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FRACTIONAL MULTIPLICATIVE OSTROWSKI-TYPE INEQUALITIES FOR MULTIPLICATIVE DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. In this manuscript, we propose a new fractional identity for multiplicative differentiable functions, based on this identity we prove some fractional Ostrowski-type inequalities for multiplicative convex functions. Some applications of the obtained results are given.

1. Introduction

Convexity theory plays a central role in several branches of applied mathematics. In particular in the classical theory of optimization where the convexity makes it possible to obtain necessary and sufficient global optimality conditions. This concept has a strong relationship in the development of the theory of inequalities, which is an important tool in the study of certain qualitative properties of the solutions of differential and integro-differential equations as well as in the error estimates of quadrature formulas.

In [22], Ostrowski showed the following inequality

$$\left|\varphi\left(\varkappa\right)-\frac{1}{\mathfrak{r}_2-\mathfrak{r}_1}\int\limits_{\mathfrak{r}_1}^{\mathfrak{r}_2}\varphi\left(u\right)du\right|\leq\frac{\mathcal{M}\left(\left(\mathfrak{r}_2-\varkappa\right)^2+\left(\varkappa-\mathfrak{r}_1\right)^2\right)}{2\left(\mathfrak{r}_2-\mathfrak{r}_1\right)},\tag{1.1}$$

where $\varphi:[\mathfrak{r}_1,\mathfrak{r}_2]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable function and $\varphi'\in L^1[\mathfrak{r}_1,\mathfrak{r}_2]$ with $|\varphi'|\leq\mathcal{M}$.

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Since its discover several paper in connection with inequality (1.1) via different types of convexity, have been appeared, we refer readers to [2, 6, 7, 12, 14, 15, 16, 17, 18, 19].

Grossman and Katz, introduced and studied the first non-Newtonian calculation system, called geometric calculation. Over the next few years they had reached an infinite family of non-Newtonian calculus, thus modifying the classical calculus introduced by Newton and Leibniz in the 17 th century each of which differed Known style from the usual calculus of Newton and Leibniz known today as the non-Newtonian calculus or the multiplicative calculus, where the ordinary product and ratio are used respectively as sum and exponential difference over the domain of positive real numbers see [13]. This calculation is useful for dealing with exponentially varying functions. It is worth noting that the complete mathematical of multiplicative calculus was given by Bashirov et al. [8].

Recently, Ali et al. [3], established the following result.

Theorem 1.1. *Let φ be a positive and multiplicative convex function on interval $[\mathfrak{r}_1, \mathfrak{r}_2]$, hen the following double inequality holds*

$$\varphi\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right) \leq \left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \varphi(\varkappa)^{d\varkappa}\right)^{\frac{1}{\mathfrak{r}_2-\mathfrak{r}_1}} \leq \sqrt{\varphi(\mathfrak{r}_1)\,\varphi(\mathfrak{r}_2)}. \tag{1.2}$$

In [5] Ali et al. gave the following Ostrowski type inequalities for multiplicative convex functions.

Theorem 1.2. *For all multiplicative differentiable and positive map φ on $[\mathfrak{r}_1, \mathfrak{r}_2]$ with $\mathfrak{r}_1 < \mathfrak{r}_2$ satisfying $|\ln \varphi^*| \leq \ln M$, we have*

$$\left|(\varphi(\varkappa))\left\{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \varphi(u)^{du}\right\}^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}}\right| \leq M^{(\mathfrak{r}_2-\mathfrak{r}_1)\left(\frac{1}{4}+\frac{\left(\varkappa-\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)^2}{(\mathfrak{r}_2-\mathfrak{r}_1)^2}\right)}.$$

Theorem 1.3. *Let $\varphi : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}^+$ be a multiplicative differentiable map on $[\mathfrak{r}_1, \mathfrak{r}_2]$ with $\mathfrak{r}_1 < \mathfrak{r}_2$. If φ is increasing on $[\mathfrak{r}_1, \mathfrak{r}_2]$ and φ^* is multiplicative convex function on*

$[\mathfrak{r}_1, \mathfrak{r}_2]$, *then for all* $\varkappa \in [\mathfrak{r}_1, \mathfrak{r}_2]$ *we have the following inequality*

$$\left| (\varphi(\varkappa)) \left\{ \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \varphi(u)^{du} \right\}^{\frac{1}{\mathfrak{r}_1 - \mathfrak{r}_2}} \right|$$
$$\leq (\varphi^*(\mathfrak{r}_1))^{\frac{(\varkappa-\mathfrak{r}_1)^2}{2(\mathfrak{r}_2-\mathfrak{r}_1)} + \frac{(\mathfrak{r}_2-\varkappa)^3-(\varkappa-\mathfrak{r}_1)^3}{3(\mathfrak{r}_2-\mathfrak{r}_1)^2}} (\varphi^*(\mathfrak{r}_2))^{\frac{(\mathfrak{r}_2-\varkappa)^2}{2(\mathfrak{r}_2-\mathfrak{r}_1)} + \frac{(\varkappa-\mathfrak{r}_1)^3-(\mathfrak{r}_2-\varkappa)^3}{3(\mathfrak{r}_2-\mathfrak{r}_1)^2}}.$$

In the same paper the authors proved some Simpson type inequalities for multiplicative convex functions. In [4], Ali et al. studied the Hermite-Hadamard type inequalities for multiplicative ϕ-convex and log-ϕ-convex functions. Özcan [24] gave generalization of the Hermite-Hadamard inequality for h-convex functions. In [23], Özcan established the Hermite-Hadamard type inequalities for multiplicative peinvex functions. In [25], the author has discussed the Hermite-Hadamard type inequalities for multiplicative h-peinvex functions. Meftah [20], showed the Maclaurin type inequalities for multiplicative convex functions. Boulares et al. [9], gave the multiplicative Bullen type inequalities. Chasreechai et al. [11], studied Simpson and Newton type inequalities for multiplicative convex functions.

