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RING ENDOMORPHISMS SATISFYING Z-SYMMETRIC PROPERTY

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Abstract. The notion of α-skew Z-symmetric rings is introduced as a generalization of Z-symmetric rings. We prove that the notions of α-skew Z-symmetric rings and Z-symmetric rings are independent, and we give some sufficient conditions over which these notions are equivalent. We investigate some basic properties of α-skew Z-symmetric rings and give a characterization of them. Moreover, we provide some characterizations of α-skew Z-symmetric rings utilizing the Dorroh extension, triangular matrix ring etc. Finally, we generalize some results of Z-symmetric rings to α-skew Z-symmetric rings.

1. Introduction

Following [7], if $ab = 0$ gives $ba = 0$ for all a and b in a ring R, then R is known as *reversible*. Recall from [6] that a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for all a and b in R. Reversible rings are also called *zero commutative* rings, and semicommutative rings are also called *zero insertive* rings by Habeb [9]. In 2009, Baser et al. [3] described reversibility of rings in terms of their endomorphisms. They called an endomorphism α of a ring R right (resp., left) *skew reversible* if $ab = 0$ gives $b\alpha(a) = 0$ for all a and b in R (resp., $\alpha(b)a = 0$ for all a and b in R). Whenever α is a right (left) skew reversible endomorphism of a ring R, then R is called right (left) α-skew reversible. When R is both right and left α-skew reversible, it is called as α-skew reversible. Recently, Bhattacharjee et al. [4] introduced a new concept of α-skew central reversible rings which generalized the class of α-skew reversible rings.

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They called α as right (left) *skew central reversible* endomorphism of R if $ab = 0$ gives $b\alpha(a) \in C(R)$, for all a and b in R, where $C(R)$ is the center of the ring R. (resp., $\alpha(b)a \in C(R)$, for all a and b in R). The ring R is referred to be (left) α-skew central reversible. When R is both right and left α-skew central reversible, then R is referred to be α-skew central reversible.

In 2021, we gave the idea of Z-symmetric rings and found that it is a proper generalization of reversible rings (see [6]). A ring R is called right (left) *Z-symmetric* if ba is a right (left) zero divisor if and only if ab is right (left) zero divisor for any a and b in R. If a ring is left as well as right Z-symmetric, it is referred to as Z-symmetric.

By this motivation, we present the previous concepts in a more general context as follows:

Definition 1.1. Let α be an endomorphism of a ring R.

(1) α is called a right *(resp., left) skew Z-symmetric* if whenever ab is right (resp., left) zero divisor, then $b\alpha(a)$ (resp., $\alpha(b)a$) is a right (resp., left) zero divisor for any $a, b \in R$.
(2) R is said to be a *right (resp., left) α-skew Z-symmetric* if α is a right (resp., left) skew Z-symmetric endomorphism of R.
(3) If R is both left and right α-skew Z-symmetric, we call it α-skew Z-symmetric.

Further, in Section 2, we go over some characterizations of α-skew Z-symmetric rings. We start by showing that the notion of a right (resp., left) α-skew Z-symmetric ring and a right (resp., left) Z-symmetric ring are independent notions (see Example 2.1). The concepts of right (resp., left) α-skew Z-symmetric rings and right (resp., left) Z-symmetric rings are then determined to be identical under suitable conditions (Proposition 2.2). Also, we determine necessary and sufficient conditions under which a right (resp., left) Z-symmetric ring is right (resp., left) α-skew Z-symmetric (see Proposition 2.3 and 2.4). Further, we conclude that a right α-skew Z-symmetric ring need not be α-skew reversible (see Remark 1) and we establish a set of reasonable conditions over which α-skew reversible ring is α-skew Z-symmetric (see Proposition 2.5). Next, we give a characterization of α-skew Z-symmetric rings (see Proposition 2.6). As a consequence, we have [6, Proposition 3.2]: A ring R is right Z-symmetric if and only if for any $a, b, c \in R$ with $abc = 0$, $acb = 0$ or $a'cb = 0$ for some $0 \neq a' \in R$. In Proposition 2.7, we show that every α-semicommutative ring is an example of right α-skew Z-symmetric rings. We also demonstrate that the idea of α-skew Z-symmetry is retained in isomorphisms and arbitrary direct products (see Proposition 2.1 and 2.8). As a consequence, we have [6, Theorem 3.3]: An arbitrary direct product of rings is right Z-symmetric if and only if every ring in the direct product is right Z-symmetric.

In Section 3, we are concerned with the study of extension rings of α-skew Z-symmetric rings. We give many characterizations of α-skew Z-symmetric rings in terms of their extension rings like Dorroh Extension, Jordan Extension, Localization ring, generalized matrix ring etc. Also, we generalize the following results: [6, Theorem 3.5, Corollary 3.6, 3.15, Proposition 3.8, 3.10, 3.12 3.13 and 3.14].

