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ON TADES OF TRANSFORMED TREE AND PATH RELATED GRAPHS

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Abstract. Given a graph G. Consider a total labeling $\xi : V \bigcup E \to \{1, 2, \ldots, k\}$. Let $e = xy$ and $f = uv$ be any two different edges of G. Let $wt(e) \neq wt(f)$ where $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. Then ξ is said to be edge irregular total absolute difference k-labeling of G. Then the total absolute difference edge irregularity strength of G, $tades(G)$, is the least number k such that there is an edge irregular total absolute difference k-labeling for G. Here, we study the $tades(G)$ of T_p-tree and path related graphs.

1. Introduction

All the graphs considered here are finite, simple and undirected. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ so that the order and size of G are $|V(G)|$ and $|E(G)|$ respectively. The numerous concepts that emerge when studying graph theory, which has received great interest, particularly in graph labeling, the labeling of graphs provides mathematical models with value for a wide variety of applications in technology (astronomy, cryptography, data security, telecommunication networks, coding theory, etc.). Consider the total labeling $\xi : V \bigcup E \to \{1, 2, \ldots, k\}$ where $wt(uv) = \xi(u) + \xi(uv) + \xi(v)$ and all the edges have distinct weights. Then ξ is total edge irregular k-labeling of G. The total edge irregularity strength, $tes(G)$, of G is the least number k for which we can construct a total edge irregular k-labeling. It was introduced by Baca et al. [1]. To know more about $tes(G)$, the reader can go through [3, 4, 14, 15, 16, 18, 19]

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Ramalakshmi and Kathiresan [17] introduced the concept of total absolute difference edge irregularity strength of graphs to reduce the edge weights. Consider a total labeling $\xi : V \bigcup E \to \{1, 2, \ldots, k\}$. Let $e = xy$ and $f = uv$ be any two different edges of G. Let $wt(e) \neq wt(f)$ where $wt(e) = |\xi(e) - \xi(x) - \xi(y)|$. Then ξ is said to be edge irregular total absolute difference k-labeling of G. Then the total absolute difference edge irregularity strength of G, $tades(G)$, is the least number k such that there is a graph G with edge irregular total absolute difference k-labeling.

Theorem 1.1. [17] The $tades(G)$ satisfies $\left\lceil \frac{|E|}{2} \right\rceil \leq tades(G) \leq |E| + 1$.

Lourdusamy et al. [7] have computed the $tades(G)$ for triangular snake, quadrilateral snake, helm, closed helm, web graph, flower graph, gear graph, lotus inside the circle and double fan graph. Also, they have obtained the $tades$ of T_p-tree graphs like $T\widehat{O}P_n$, $T\widehat{O}K_{1,n}$, $T\widehat{O}C_n$ and $T \odot nK_1$ in [8]. Lourdusamy et al. [9] discussed the $tades(G)$ for super subdivision of comb, super subdivision of bistar, super subdivision of ladder, $P_n \odot mK_1$, $L_n \odot mK_1$, zigzag graph and grid graph. Also they have obtained the $tades$ of staircase graph, disjoint union of grid graph and disjoint union of zigzag graph in [10].

Definition 1.1. [2] Consider a tree T with two adjacent vertices u_0 and v_0. Assume that there are two pendant vertices u and v in T with the property that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. An elementary parallel transformation (ept) is defined as the removal of the edge u_0v_0 from T and adding the edge uv in T. Here the edge u_0v_0 is called transformable edge.

If T can be transformed to a path by a sequence of ept's, then T is called a T_p-tree (transformed tree) and the sequence of ept's is a composition of mappings (ept's) denoted by P which is called a parallel transformation of T. Here $P(T)$ is the path which is nothing but the image of T under P.

Definition 1.2. [13] Assume G_1 and G_2 be two graphs. A graph $G_1\widehat{O}G_2$ is derived from G_1 and $|V(G_1)|$ copies of G_2 with the operation that one vertex of i^{th} copy of G_2 is identifying with i^{th} vertex of G_1.