Recently, Abdeljawad and Grossman [1] introduced the multiplicative Riemann-Liouville fractional integrals as follows:

Definition 1.1. The relation with the multiplicative left and right Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ where $Re(\alpha) > 0$, and the left and right Riemann-Liouville fractional integral is as follows:

$$\left({}_{\mathfrak{r}_1}I_*^{\alpha}\varphi\right)(\varkappa) = e^{\left(J_{\mathfrak{r}_1^+}^{\alpha}(\ln \circ \varphi)\right)(\varkappa)}$$

and

$$\left({}_*I_{\mathfrak{r}_2}^{\alpha}\varphi\right)(\varkappa) = e^{\left(J_{\mathfrak{r}_2^-}^{\alpha}(\ln \circ \varphi)\right)(\varkappa)},$$

where $J_{\mathfrak{r}_1^+}^{\alpha}$ and $J_{\mathfrak{r}_2^-}^{\alpha}$ is the left and right Riemann-Liouville fractional integral, defined by

$$\left(J_{\mathfrak{r}_1^+}^{\alpha}\varphi\right)(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{r}_1}^{\xi} (\xi - \mu)^{\alpha-1} \varphi(\mu)\, d\mu, \ \mathfrak{r}_1 < \xi$$

and

$$\left(J_{\mathfrak{r}_2^-}^{\alpha}\varphi\right)(\xi)=\frac{1}{\Gamma(\alpha)}\int_{\xi}^{\mathfrak{r}_2}(\mu-\xi)^{\alpha-1}\varphi(\mu)\,d\mu,\ \xi<\mathfrak{r}_2.$$

The above relations can be considered as definitions of the multiplicative the Riemann-Liouville fractional integral by abuse of language.

Budak and Özçelik [10], used the above operator and prove some Hermite-Hadamard type inequalities for multiplicative fractional integrals. Moumen et al.[21], established the multiplicative Simpson type inequalitie.

Motivated by paper [10, 21] and some existing litarature, in this study we propose a new fractional identity for multiplicative differentiable functions, based on this identity we establish some fractional Ostrowski type inequalities for multiplicative convex functions. Some applications of the obtained results are provided at the end.

2. Preliminaries

In this section we begin by recalling some definitions, properties and notions of derivation as well as multiplicative integration.

Definition 2.1. [8] The multiplicative derivative of the function φ, where $\varphi:\mathbb{R}\to\mathbb{R}^+$, noted by φ^* is defined as follows:

$$\frac{d^*\varphi}{dt}=\varphi^*\left(\widetilde{\xi}\right)=\lim_{h\to0}\left\{\frac{\varphi\left(\widetilde{\xi}+h\right)}{\varphi\left(\widetilde{\xi}\right)}\right\}^{\frac{1}{h}}.$$

Remark 1. For positive and differentiable function φ we have the following relation

$$\varphi^*\left(\widetilde{\xi}\right)=e^{\left(\ln\varphi\left(\widetilde{\xi}\right)\right)'}=e^{\frac{\varphi'\left(\widetilde{\xi}\right)}{\varphi\left(\widetilde{\xi}\right)}},$$

where φ' is the ordinary derivative.

The multiplicative derivative admits the following properties:

Theorem 2.1. [8] *Let φ and ϑ be two multiplicative differentiable functions. Then functions $c\varphi,\varphi\vartheta,\varphi+\vartheta,\varphi/\vartheta$ and φ^{ϑ}, where c is an arbitrary constant, are * differentiable and we have*

- $(c\varphi)^*\left(\widetilde{\xi}\right)=\varphi^*\left(\widetilde{\xi}\right)$,

- $(\varphi\vartheta)^{*}\left(\widetilde{\xi}\right)=\varphi^{*}\left(\widetilde{\xi}\right)\vartheta^{*}\left(\widetilde{\xi}\right)$,
- $(\varphi+\vartheta)^{*}\left(\widetilde{\xi}\right)=\varphi^{*}\left(\widetilde{\xi}\right)^{\frac{\varphi(\widetilde{\xi})}{\varphi(\widetilde{\xi})+\vartheta(\widetilde{\xi})}}\vartheta^{*}\left(\mathfrak{t}\right)^{\frac{\vartheta(\widetilde{\xi})}{\varphi(\widetilde{\xi})+\vartheta(\widetilde{\xi})}}$,
- $\left(\frac{\varphi}{\vartheta}\right)^{*}\left(\widetilde{\xi}\right)=\frac{\varphi^{*}\left(\widetilde{\xi}\right)}{\vartheta^{*}\left(\widetilde{\xi}\right)}$,
- $\left(\varphi^{\vartheta}\right)^{*}\left(\widetilde{\xi}\right)=\varphi^{*}\left(\widetilde{\xi}\right)^{\vartheta\left(\widetilde{\xi}\right)}\varphi\left(\widetilde{\xi}\right)^{\vartheta'\left(\widetilde{\xi}\right)}$.