Here, R denotes an associative ring with identity. $End(R)$ denotes the ring of all endomorphisms of R. The $M_n(R)$ stands for the ring of all n-square matrices over R, whereas $T_n(R)$ stands for the ring of all n-square upper triangular matrices; and the symbol E_{ij} stands for a matrix of $M_n(R)$ with $(i, j)^{th}$ entry as 1, the identity of R, and all other entries are equal to 0, the zero of R. We denote the collection of all right zero divisors, left zero divisors, zero divisors, central elements (i.e. elements that commute with all elements of the ring) and the set of all idempotent elements of a ring R by $Z_r(R)$, $Z_l(R)$, $Z(R)$, $C(R)$ and $I(R)$, respectively. For all other undefined terminologies, readers are refered to [5] and [10].

2. Some properties

We demonstrate in the following Example that the notions of α-skew Z-symmetric ring and a Z-symmetric ring are independent.

Example 2.1. Consider a ring R which is not right Z-symmetric (e.g.-any non Dedekind-finite ring). Then, by [6, Proposition 3.10], $T_2(R)$ is not right Z-symmetric. But, $T_2(R)$ is right α-skew Z-symmetric with respect to $\alpha \in End(T_2(R))$ given by $\alpha(\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}) = a_1E_{11}$ as for any $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, B = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} \in T_2(R)$, we have $E_{22}B\alpha(A) = E_{22}BxE_{11} = 0$ which implies that $B\alpha(A) \in Z_r(T_2(R))$ for any $A, B \in T_2(R)$.

On the other hand, let $R = \mathbb{Z} \times \mathbb{Z}$. Then R is a right Z-symmetric ring being a commutative ring. But, it is not right α-skew Z-symmetric with respect to the endomorphism α of R given by $\alpha(u, v) = (u, u)$ because for $X = (1, 0), Y = (1, 1) \in R$, $XY = (1, 0) \in Z_r(R)$ while $Y\alpha(X) = (1, 1) \notin Z_r(R)$.

We observe that every commutative ring is Z-symmetric. But the above example shows that commutative rings are not necessarily α-skew Z-symmetric for certain endomorphisms α.

Note that the class of right Z-symmetric rings is the same as the class of 1_R-skew Z-symmetric rings, where 1_R denotes the identity map on the ring R. Analogously, the class of Z-symmetric rings is equivalent to that of 1_R-skew Z-symmetric rings, where 1_R denotes the identity map on the ring R.

Consider an endomorphism α of the subring $S = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} |x, y \in R \right\} \subseteq M_2(R)$ sends $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ to $\begin{bmatrix} x & -y \\ 0 & x \end{bmatrix}$, where R is a ring. In [4, Example 1.4], authors have demonstrated that for a central reversible ring R, S need not be α-skew central reversible. However, in the following, we show that it is a right α-skew Z-symmetric ring.

Proposition 2.1. The ring $S = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} |x, y \in R \right\}$ is right α-skew Z-symmetric with respect to $\alpha \in End(S)$ which sends $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ to $\begin{bmatrix} x & -y \\ 0 & x \end{bmatrix}$ for any right Z-symmetric ring R.

Proof. Let $U = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}, V = \begin{bmatrix} u & v \\ 0 & u \end{bmatrix} \in S$. Assume $UV \in Z_r(S)$. Then, there exists a nonzero element $W = \begin{bmatrix} p & q \\ 0 & p \end{bmatrix} \in S$ such that $WUV = \begin{bmatrix} pxu & pyu + qxu + pxv \\ 0 & pxu \end{bmatrix}$ $= 0$. So, $pxu = 0$ and $pyu + qxu + pxv = 0$. We have the following two possiblities:

Case-I: Assume that $p = 0$. Then, $qxu = 0$ and $q \neq 0$ as $W \neq 0$. Hence $xu \in Z_r(R)$ and so $ux \in Z_r(R)$. Thus, there is a nonzero element $r \in R$ such that $rux = 0$.

Case-II: Assume that $p \neq 0$. Then, again $xu \in Z_r(R)$ as we have $pxu = 0$. So, $ux \in Z_r(R)$ because R is right Z-symmetric. Thus, there is a nonzero element $s \in R$ such that $sux = 0$.