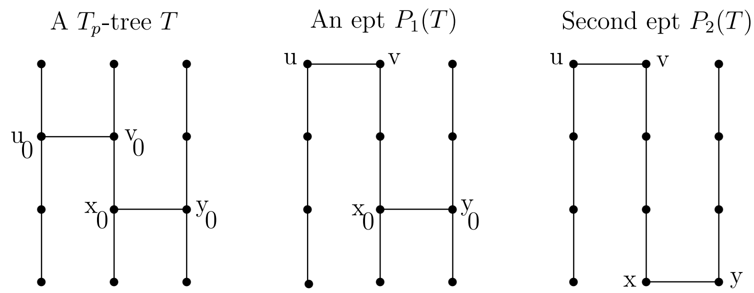

FIGURE 1. A T_p-tree and a sequence of two ept's reducing it to a path

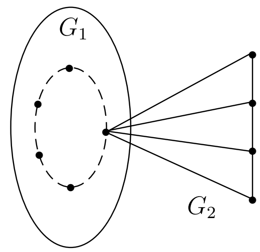

FIGURE 2. $G_1 \widehat{O} G_2$

Definition 1.3. [12] An armed crown is a cycle attached with paths of equal length at each vertex of the cycle. It is denoted by $C_m \ominus P_n$ is a path of length $n-1$.

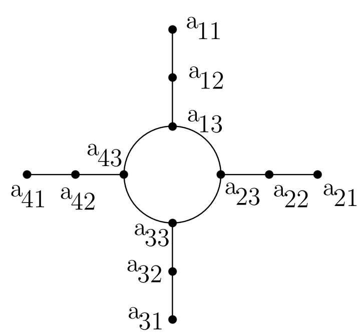

FIGURE 3. $C_4 \ominus P_4$

Definition 1.4. [12] A quadrilateral snake Q_n is obtained from a path $v_1, v_2, \cdots, v_n$ by joining v_i, v_{i+1} to new vertices u_i, w_i for every $i = 1, 2, \cdots, n-1$ respectively and then joining u_i and w_i. That is every edge of the path is replaced by a cycle C_4.

Definition 1.5. [12] Duplication of a vertex v_k by a new edge $e = u_k w_k$ in a graph G produces a new graph G' such that $N(u_k) \cap N(w_k) = v_k$.

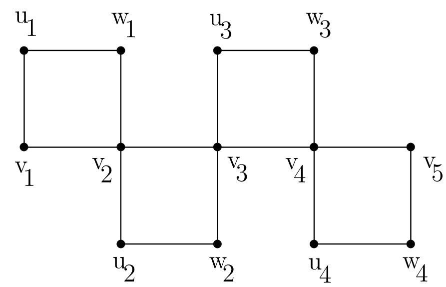

FIGURE 4. Q_5

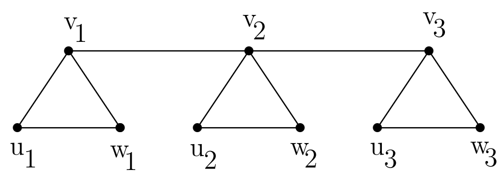

FIGURE 5. Duplication of a vertex by an edge

Definition 1.6. [11] Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

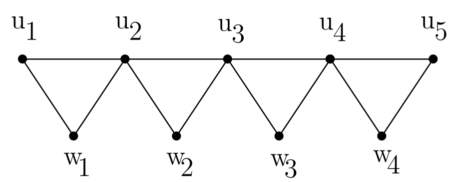

FIGURE 6. Duplication of an edge by a vertex

Definition 1.7. [5] The key graph is a graph obtained from K_2 by appending one vertex of C_m to one end point and comb graph $P_n \odot K_1$ to the other end of K_2. It is denoted as $KY(m, n)$.

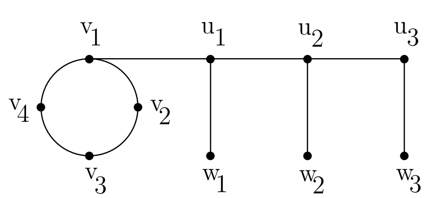

FIGURE 7. $KY(4, 3)$

Definition 1.8. [6] The H-graph of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$ by joining the vertices if $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ by an edge if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ by an edge if n is even.

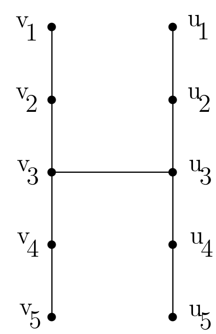

FIGURE 8. H graph

2. Main Results

In this section, we discuss the total absolute difference edge irregularity strength of T_p-tree related graphs and H graph.

Theorem 2.1. Let m be an even integer. Let T be a T_p-tree on m vertices. Then $tades(T) = \frac{m}{2}$.