In [8], Bashirov et al. introduced the concept of the * integral called multiplicative integral which is written as $\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}$. It is clear that the sum in the classical Riemann integral of φ over $[\mathfrak{r}_1,\mathfrak{r}_2]$, is replaced in the multiplicative integral of φ over $[\mathfrak{r}_1,\mathfrak{r}_2]$ by the product. However, the product is represented by the raising to power.

The relationship between the Riemann integral and the multiplicative integral is as follows:

Proposition 2.1 ([8])**.** *If φ is Riemann integrable on $[\mathfrak{r}_1,\mathfrak{r}_2]$, then φ is multiplicative integrable on $[\mathfrak{r}_1,\mathfrak{r}_2]$ and*

$$\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}=\exp\left\{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\ln\left(\varphi\left(\widetilde{\xi}\right)\right)d\widetilde{\xi}\right\}.$$

Moreover, Bashirov et al. showed that multiplicative integral has the following results and properties:

Theorem 2.2. [8] *Let φ be a positive and Riemann integrable on $[\mathfrak{r}_1,\mathfrak{r}_2]$, then φ is multiplicative integrable on $[\mathfrak{r}_1,\mathfrak{r}_2]$ and*

- $\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\left(\varphi\left(\widetilde{\xi}\right)\right)^{p}\right)^{d\widetilde{\xi}}=\left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}\right)^{p}$,
- $\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\vartheta\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}=\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\vartheta\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}$,
- $\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\frac{\varphi\left(\widetilde{\xi}\right)}{\vartheta\left(\widetilde{\xi}\right)}\right)^{d\widetilde{\xi}}=\frac{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}}{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\vartheta\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}}$,
- $\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}=\int_{\mathfrak{r}_1}^{c}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}\int_{c}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}$, $\mathfrak{r}_1<c<\mathfrak{r}_2$,
- $\int_{\mathfrak{r}_1}^{\mathfrak{r}_1}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}=1$ and $\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}=\left(\int_{\mathfrak{r}_2}^{\mathfrak{r}_1}\left(\varphi\left(\widetilde{\xi}\right)\right)^{d\widetilde{\xi}}\right)^{-1}$.

Lemma 2.1. [8, Multiplicative Integration by Parts] *Let $\varphi : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}$ be multiplicative differentiable, let $\vartheta : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}$ be differentiable so the function φ^{ϑ} is multiplicative integrable, and*

$$\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \left(\varphi^* \left(\widetilde{\xi}\right)^{\vartheta\left(\widetilde{\xi}\right)}\right)^{d\widetilde{\xi}} = \frac{\varphi(\mathfrak{r}_2)^{\vartheta(\mathfrak{r}_2)}}{\varphi(\mathfrak{r}_1)^{\vartheta(\mathfrak{r}_1)}} \times \frac{1}{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \left(\varphi\left(\widetilde{\xi}\right)^{\vartheta'\left(\widetilde{\xi}\right)}\right)^{d\widetilde{\xi}}}.$$

Lemma 2.2. [5] *Let $\varphi : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}$ be multiplicative differentiable, let $\zeta : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}$ and let $\vartheta : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then we have*

$$\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \left(\varphi^* \left(\zeta\left(\widetilde{\xi}\right)\right)^{\zeta'(\mathfrak{t})\vartheta\left(\widetilde{\xi}\right)}\right)^{d\widetilde{\xi}} = \frac{\varphi(\zeta(\mathfrak{r}_2))^{\vartheta(\mathfrak{r}_2)}}{\varphi(\zeta(\mathfrak{r}_1))^{\vartheta(\mathfrak{r}_1)}} \times \frac{1}{\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \left(\varphi\left(\zeta\left(\widetilde{\xi}\right)\right)^{\vartheta'\left(\widetilde{\xi}\right)}\right)^{d\widetilde{\xi}}}.$$

Definition 2.2. [10] Let $\varphi : I \to [0, +\infty)$, $\varkappa, y \in I$ and $\mathfrak{t} \in [0, 1]$, if

$$\varphi\left(\mathfrak{t}\varkappa + (1-\mathfrak{t})\, y\right) \leq \left[\varphi\left(\varkappa\right)\right]^{\mathfrak{t}} \left[\varphi(y)\right]^{1-\mathfrak{t}}.$$

Then φ is called multiplicative convex or log-convex function.

3. Main results

We now list the following lemma that is necessary to reach the desired results.

