligned} \tag{2.26}$$

Next, substituting (2.21) in (2.26) and we find that

$$\begin{aligned}n(n-1)\Gamma(u^2) &= n(n-1)\Gamma(e)u^2 + n(n-1)h(u)u + n(n-1)uh(u)\\ &+ n(n-1)\Gamma(e)h(u)u + n(n-1)\Gamma(e)uh(u)\\ &+ n(n-1)\Big(\Gamma(e)+e\Big)h^2(u) \text{ for all } u \in \mathfrak{R}.\end{aligned} \tag{2.27}$$

Utilizing the fact that $\mathfrak{R}$ is a $n!$-torsion free semi-prime ring we arrive at

$$\begin{aligned}\Gamma(u^2) &= \Gamma(e)u^2 + \Gamma(e)h(u)u + h(u)u + uh(u) + \Gamma(e)uh(u)\\ &+ \Big(\Gamma(e)+e\Big)h^2(u) \text{ for all } u \in \mathfrak{R}.\end{aligned} \tag{2.28}$$

Employing again (2.20) in (2.27) we reach

$$\Gamma(u^2) = \Gamma(u)u + uh(u) + \Gamma(e)uh(u) + \Big(\Gamma(e)+e\Big)h^2(u) \text{ for all } u \in \mathfrak{R}. \tag{2.29}$$

Finally, applying (2.21) in (2.29) we attain

$$\Gamma(u^2) = \Gamma(u)u + uh(u) + \Gamma(u)h(u) \text{ for all } u \in \mathfrak{R}. \tag{2.30}$$

Equating (2.19), (2.20) and (2.21) simultaneously in (2.30) we deduce that

$$\begin{aligned}&\Gamma(e)\mathfrak{h}(u^2) + \Gamma(e)u^2 + \mathfrak{h}(u^2) = \Gamma(e)\mathfrak{h}(u)u + \Gamma(e)u^2 +\\ &\mathfrak{h}(u)u + \Gamma(e)\mathfrak{h}^2(u) + \Gamma(e)uh(u) + \mathfrak{h}^2(u) + uh(u) \text{ for all } u \in \mathfrak{R}.\end{aligned}$$

This can be reshaped in the following manner

$$\Big[\Gamma(e)+e\Big]\mathfrak{h}(u^2) = \Big[\Gamma(e)+e\Big]\Big(\mathfrak{h}(u)u + uh(u) + \mathfrak{h}^2(u)\Big) \text{ for all } u \in \mathfrak{R}.$$

Or equivalentely

$$\Big[\Gamma(e)+e\Big]\Big(\mathfrak{h}(u^2) - \mathfrak{h}(u)u - uh(u) - \mathfrak{h}^2(u)\Big) = 0 \text{ for all } u \in \mathfrak{R}.$$

Consequently, since $\Gamma(e) \neq e$ and by the primeness of $\mathfrak{R}$ we arrive at

$$h(u^2) = h^2(u) + h(u)u + uh(u) \text{ for all } u \in \mathcal{R}.$$

Hence h is a Jordan homoderivation on $\mathcal{R}$. So, according to Lemma 2.2 our result is proved. This completes the proof. □

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