Proof. Let T be a T_p-tree T on even m vertices. We can find a parallel transformation P of T which will satisfy the following
(i) $V(P(T)) = V(T)$
(ii) $E(P(T)) = (E(T) - E_d) \bigcup E_p$.
Here $P(T)$ is the path; E_d is a collection of edges removed from T; The E_p is a collection of edges newly introduced by the sequence $P = (P_1, P_2, \cdots, P_k)$ of *epts* P that have been used to reach path $P(T)$. Obviously, E_d and E_p have the same number of edges. We use the label $\alpha_1, \alpha_2, \cdots, \alpha_m$ successively beginning at a pendant vertex of $P(T)$ and proceeding to the right up to the other pendant vertex to write the vertices of $P(T)$.

By Theorem 1.1, we have $tades(T) \geq \frac{m}{2}$. Let us now prove the converse part. Define $\xi : V \bigcup E \rightarrow \{1, 2, 3, \ldots, \frac{m}{2}\}$ as follows:

For $1 \leq r \leq m$ $\xi(\alpha_r) = \begin{cases} \frac{r+1}{2} & \text{if } r \text{ is odd} \\ \frac{r}{2} & \text{if } r \text{ is even; } . \end{cases}$

$$\xi(\alpha_r\alpha_{r+1}) = 2,\ 1 \leq r \leq m-1.$$

For $1 \leq r < s \leq m$, $\alpha_r\alpha_s$ be a transformed edge in T. Consider P_1 to be the *ept* obtained by removing $\alpha_r\alpha_s$ and including $\alpha_{r+t}\alpha_{s-t}$ where $t = d(\alpha_r, \alpha_{r+t}) = d(\alpha_s, \alpha_{s-t})$. Take P as a parallel transformation of T where P_1 as one of the constituent *epts*. Obviously the edge $\alpha_{r+t}\alpha_{s-t}$ is in $P(T)$. So $r+t+1 = s-t$ and thus $s = r+2t+1$. Clearly, s and t have opposite parity.

The weight of $\alpha_r\alpha_s$ is

$$\begin{aligned} wt(\alpha_r\alpha_s) &= wt(\alpha_r\alpha_{r+2t+1}) \\ &= |\xi(\alpha_r\alpha_{r+2t+1}) - \xi(\alpha_r) - \xi(\alpha_{r+2t+1})| \\ &= r+t-1. \end{aligned}$$

The weight of $\alpha_{r+t}\alpha_{s-t}$ is

$$\begin{aligned} wt(\alpha_{r+t}\alpha_{s-t}) &= wt(\alpha_{r+t}\alpha_{r+t+1}) \\ &= |\xi(\alpha_{r+t}\alpha_{r+t+1}) - \xi(\alpha_{r+t}) - \xi(\alpha_{r+t+1})| \\ &= r+t-1. \end{aligned}$$

The above argument implies that $wt(\alpha_r\alpha_s) = wt(\alpha_{r+t}\alpha_{s-t})$.

The edge weight is

$$wt(\alpha_r\alpha_{r+1}) = r-1,\ 1 \leq r \leq m-1;$$

So, $tades(T) \leq \frac{m}{2}$. Note that the edge weights are distinct. Hence $tades(T) = \frac{m}{2}$. □

Theorem 2.2. For a T_p-tree T on m vertices, we have $tades(T\widehat{O}Q_n) = \lceil \frac{4mn+m-1}{2} \rceil$.

Proof. Consider a T_p-tree with m vertices. Then there is a parallel transformation P in T,

(*i*) $V(P(T)) = V(T)$

(*ii*) $E(P(T)) = (E(T) - E_d) \bigcup E_p$.

Here E_d is the collection of edges deleted from T; E_p is the collection of edges newly added using the sequence $P = (P_1, P_2, \cdots, P_k)$ of the *epts* P which have been used to form $P(T)$. Obviously, we have the same number of edges for E_d and E_p. We take $b_1, b_2, \cdots, b_m$ successively beginning at a pendant vertex of $P(T)$ and ending at other pendant vertex as the vertices of $P(T)$. Let $a_1^s, a_2^s, \cdots, a_n^s, a_{n+1}^s (1 \leq s \leq m)$ be the vertices of s^{th} copy of Q_n with $a_{n+1}^s = b_s$. Then $V(T\widehat{O}Q_n) = \{a_r^s : 1 \leq r \leq n+1, 1 \leq s \leq m\} \bigcup \{x_r^s, y_r^s : 1 \leq r \leq n, 1 \leq s \leq m\}$ and $E(T\widehat{O}Q_n) = E(T) \bigcup E(Q_n)$. Note that $\left|V(T\widehat{O}Q_n)\right| = 3nm + m$ and $\left|E(T\widehat{O}Q_n)\right| = 4mn + m - 1$.