Lemma 3.1. *Let $\varphi : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}^+$ be a multiplicative differentiable mapping on $[\mathfrak{r}_1, \mathfrak{r}_2]$ with $\mathfrak{r}_1 < \mathfrak{r}_2$. If φ^* is multiplicative integrable on $[\mathfrak{r}_1, \mathfrak{r}_2]$, then we have*

$$\left(\varphi\left(\varkappa\right)\right)^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha}+(\varkappa-\mathfrak{r}_1)^{\alpha}}{\mathfrak{r}_2-\mathfrak{r}_1}} \left(\left({}_{*}I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_1)\left({}_{*}I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}$$

$$= \left(\int_0^1 \left(\varphi^*\left((1-\mathfrak{t})\,\mathfrak{r}_1 + \mathfrak{t}\varkappa\right)^{\mathfrak{t}^{\alpha}}\right)^{dt}\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}} \left(\int_0^1 \left(\varphi^*\left((1-\mathfrak{t})\,\varkappa + \mathfrak{t}\mathfrak{r}_2\right)^{(1-\mathfrak{t})^{\alpha}}\right)^{dt}\right)^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}}.$$

Proof. Let

$$I_1 = \left(\int_0^1 \left(\varphi^*\left((1-\mathfrak{t})\,\mathfrak{r}_1 + \mathfrak{t}\varkappa\right)^{\mathfrak{t}^{\alpha}}\right)^{dt}\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}}$$

and

$$I_2 = \left(\int_0^1 \left(\varphi^*\left((1-\mathfrak{t})\,\varkappa + \mathfrak{t}\mathfrak{r}_2\right)^{(1-\mathfrak{t})^{\alpha}}\right)^{dt}\right)^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}}.$$

Using the integration by parts for multiplicative integrals and Lemma 2.2, I_1 gives

$$
\begin{aligned}
I_1 &= \left(\int_0^1 \left(\varphi^* \left((1-\mathfrak{t}) \mathfrak{r}_1 + \mathfrak{t}\varkappa \right)^{\mathfrak{t}^\alpha} \right)^{dt} \right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}} \\
&= \left(\int_0^1 \left(\varphi^* \left((1-\mathfrak{t}) \mathfrak{r}_1 + \mathfrak{t}\varkappa \right)^{(\varkappa-\mathfrak{r}_1)\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}\mathfrak{t}^\alpha} \right)^{dt} \right) \\
&= \frac{(\varphi(\varkappa))^{\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}}{1} . \frac{1}{\int_0^1 \left(\varphi(((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa))^{\frac{\alpha(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}\mathfrak{t}^{\alpha-1}} \right)^{dt}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} \frac{1}{\exp\left\{ \int_0^1 \frac{\alpha(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1} \mathfrak{t}^{\alpha-1} \ln(\varphi((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa)) dt \right\}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} \frac{1}{\exp\left\{ \frac{\Gamma(\alpha+1)}{\mathfrak{r}_2-\mathfrak{r}_1} \left(\frac{1}{\Gamma(\alpha)} \int_{\mathfrak{r}_1}^{\varkappa} (u-\mathfrak{r}_1)^{\alpha-1} \ln(\varphi(u)) du \right) \right\}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} \left(\exp\left\{ \left(\frac{1}{\Gamma(\alpha)} \int_{\mathfrak{r}_1}^{\varkappa} (u-\mathfrak{r}_1)^{\alpha-1} \ln\left(\varphi\left(u \right) \right) du \right) \right\} \right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} \left(\left({}_{*}I_{\varkappa}^{\alpha}\varphi \right) (\mathfrak{r}_1) \right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}} .
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
I_2 &= \left(\int_0^1 \left(\varphi^* \left((1-\mathfrak{t}) \varkappa + \mathfrak{t}\mathfrak{r}_2 \right)^{(1-\mathfrak{t})^\alpha} \right)^{dt} \right)^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}} \\
&= \left(\int_0^1 \left(\varphi^* \left((1-\mathfrak{t}) \varkappa + \mathfrak{t}\mathfrak{r}_2 \right)^{-(\mathfrak{r}_2-\varkappa)\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}(1-\mathfrak{t})^\alpha} \right)^{dt} \right) \\
&= \frac{1}{(\varphi(\varkappa))^{-\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}} . \frac{1}{\int_0^1 \left(\varphi((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2)^{\frac{\alpha(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}(1-\mathfrak{t})^{\alpha-1}} \right)^{dt}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} . \frac{1}{\exp\left\{ \frac{\alpha(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1} \int_0^1 (1-\mathfrak{t})^{\alpha-1} \ln \varphi((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2) dt \right\}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} . \frac{1}{\exp\left\{ \frac{\alpha(\mathfrak{r}_2-\varkappa)^{\alpha-1}}{\mathfrak{r}_2-\mathfrak{r}_1} \int_{\varkappa}^{\mathfrak{r}_2} \left(1-\frac{u-\varkappa}{\mathfrak{r}_2-\varkappa} \right)^{\alpha-1} \ln \varphi(u) du \right\}} \\
&= \left(\varphi\left(\varkappa \right) \right)^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} . \frac{1}{\exp\left\{ \frac{\Gamma(\alpha+1)}{\mathfrak{r}_2-\mathfrak{r}_1} \left(\frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\mathfrak{r}_2} (\mathfrak{r}_2-u)^{\alpha-1} \ln \varphi(u) du \right) \right\}}
\end{aligned}
$$

$$= (\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} . \left(\exp\left\{\left(\frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\mathfrak{r}_2}(\mathfrak{r}_2-u)^{\alpha-1}\ln\varphi(u)\,du\right)\right\}\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}$$
$$= (\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} . \left(\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}} .$$

Multiplying above equalities, we get

$$I_1\times I_2 = (\varphi(\varkappa))^{\frac{(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}\left(\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_1)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}(\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}} . \left(\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}$$
$$= (\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha+(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}\left(\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_1)\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}} .$$