By Theorem 1.1, we have $tades(T\widehat{O}Q_n) \geq \lceil \frac{4mn+m-1}{2} \rceil$. For the reverse inequality, we show that $tades(T\widehat{O}Q_n) \leq \lceil \frac{4mn+m-1}{2} \rceil$. Define $\xi : V(T\widehat{O}Q_n) \bigcup E(T\widehat{O}Q_n) \rightarrow \left\{1, 2, 3, \ldots, \lceil \frac{4mn+m-1}{2} \rceil \right\}$ as follows:

$$\xi(a_r^1) = \begin{cases} 1 & \text{if } \ r = 1 \\ (r-1)2 & \text{if } \ 2 \leq r \leq n+1; \end{cases}$$

For $1 \leq r \leq n+1$,

$$\xi(a_r^s) = \begin{cases} \frac{(4n+1)(s-1)}{2} + 2(r-1) & \text{if } \ s \text{ is odd and } 2 \leq s \leq m \\ \frac{(4n+1)s}{2} - 2(r-1) & \text{if } \ s \text{ is even and } 2 \leq s \leq m. \end{cases}$$

$$\xi(b_s) = \xi(a_{n+1}^s).$$

For $1 \leq r \leq n$,

$$\xi(x_r^s) = \begin{cases} \frac{(4n+1)(s-1)}{2} + 2r & \text{for } \ s \text{ odd and } 1 \leq s \leq m \\ \frac{(4n+1)s}{2} - 2r & \text{for } s \text{ even and } 1 \leq s \leq m; \end{cases}$$

$$\xi(y_r^s) = \begin{cases} \frac{(4n+1)(s-1)}{2} + 2(r-1) + 1 & \text{if } \ s \text{ is odd and } 1 \leq s \leq m \\ \frac{(4n+1)s}{2} - 2r + 1 & \text{if } \ s \text{ is even and } 1 \leq s \leq m; \end{cases}$$

$$\xi(b_s b_{s+1}) = 1, \ 1 \leq s \leq m-1;$$

$$\xi(a_r^1 x_r^1) = \begin{cases} 2 & \text{if } \ r = 1 \\ 1 & \text{if } \ 2 \leq r \leq n; \end{cases}$$

$$\xi(a_r^s x_r^s) = 1, \ 2 \leq s \leq m \text{ and } 1 \leq r \leq n;$$

$$\xi(a_r^1 y_r^1) = \begin{cases} 2 & \text{if } \ r = 1 \\ 1 & \text{if } \ 2 \leq r \leq n \end{cases}$$

$$\xi(a_r^s y_r^s) = 1, \ 2 \leq s \leq m \text{ and } 1 \leq r \leq n \ ;$$

$$\xi(x_r^s a_{r+1}^s) = \xi(y_r^s a_{r+1}^s) = 1, \ 1 \leq s \leq m \text{ and } 1 \leq r \leq n.$$

Let b_rb_s be an edge which is transformed in T, $1 \leq r < s \leq m$. Let P_1 be the *ept* obtained by deleting b_rb_s and adding $b_{r+t}b_{s-t}$ where $t = d(b_r, b_{r+t}) = d(b_s, b_{s-t})$. Let P be a parallel transformation in T which has P_1 as one of the constituent *epts*. Note that the edge $b_{r+t}b_{s-t}$ is in $P(T)$. So $r+t+1 = s-t$ and so $s = r+2t+1$. Clearly, r and s have opposite parity.

The weight of b_rb_s is given by

$$\begin{aligned} wt(b_rb_s) &= wt(b_rb_{r+2t+1}) \\ &= |\xi(b_rb_{r+2t+1}) - \xi(b_r) - \xi(b_{r+2t+1})| \\ &= (4n+1)(r+t) - 1. \end{aligned}$$

The weight of edge $b_{r+t}b_{s-t}$ is given by

$$\begin{aligned} wt(b_{r+t}b_{s-t}) &= wt(b_{r+t}b_{r+t+1}) \\ &= |\xi(b_{r+t}b_{r+t+1}) - \xi(b_{r+t}) - \xi(b_{r+t+1}) \\ &= (4n+1)(r+t) - 1. \end{aligned}$$

Therefore, $wt(b_rb_s) = wt(b_{r+t}b_{s-t})$.

The edge weights are calculated below.

for $1 \leq r \leq n$,

$$wt(b_sb_{s+1}) = (4n+1)s - 1, \ 1 \leq s \leq m-1;$$

$$wt(a_r^s x_r^s) = \begin{cases} (4n+1)(s-1) + 4r - 3 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ (4n+1)s - 4r + 1 & \text{if } s \text{ is even and } 1 \leq s \leq m; \end{cases}$$

$$wt(a_r^s y_r^s) = \begin{cases} (4n+1)(s-1) + 4r - 4 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ (4n+1)s - 4r + 2 & \text{if } s \text{ is even and } 1 \leq s \leq m; \end{cases}$$

$$wt(x_r^s a_{r+1}^s) = \begin{cases} (4n+1)(s-1) + 4r - 1 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ (4n+1)s - 4r - 1 & \text{if } s \text{ is even and } 1 \leq s \leq m; \end{cases}$$

$$wt(y_r^s a_{r+1}^s) = \begin{cases} (4n+1)(s-1) + 4r - 2 & \text{if } s \text{ is odd and } 1 \leq s \leq m \\ (4n+1)s - 4r & \text{if } s \text{ is even and } 1 \leq s \leq m. \end{cases}$$

Hence $tades(T\widehat{O}Q_n)) = \lceil \frac{4mn+m-1}{2} \rceil$. □

Theorem 2.3. For the H-graph G, we have $tades(G) = n$

Proof. Let $V(G) = \{\alpha_r, \beta r : 1 \leq r \leq n\}$ and

$$E(G) = \begin{cases} \{\alpha_r\alpha_{r+1}, \beta_r\beta_{r+1} : 1 \leq r \leq n-1\} \cup \{\alpha_{\frac{n+1}{2}}\beta_{\frac{n+1}{2}}\} & \text{if } n \text{ is odd} \\ \{\alpha_r\alpha_{r+1}, \beta r\beta r+1 : 1 \leq r \leq n-1\} \cup \{\alpha_{\frac{n}{2}+1}\beta_{\frac{n}{2}}\} & \text{if } n \text{ is even .} \end{cases}$$

By Theorem 1.1, $tades(G) \geq n$. We now prove the reverse inequality. The labeling $\xi : V \bigcup E \rightarrow \{1, 2, 3, \ldots, n\}$ is defined as follows:

For $1 \leq r \leq n$,

$$\xi(\alpha_r) = \begin{cases} \frac{r+1}{2} & \text{for } r \text{ odd} \\ \frac{r}{2} & \text{for } r \text{ even ;} \end{cases}$$

$$\xi(\beta_r) = n - \left\lfloor \frac{r-1}{2} \right\rfloor;$$

$$\xi(\alpha_r\alpha_{r+1}) = 2,\ 1 \leq r \leq n-1;$$

$$\xi(\beta_r\beta_{r+1}) = 2,\ 1 \leq r \leq n-1;$$

Fix $\xi(\alpha_{\frac{n+1}{2}}\beta_{\frac{n+1}{2}}) = 2$, for odd n.

Fix $\xi(\alpha_{\frac{n}{2}+1}\beta_{\frac{n}{2}}) = \begin{cases} 2 & \text{if } n \not\equiv 0(mod\ 4) \\ 3 & \text{if } n \equiv 0(mod\ 4), \end{cases}$ for even n.

The edge weights are

$$wt(\alpha_r\alpha_{r+1}) = r - 1 \text{ for } 1 \leq r \leq n-1;$$

$$wt(\beta_r\beta_{r+1}) = 2n - r - 1 \text{ for } 1 \leq r \leq n-1.$$

Clearly, $wt(\alpha_{\frac{n+1}{2}}\beta_{\frac{n+1}{2}}) = n - 1$, for n odd.

And $wt(\alpha_{\frac{n}{2}+1}\beta_{\frac{n}{2}}) = n - 1$, for n even.

Clearly, $tades(G) \leq n$. Note that the edge weights are different. Hence $tades(G) = n$. □

3. Main Results For Path Related Graphs

In this section, we investigate total absolute difference edge irregularity strength of path related graphs.

Theorem 3.1. Form a graph G by duplicating each vertex by an edge in the path P_n. Then $tades(G) = \left\lceil \frac{4n-1}{2} \right\rceil$.