Desired result. □

Theorem 3.1. *Let $\varphi : [\mathfrak{r}_1, \mathfrak{r}_2] \to \mathbb{R}^+$ be a multiplicative differentiable mapping on $[\mathfrak{r}_1, \mathfrak{r}_2]$ with $\mathfrak{r}_1 < \mathfrak{r}_2$. If $|\ln\varphi^*| \le \ln\mathcal{M}$ on $[\mathfrak{r}_1, \mathfrak{r}_2]$, then we have*

$$\left|(\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha+(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}\left(\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_1)\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right| \le \mathcal{M}^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}+(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\alpha+1)(\mathfrak{r}_2-\mathfrak{r}_1)}} .$$

Proof. From Lemma 3.1, properties of multiplicative integral and using the fact that $|\ln f^*| \le \ln\mathcal{M}$, we get

$$\left|(\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha+(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}\left(\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_1)\left({}_*I_{\varkappa}^{\alpha}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|$$
$$= \left|\left(\int_0^1\left(\varphi^*((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa)^{\mathfrak{t}^\alpha}\right)^{d\mathfrak{t}}\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}}\right|$$
$$\times\left|\left(\int_0^1\left(\varphi^*((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2)^{(1-\mathfrak{t})^\alpha}\right)^{d\mathfrak{t}}\right)^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}}\right|$$
$$= \left|\left(\int_0^1\left|(\varphi^*((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa))^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}\mathfrak{t}^\alpha}\right|^{d\mathfrak{t}}\right)\right|$$
$$\times\left|\left(\int_0^1\left(\left|(\varphi^*((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2))^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}(1-\mathfrak{t})^\alpha}\right|\right)^{d\mathfrak{t}}\right)\right|$$
$$\le \left(\exp\left\{\int_0^1\left|\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}\mathfrak{t}^\alpha\ln(\varphi^*((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa))\right|d\mathfrak{t}\right\}\right)$$

$$\times\left(\exp\left\{\int_0^1\left|-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}(1-\mathfrak{t})^{\alpha}\ln\left(\varphi^*\left((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2\right)\right)\right|d\mathfrak{t}\right\}\right)$$
$$=\left(\exp\left\{\int_0^1\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}\mathfrak{t}^{\alpha}\left|\ln\left(\varphi^*\left((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa\right)\right)\right|d\mathfrak{t}\right\}\right)$$
$$\times\left(\exp\left\{\int_0^1\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}(1-\mathfrak{t})^{\alpha}\left|\ln\left(\varphi^*\left((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2\right)\right)\right|d\mathfrak{t}\right\}\right)$$
$$\leq\left(\exp\left\{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}\ln\mathcal{M}\int_0^1\mathfrak{t}^{\alpha}d\mathfrak{t}\right\}\right)\left(\exp\left\{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\mathfrak{r}_2-\mathfrak{r}_1)}\ln\mathcal{M}\int_0^1(1-\mathfrak{t})^{\alpha}d\mathfrak{t}\right\}\right)$$
$$=\left(\exp\left\{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\alpha+1)(\mathfrak{r}_2-\mathfrak{r}_1)}\ln\mathcal{M}\right\}\right)\left(\exp\left\{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\mathfrak{r}_2-\mathfrak{r}_1)}\ln\mathcal{M}\right\}\right)$$
$$=\left(\exp\left\{\ln\mathcal{M}^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\alpha+1)(\mathfrak{r}_2-\mathfrak{r}_1)}}\right\}\right)\left(\exp\left\{\ln\mathcal{M}^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\mathfrak{r}_2-\mathfrak{r}_1)}}\right\}\right)$$
$$=\mathcal{M}^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}+(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\mathfrak{r}_2-\mathfrak{r}_1)}}.$$

The proof is completed. □

Remark 2. Theorem 3.1 will be reduced to Theorem 1 from [5], if we take $\alpha=1$.

Corollary 3.1. *In Theorem 3.1, if we choose* $\varkappa=\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}$, *we obtain*

$$\left|\left(\varphi\left(\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\right)^{\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\alpha-1}}{2^{\alpha-1}}}\left(\left({}_{*}I^{\alpha}_{\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}}\varphi\right)(\mathfrak{r}_1)\left({}_{*}I^{\alpha}_{\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|\leq\mathcal{M}^{\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\alpha}}{2^{\alpha}(\alpha+1)}}.$$

Corollary 3.2. *In Corollary 3.1, if we take* $\alpha=1$, *we obtain*

$$\left|\varphi\left(\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\varphi(u)^{du}\right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|\leq\mathcal{M}^{\frac{\mathfrak{r}_2-\mathfrak{r}_1}{4}}.$$

Theorem 3.2. *Let* $\varphi:[\mathfrak{r}_1,\mathfrak{r}_2]\to\mathbb{R}^+$ *be a multiplicative differentiable mapping on* $[\mathfrak{r}_1,\mathfrak{r}_2]$ *with* $\mathfrak{r}_1<\mathfrak{r}_2$. *If* φ^* *is multiplicative convex function on* $[\mathfrak{r}_1,\mathfrak{r}_2]$, *then we have*

$$\left|(\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha}+(\varkappa-\mathfrak{r}_1)^{\alpha}}{\mathfrak{r}_2-\mathfrak{r}_1}}\left(\left({}_{*}I^{\alpha}_{\varkappa}\varphi\right)(\mathfrak{r}_1)\left({}_{*}I^{\alpha}_{\varkappa}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|$$
$$\leq(\varphi^*(\mathfrak{r}_1))^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\alpha+1)(\alpha+2)(\mathfrak{r}_2-\mathfrak{r}_1)}}(\varphi^*(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}+(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\alpha+2)(\mathfrak{r}_2-\mathfrak{r}_1)}}(\varphi^*(\mathfrak{r}_2))^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\alpha+2)(\mathfrak{r}_2-\mathfrak{r}_1)}}.$$