Proof. Let $V(G) = \{\alpha_r, \alpha_r^{'}, \alpha_r^{''} : 1 \leq r \leq n\}$ and
$E(G) = \{\alpha_r\alpha_r^{'}, \alpha_r\alpha_r^{''}, \alpha_r^{'}\alpha_r^{''} : 1 \leq r \leq n\} \cup \{\alpha_r\alpha_{r+1} : 1 \leq r \leq n-1\}$.
By Theorem 1.1, we have $tades(G) \geq \lceil \frac{4n-1}{2} \rceil$. Let us now prove the reverse inequality.The labeling $\xi : V \bigcup E \to \{1, 2, 3, \ldots, \lceil \frac{4n-1}{2} \rceil\}$ is defined below.
For $1 \leq r \leq n$,

$$\xi(\alpha_r) = \begin{cases} 2r & \text{(when } r \text{ odd)} \\ 2r-1 & \text{(when } r \text{ even) ;} \end{cases}$$

$$\xi(\alpha_r^{'}) = 2r - 1;$$

$$\xi(\alpha_r^{''}) = \begin{cases} 2r-1 & \text{(when } r \text{ odd)} \\ 2r & \text{(when } r \text{ even) .} \end{cases}$$

$$\xi(\alpha_r\alpha_{r+1}) = 2,\ 1 \leq r \leq n-1;$$

for $1 \leq r \leq n$,

$$\xi(\alpha_r\alpha_r^{'}) = 2,$$

$$\xi(\alpha_r\alpha_r^{''}) = \begin{cases} 1 & \text{(when } r \text{ odd)} \\ 2 & \text{(when } r \text{ even) ;} \end{cases}$$

$$\xi(\alpha_r^{'}\alpha_r^{''}) = \begin{cases} 2 & \text{(when } r \text{ odd)} \\ 1 & \text{(when } r \text{ even) .} \end{cases}$$

We arrive at the weight of the edges:

$$wt(\alpha_r\alpha_{r+1}) = 4r - 1,\ 1 \leq r \leq n-1;$$

for $1 \leq r \leq n$

$$wt(\alpha_r\alpha_r^{'}) = \begin{cases} 4r-3 & \text{for } r \text{ odd} \\ 4r-4 & \text{for } r \text{ even ;} \end{cases}$$

$$wt(\alpha_r\alpha_r^{''}) = \begin{cases} 4r-2 & \text{if } r \text{ is odd} \\ 4r-3 & \text{if } r \text{ is even ;} \end{cases}$$

$$wt(\alpha_r^{'}\alpha_r^{''}) = \begin{cases} 4r-4 & \text{if } r \text{ is odd} \\ 4r-2 & \text{if } r \text{ is even .} \end{cases}$$

Clearly, $tades(G) \leq \lceil \frac{4n-1}{2} \rceil$. Hence $tades(G) = \lceil \frac{4n-1}{2} \rceil$ as the edge weights are distinct. □

Theorem 3.2. Form a graph G by duplicating each edge by a vertex in path P_n. Then $tades(G) = \left\lceil \frac{3n-3}{2} \right\rceil$.

Proof. Let $V(G) = \{\alpha_r : 1 \leq r \leq n\} \cup \{\beta_r : 1 \leq r \leq n-1\}$ and
$E(G) = \{\alpha_r\beta_r, \alpha_{r+1}\beta_r, \alpha_r\alpha_{r+1} : 1 \leq r \leq n-1\}$.
Clearly $tades(G) \geq \left\lceil \frac{3n-3}{2} \right\rceil$ (Theorem 1.1). Let us proceed to derive the reverse inequality. We define the labeling $\xi : V \bigcup E \to \{1, 2, 3, \ldots, \left\lceil \frac{3n-3}{2} \right\rceil\}$ as follows.

Case 1. n is odd.

$$\xi(\alpha_r) = \begin{cases} 1 & (r = 1) \\ \frac{3r-2}{2} & \text{for } r \text{ even and } 2 \leq r \leq n \\ \frac{3r-3}{2} & \text{for } r \text{ odd and } 2 \leq r \leq n \ ; \end{cases}$$

$$\xi(\beta_r) = \begin{cases} \left\lceil \frac{3r-2}{2} \right\rceil & \text{for } r \text{ odd and } 1 \leq r \leq n-1 \\ \frac{3r}{2} & \text{for } r \text{ even and } 1 \leq r \leq n-1 \ ; \end{cases}$$

$$\xi(\alpha_r\beta_r) = \begin{cases} 2 & (r = 1) \\ 2 & \text{for } r \text{ even } 2 \leq r \leq n-1 \\ 1 & \text{for } r \text{ odd and } \ 2 \leq r \leq n-1 \ ; \end{cases}$$

$$\xi(\alpha_r\alpha_{r+1}) = \begin{cases} 2 & (r = 1) \\ 1 & (2 \leq r \leq n-1); \end{cases}$$

$$\xi(\alpha_r\beta_r) = 1, 1 \leq r \leq n-1$$

Case 2. n is even.

$$\xi(\alpha_r) = \left\lceil \frac{3r-2}{2} \right\rceil, 1 \leq r \leq n;$$

$$\xi(\beta_r) = \begin{cases} \left\lceil \frac{3r-2}{2} \right\rceil & (r \text{ is odd and } 1 \leq r \leq n-1) \\ \frac{3r}{2} & (r \text{ is even and } 1 \leq r \leq n-1) \ ; \end{cases}$$

$$\xi(\alpha_r\beta_r) = 2, \ 1 \leq r \leq n-1;$$

$$\xi(\alpha_r\alpha_{r+1}) = 2, \ 1 \leq r \leq n-1;$$

$$\xi(\alpha_r\beta_r) = \begin{cases} 1 & (r \text{ is odd and } 1 \leq r \leq n-1) \\ 2 & (r \text{is even and } 1 \leq r \leq n-1). \end{cases}$$

Below we give the calculation of the weight of the edges.

For $1 \leq r \leq n-1$

$$wt(\alpha_r\alpha_{r+1}) = 3r - 2;$$

$$wt(\alpha_r\beta_r) = 3r - 3;$$
$$wt(\alpha_{r+1}\beta_r) = 3r - 1.$$

Clearly, $tades(G) \leq \lceil \frac{3n-3}{2} \rceil$. Note that the edge weights are distinct. Hence $tades(G) = \lceil \frac{3n-3}{2} \rceil$. □

Theorem 3.3. For Key graph $KY(m,n)$, $tades(KY(m,n)) = \lceil \frac{m+2n}{2} \rceil$.

Proof. Let $H_1 = C_m$, $H_2 = P_n \odot K_1$. The vertex set of $KY(m,n)$ is $\{v_r : 1 \leq r \leq m\} \bigcup \{u_s, w_s : 1 \leq s \leq n\}$ and edge set of $KY(m,n)$ is $\{v_r v_{r+1}, v_m v_1, v_m u_1 : 1 \leq r \leq m-1\} \bigcup \{u_s u_{s+1} : 1 \leq s \leq n-1\} \bigcup \{u_s w_s : 1 \leq s \leq n\}$. The graph H_1 has m edges and H_2 has $2n-1$ edges. Therefore $KY(m,n)$ has $m+2n$ edges. By Theroem 1.1, we have $tades(KY(m,n)) \geq \lceil \frac{m+2n}{2} \rceil$. We now proceed to derive the reverse inequality. We construct $\xi : V \bigcup E \to \{1, 2, \cdots \lceil \frac{m+2n}{2} \rceil\}$ as follows:

for $1 \leq r \leq m$,

$$\xi(v_r) = \begin{cases} \frac{r+1}{2} & \text{for } r \text{ odd} \\ \frac{r}{2} & \text{for } r \text{ even .} \end{cases}$$

Case 1: m is odd.

For $1 \leq s \leq n$,

$$\xi(u_s) = \begin{cases} \lceil \frac{m}{2} \rceil + s - 1 & \text{for } r \text{ odd} \\ \lceil \frac{m}{2} \rceil + s & \text{for } r \text{ even ;} \end{cases}$$

$$\xi(w_s) = \begin{cases} \lceil \frac{m}{2} \rceil + s & \text{for } r \text{ odd} \\ \lceil \frac{m}{2} \rceil + s - 1 & \text{for } r \text{ even .} \end{cases}$$

Case 2: m is even.

$$\xi(u_s) = \xi(w_s) = \tfrac{m}{2} + s, \ 1 \leq s \leq n.$$

In both the cases the edge labelings are,

$$\xi(v_r v_{r+1}) = \begin{cases} 2 & \text{if } \ 1 \leq r \leq \lceil \frac{m}{2} \rceil \\ 1 & \text{if } \ \lceil \frac{m}{2} \rceil + 1 \leq r \leq m - 1 \ ; \end{cases}$$

$$\xi(v_m v_1) = \xi(v_m u_1) = 1;$$
$$\xi(u_s u_{s+1}) = 1, \ 1 \leq s \leq n - 1;$$
$$\xi(u_s w_s) = 1, \ 1 \leq s \leq n.$$

We arrive at the following edge weights.

$$wt(v_r v_{r+1}) = \begin{cases} r-1 & \text{if } 1 \leq r \leq \lceil \frac{m}{2} \rceil \\ r & \text{if } \lceil \frac{m}{2} \rceil + 1 \leq r \leq m-1 \end{cases};$$

$wt(v_m v_1) = \lceil \frac{m}{2} \rceil$;

$wt(v_m u_1) = m$;

$wt(u_s u_{s+1}) = m + 2s,\ 1 \leq s \leq n-1$;

$wt(u_s w_s) = m + 2s - 1,\ 1 \leq s \leq n$.