Proof. From Lemma 3.1, properties of multiplicative integral and the multiplicative convexity of φ^*, we have

$$\begin{aligned}
&\left|(\varphi(\varkappa))^{\frac{(\mathfrak{r}_2-\varkappa)^\alpha+(\varkappa-\mathfrak{r}_1)^\alpha}{\mathfrak{r}_2-\mathfrak{r}_1}}\left(\left({}_*I_\varkappa^\alpha\varphi\right)(\mathfrak{r}_1)\left({}_*I_\varkappa^\alpha\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right| \\
&= \left|\left(\int_0^1\left(\varphi^*\left((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa\right)^{\mathfrak{t}^\alpha}\right)^{d\mathfrak{t}}\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}}\right| \\
&\quad\times\left|\left(\int_0^1\left(\varphi^*\left((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2\right)^{(1-\mathfrak{t})^\alpha}\right)^{d\mathfrak{t}}\right)^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}}\right| \\
&= \left|\left(\int_0^1\left|\left(\varphi^*\left((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa\right)\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\mathfrak{t}^\alpha}\right|^{d\mathfrak{t}}\right)\right| \\
&\quad\times\left|\left(\int_0^1\left(\left|\left(\varphi^*\left((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2\right)\right)^{-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}(1-\mathfrak{t})^\alpha}\right|\right)^{d\mathfrak{t}}\right)\right| \\
&\le \left(\exp\left\{\int_0^1\left|\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\mathfrak{t}^\alpha\ln\left(\varphi^*\left((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa\right)\right)\right|d\mathfrak{t}\right\}\right) \\
&\quad\times\left(\exp\left\{\int_0^1\left|-\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}(1-\mathfrak{t})^\alpha\ln\left(\varphi^*\left((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2\right)\right)\right|d\mathfrak{t}\right\}\right) \\
&= \left(\exp\left\{\int_0^1\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\mathfrak{t}^\alpha\left|\ln\left(\varphi^*\left((1-\mathfrak{t})\mathfrak{r}_1+\mathfrak{t}\varkappa\right)\right)\right|d\mathfrak{t}\right\}\right) \\
&\quad\times\left(\exp\left\{\int_0^1\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}(1-\mathfrak{t})^\alpha\left|\ln\left(\varphi^*\left((1-\mathfrak{t})\varkappa+\mathfrak{t}\mathfrak{r}_2\right)\right)\right|d\mathfrak{t}\right\}\right) \\
&\le \left(\exp\left\{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\int_0^1\mathfrak{t}^\alpha\left|\ln\left(\varphi^*(\mathfrak{r}_1)\right)^{(1-\mathfrak{t})}\left(f^*(\varkappa)\right)^{\mathfrak{t}}\right|d\mathfrak{t}\right\}\right) \\
&\quad\times\left(\exp\left\{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\int_0^1(1-\mathfrak{t})^\alpha\left|\ln\left(\varphi^*(\varkappa)\right)^{(1-\mathfrak{t})}\left(f^*(\mathfrak{r}_2)\right)^{\mathfrak{t}}\right|d\mathfrak{t}\right\}\right) \\
&= \left(\exp\left\{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\int_0^1\mathfrak{t}^\alpha\left((1-\mathfrak{t})\ln\left(\varphi^*(\mathfrak{r}_1)\right)+\mathfrak{t}\ln\left(\varphi^*(\varkappa)\right)\right)d\mathfrak{t}\right\}\right)
\end{aligned}$$

$$\times\left(\exp\left\{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\int_0^1(1-\mathfrak{t})^{\alpha}\left((1-\mathfrak{t})\ln\left(\varphi^*(\varkappa)\right)+\mathfrak{t}\ln\left(\varphi^*(\mathfrak{r}_2)\right)\right)d\mathfrak{t}\right\}\right)$$

$$=\left(\exp\left\{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\left(\ln\left(\varphi^*(\mathfrak{r}_1)\right)\int_0^1\mathfrak{t}^{\alpha}(1-\mathfrak{t})\,d\mathfrak{t}+\ln\left(\varphi^*(\varkappa)\right)\int_0^1\mathfrak{t}^{\alpha+1}d\mathfrak{t}\right)\right\}\right)$$

$$\times\left(\exp\left\{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\left(\ln\left(\varphi^*(\varkappa)\right)\int_0^1(1-\mathfrak{t})^{\alpha+1}d\mathfrak{t}+\ln\left(\varphi^*(\mathfrak{r}_2)\right)\int_0^1(1-\mathfrak{t})^{\alpha}\mathfrak{t}d\mathfrak{t}\right)\right\}\right)$$