Note that the edge weights are distinct. Hence $tades(KY(m,n)) = \lceil \frac{m+2n}{2} \rceil$. □

Theorem 3.4. For $C_m \ominus P_n$, $tades(C_m \ominus P_n) = \lceil \frac{mn}{2} \rceil$.

Proof. Let $a_{1n}, a_{2n} \cdots a_{mn}$ be the vertices of the cycle C_n and $a_{11}a_{12} \cdots a_{1n}, a_{21}a_{22} \cdots a_{2n} \cdots a_{m1}a_{m2} \cdots a_{mn}$ be the vertices of the path P_n attached with a_{rn} by identifying a_{rs} with a_{rn} for $1 \leq r \leq m,\ 1 \leq s \leq n$. Therefore, $C_m \ominus P_n$ have mn edges and mn vertices. By Theorem 1.1, $tades(C_m \ominus P_n) \geq \lceil \frac{mn}{2} \rceil$. Define $\xi : V \bigcup E \to \{1, 2 \cdots \lceil \frac{mn}{2} \rceil\}$ as follows:

Let $1 \leq r \leq m$,

Case 1. n is odd,

$$\xi(a_{rs}) = \begin{cases} \frac{nr+1}{2} - \lceil \frac{s-1}{2} \rceil & \text{for } r \text{ odd and } 1 \leq s \leq n, \\ \lceil \frac{n(r-1)}{2} \rceil + \lfloor \frac{s}{2} \rfloor & \text{for } r \text{ even and } 1 \leq s \leq n; \end{cases}$$

Case 2. n is even,

$$\xi(a_{rs}) = \begin{cases} \frac{nr}{2} - \lfloor \frac{s-1}{2} \rfloor & \text{for } r \text{ odd and } 1 \leq s \leq n, \\ \frac{n(r-1)}{2} + \lceil \frac{s}{2} \rceil & \text{for } r \text{ even and } 1 \leq s \leq n; \end{cases}$$

Now we assign the labels for edges.

$\xi(a_{rn}a_{r+1n}) = 2,\ 1 \leq r \leq \lfloor \frac{m-1}{2} \rfloor$;

$\xi(a_{rn}a_{r+1n}) = 1, \lfloor \frac{m+1}{2} \rfloor \leq r \leq m-1$;

$\xi(a_{mn}a_{1n}) = 2$;

$\xi(a_{rs}a_{rs+1}) = 2,\ 1 \leq r \leq \lfloor \frac{m}{2} \rfloor$ and $1 \leq s \leq n-1$;

$\xi(a_{rs}a_{rs+1}) = 1, \lceil \frac{m+1}{2} \rceil \leq r \leq m-1$ and $1 \leq s \leq n-1$.

Below we give the calculation for the weight of the edges:

$$wt(a_{rn}a_{r+1n}) = \begin{cases} nr-1 & \text{for } 1 \leq r \leq \lfloor \frac{m-1}{2} \rfloor \\ nr & \text{for } \lfloor \frac{m+1}{2} \rfloor \leq r \leq m-1; \end{cases}$$

$$wt(a_{mn}a_{1n}) = \begin{cases} \frac{nm-n}{2} & \text{for } m \text{ odd} \\ \frac{nm}{2} - 1 & \text{for } m \text{ even;} \end{cases}$$

for $1 \leq r \leq \lfloor \frac{m}{2} \rfloor$ and $1 \leq s \leq n-1$,

$$wt(a_{rs}a_{rs+1}) = \begin{cases} nr - s - 1 & \text{for } r \text{ odd} \\ n(r-1) + s - 1 & \text{for } r \text{ even;} \end{cases}$$

for $\lceil \frac{m+1}{2} \rceil \leq r \leq m-1$ and $1 \leq s \leq n-1$,

$$wt(a_{rs}a_{rs+1}) = \begin{cases} nr - s & \text{for } r \text{ odd} \\ n(r-1) + s & \text{for } r \text{ even.} \end{cases}$$

Hence $tades(C_m \ominus P_n) = \lceil \frac{mn}{2} \rceil$ as the the edge weights are distinct. □

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