$$=\left(\exp\left\{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\left(\frac{1}{(\alpha+1)(\alpha+2)}\ln\left(\varphi^*(\mathfrak{r}_1)\right)+\frac{1}{\alpha+2}\ln\left(\varphi^*(\varkappa)\right)\right)\right\}\right)$$

$$\times\left(\exp\left\{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{\mathfrak{r}_2-\mathfrak{r}_1}\left(\frac{1}{\alpha+2}\ln\left(\varphi^*(\varkappa)\right)+\frac{1}{(\alpha+1)(\alpha+2)}\ln\left(\varphi^*(\mathfrak{r}_2)\right)\right)\right\}\right)$$

$$=\left(\varphi^*(\mathfrak{r}_1)\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}}{(\alpha+1)(\alpha+2)(\mathfrak{r}_2-\mathfrak{r}_1)}}\left(\varphi^*(\varkappa)\right)^{\frac{(\varkappa-\mathfrak{r}_1)^{\alpha+1}+(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\alpha+2)(\mathfrak{r}_2-\mathfrak{r}_1)}}\left(\varphi^*(\mathfrak{r}_2)\right)^{\frac{(\mathfrak{r}_2-\varkappa)^{\alpha+1}}{(\alpha+1)(\alpha+2)(\mathfrak{r}_2-\mathfrak{r}_1)}}.$$

The proof is completed. □

Corollary 3.3. *In Theorem 3.2, if we choose* $\varkappa=\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}$, *we obtain*

$$\left|\left(\varphi\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\right)^{\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\alpha-1}}{2^{\alpha-1}}}\left(\left({}_*I^{\alpha}_{\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}}\varphi\right)(\mathfrak{r}_1)\left({}_*I^{\alpha}_{\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|$$

$$\leq\left(\left(\varphi^*(\mathfrak{r}_1)\right)\left(\varphi^*\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\right)^2\left(\varphi^*(\mathfrak{r}_2)\right)\right)^{\frac{(\mathfrak{r}_1-\mathfrak{r}_2)^{\alpha}}{2^{\alpha+1}(\alpha+1)(\alpha+2)}}.$$

Corollary 3.4. *In Corollary 3.3, using the multiplicative convexity of* φ^* *i.e.* $f^*\left(\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\leq\sqrt{\varphi^*(\mathfrak{r}_1)\varphi^*(\mathfrak{r}_2)}$, *we obtain*

$$\left|\left(\varphi\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\right)^{\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\alpha-1}}{2^{\alpha-1}}}\left(\left({}_*I^{\alpha}_{\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}}\varphi\right)(\mathfrak{r}_1)\left({}_*I^{\alpha}_{\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}}\varphi\right)(\mathfrak{r}_2)\right)^{\frac{\Gamma(\alpha+1)}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|$$

$$\leq\left(\left(\varphi^*(\mathfrak{r}_1)\right)\left(\varphi^*(\mathfrak{r}_2)\right)\right)^{\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^{\alpha}}{2^{\alpha}(\alpha+1)(\alpha+2)}}.$$

Corollary 3.5. *In Theorem 3.2, if we take* $\alpha=1$, *we obtain*

$$\left|\varphi(\varkappa)\left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\varphi(u)^{du}\right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}}\right|$$

$$\leq\left(\varphi^*(\mathfrak{r}_1)\right)^{\frac{(\varkappa-\mathfrak{r}_1)^2}{6(\mathfrak{r}_2-\mathfrak{r}_1)}}\left(\varphi^*(\varkappa)\right)^{\frac{(\varkappa-\mathfrak{r}_1)^2+(\mathfrak{r}_2-\varkappa)^2}{3(\mathfrak{r}_2-\mathfrak{r}_1)}}\left(\varphi^*(\mathfrak{r}_2)\right)^{\frac{(\mathfrak{r}_2-\varkappa)^2}{6(\mathfrak{r}_2-\mathfrak{r}_1)}}.$$

Corollary 3.6. *In Corollary 3.5, if we choose* $\varkappa = \frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}$, *we obtain*

$$\left| \varphi\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right) \left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \varphi(u)^{du} \right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}} \right| \leq \left(\left(\varphi^*(\mathfrak{r}_1)\right) \left(\varphi^*\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right)\right)^4 \left(\varphi^*(\mathfrak{r}_2)\right) \right)^{\frac{\mathfrak{r}_2-\mathfrak{r}_1}{24}}.$$

Corollary 3.7. *In Corollary 3.6, using the multiplicative convexity of* f^*, *we get*

$$\left| \varphi\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right) \left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \varphi(u)^{du} \right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}} \right| \leq \left(\left(\varphi^*(\mathfrak{r}_1)\right) \left(\varphi^*(\mathfrak{r}_2)\right) \right)^{\frac{\mathfrak{r}_2-\mathfrak{r}_1}{8}}.$$

4. Examples

In this section, we assume some particular functions and give mathematical examples to show the validation of the newly established inequalities.

Example 4.1. *We consider the function* $\varphi(u) = u^3$ *and from Corollary 3.2 for the interval* $[2, 4]$

$$\left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \varphi(u)^{du} \right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}} = \left(\int_2^4 \left(u^3\right)^{du} \right)^{-\frac{1}{2}} = \left(e^{\int_2^4 \ln u^3 du} \right)^{-\frac{1}{2}} \simeq \left(e^{6.4758}\right)^{-\frac{1}{2}} \simeq 0.0392$$

and

$$\varphi\left(\tfrac{\mathfrak{r}_1+\mathfrak{r}_2}{2}\right) = \varphi(3) = 27.$$

Then for the left hand side of the inequality given in Corollary 3.2, we have

$$(\varphi(3)) \left(\int_2^4 \varphi(u)^{du} \right)^{-\frac{1}{2}} = 1.0584.$$

Clearly, $\varphi^*(u) = e^{(3\ln u)'} = e^{\frac{3}{u}}$. *Since* $2 \leq u \leq 4$. *Then* $\frac{3}{4} \leq \frac{3}{u} \leq \frac{3}{2}$. *So we can choose* $\mathcal{M} = \sup_{u\in[2,4]} \varphi^*(u) = e^{\frac{3}{2}} = 4.8116$, *which gives* $\mathcal{M}^{\frac{1}{2}} = 2.1935$.

Thus, the inequality given in Corollary 3.2 is valid.

Example 4.2. *We consider the function* $\varphi(u) = u^3$ *and from Corollary 3.7 for the interval* $[2, 4]$, *the left hand side of the inequality given in Corollary 3.7, gives*

$$(\varphi(3)) \left(\int_2^4 \varphi(u)^{du} \right)^{-\frac{1}{2}} = 1.0584.$$

Since $\varphi^{*}(u)=e^{\frac{3}{u}}$*, then we have* $\varphi^{*}(2)=4.8116=$ *and* $\varphi^{*}(4)=2.1170$*. So the right hand side of the inequality given in Corollary 3.7* $\left((\varphi^{*}(2))(\varphi^{*}(4))\right)^{\frac{1}{4}}=(10.1861)^{\frac{1}{4}}=1.7864$.

Thus, the inequality given in Corollary 3.7 is valid.

5. Applications

The Arithmetic mean: $A(\mathfrak{r}_1,\mathfrak{r}_2)=\frac{\mathfrak{r}_1+\mathfrak{r}_2}{2}$.

The logarithmic means: $L(\mathfrak{r}_1,\mathfrak{r}_2)=\frac{\mathfrak{r}_2-\mathfrak{r}_1}{\ln\mathfrak{r}_2-\ln\mathfrak{r}_1}$, $\mathfrak{r}_1,\mathfrak{r}_2>0$ and $\mathfrak{r}_1\neq\mathfrak{r}_2$.

The p-Logarithmic mean: $L_p(\mathfrak{r}_1,\mathfrak{r}_2)=\left(\frac{\mathfrak{r}_2^{p+1}-\mathfrak{r}_1^{p+1}}{(p+1)(\mathfrak{r}_2-\mathfrak{r}_1)}\right)^{\frac{1}{p}}$, $\mathfrak{r}_1,\mathfrak{r}_2>0,\mathfrak{r}_1\neq\mathfrak{r}_2$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 5.1. *Let* $\mathfrak{r}_1,\mathfrak{r}_2\in\mathbb{R}$ *with* $0<\mathfrak{r}_1<\mathfrak{r}_2$*, then we have*

$$e^{A^{-1}(\mathfrak{r}_1,\mathfrak{r}_2)-L^{-1}(\mathfrak{r}_1,\mathfrak{r}_2)}\leq e^{-\frac{\mathfrak{r}_2-\mathfrak{r}_1}{4\mathfrak{r}_2^2}}.$$

Proof. To confirm that from Corollary 3.2 applied to the function $\varphi(\mathfrak{t})=e^{\frac{1}{\mathfrak{t}}}$ whose $\varphi^{*}(\mathfrak{t})=e^{-\frac{1}{\mathfrak{t}^2}}$, $\mathcal{M}=e^{-\frac{1}{b^2}}$ and $\left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\varphi(u)^{du}\right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}}=\exp\{-L^{-1}(\mathfrak{r}_1,\mathfrak{r}_2)\}$. □

Proposition 5.2. *Let* $\mathfrak{r}_1,\mathfrak{r}_2\in\mathbb{R}$ *with* $0<\mathfrak{r}_1<\mathfrak{r}_2$*, then we have*

$$e^{A^{p}\left(\frac{3}{2}\mathfrak{r}_1,\frac{1}{2}\mathfrak{r}_2\right)-L_p^{p}(\mathfrak{r}_1,\mathfrak{r}_2)}\leq\left(e^{\mathfrak{r}_1^{p-1}+20\left(\frac{3\mathfrak{r}_1+\mathfrak{r}_2}{4}\right)^{p-1}+9\mathfrak{r}_2^{p-1}}\right)^{p\frac{\mathfrak{r}_2-\mathfrak{r}_1}{96}}.$$

Proof. To confirm that from Corollary 3.5 by trying $\varkappa=\frac{3\mathfrak{r}_1+\mathfrak{r}_2}{4}$, applied to the function $\varphi(\mathfrak{t})=e^{\mathfrak{t}^p}$ with $p\geq 2$ whose $\varphi^{*}(\mathfrak{t})=e^{p\mathfrak{t}^{p-1}}$ and $\left(\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\varphi(u)^{du}\right)^{\frac{1}{\mathfrak{r}_1-\mathfrak{r}_2}}=\exp\left\{-L_p^{p}(\mathfrak{r}_1,\mathfrak{r}_2)\right\}$. □

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