If we take $A = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}$ in S, then $A \neq 0$, $B \neq 0$, $AV\alpha(U) = 0$ and $BV\alpha(U) = 0$ which implies that $V\alpha(U) \in Z_r(S)$. Thus, S is right α-skew Z-symmetric. □

Recall from [1], $\alpha \in End(R)$ is called *compatible* whenever $ab = 0$ if and only if $a\alpha(b) = 0$, for all a, b in R. Example 2.1 demonstrates how the notion of α-skew Z-symmetric rings and Z-symmetric rings are two independent notions. A sufficient condition for these two notions being equivalent is given in the following:

Proposition 2.2. If α is a compatible endomorphism of a ring R, then the following are equivalent:

(1) R is right (resp., left) Z-symmetric;
(2) R is right (resp., left) α-skew Z-symmetric;
(3) For any two elements a, b in R, if $ab \in Z_r(R)$, then $\alpha(b)a \in Z_r(R)$ (resp., if $ab \in Z_l(R)$, then $b\alpha(a) \in Z_l(R)$).

Proof. (1) $\Longrightarrow$ (2). Let $a, b \in R$. Suppose that $ab \in Z_r(R)$. Then, we have $ba \in Z_r(R)$ because R is right Z-symmetric. There is a nonzero element c in R for which $cba = 0$. So, $cb\alpha(a) = 0$ because α is compatible. Hence, $b\alpha(a) \in Z_r(R)$. Thus, we have shown that (2) holds.

(2) $\Longrightarrow$ (3). Let $a, b \in R$. Suppose that $ab \in Z_r(R)$. Then, we have $b\alpha(a) \in Z_r(R)$ because R is right α-skew Z-symmetric. There is a nonzero element c in R for which $cb\alpha(a) = 0$. So, $\alpha(cb)a = \alpha(c)\alpha(b)a = 0$ because α is compatible. Since $c \neq 0$ and α is compatible, $\alpha(c) \neq 0$ by [1, Lemma 2.1(i)]. Hence, $\alpha(b)a \in Z_r(R)$.

(3) $\Longrightarrow$ (1). Assume condition (3). Let a and b be two elements in R and $ab \in Z_r(R)$. Then, $\alpha(b)a \in Z_r(R)$ by the assumption. There is a nonzero element c in R such that $c\alpha(b)a = 0$. As α is compatible, $(c\alpha(b))\alpha(a) = c\alpha(ba) = 0$ and so $cba = 0$. Since $c \neq 0$, it follows that $ba \in Z_r(R)$. Thus, we have the condition (1). □

We give necessary and sufficient criteria for a Z-symmetric ring to be α-skew Z-symmetric in the following two results.

Proposition 2.3. If a ring R is right Z-symmetric, then the following are equivalent for any endomorphism α of R:

(1) R is right α-skew Z-symmetric;
(2) For any two elements a and b in R, if $ab \in Z_r(R)$, then $\alpha(b)a \in Z_r(R)$;
(3) For any two elements a and b in R, if $ab \in Z_r(R)$, then $R\alpha^n(b)a \subseteq Z_r(R)$ and $Rb\alpha^n(a) \subseteq Z_r(R)$ for any $n \geq 0$.
(4) For any two elements a and b in R, if $ab \in Z_r(R)$, then $aR\alpha^n(b) \subseteq Z_r(R)$ and $\alpha^n(a)Rb \subseteq Z_r(R)$ for any $n \geq 0$.

Proof. (1) $\Longrightarrow$ (2) Let $ab \in Z_r(R)$ for two elements a, b in R. Since R is right Z-symmetric, $ba \in Z_r(R)$. It gives $a\alpha(b) \in Z_r(R)$ as R is right α-skew Z-symmetric. Therefore, $\alpha(b)a \in Z_r(R)$ as R is right Z-symmetric.

(2) $\Longrightarrow$ (3) Let $ab \in Z_r(R)$ for two elements a, b in R. Then, $\alpha(b)a \in Z_r(R)$ by the assumption. So, $a\alpha(b) \in Z_r(R)$ as R is right Z-symmetric. Hence, it folows that $\alpha^2(b)a \in Z_r(R)$. Proceeding inductivily, we get $\alpha^n(b)a \in Z_r(R)$ for any $n \geq 0$. Since we know that $r \in Z_r(R)$ imples that $sr \in Z_r(R)$ for any $s \in R$, $R\alpha^n(b)a \in Z_r(R)$ for any $n \geq 0$.

Next, since $ab \in Z_r(R)$, $ba \in Z_r(R)$ as R is right Z-symmetric. Hence by the assumption, $\alpha(a)b \in Z_r(R)$ and so, $b\alpha(a) \in Z_r(R)$ as R is right Z-symmetric. On proceeding inductively, we get $b\alpha^n(a) \in Z_r(R)$ for any $n \geq 0$. Since $r \in Z_r(R)$ imples that $sr \in Z_r(R)$ for any $s \in R$, $Rb\alpha^n(a) \in Z_r(R)$ for any $n \geq 0$.

(3) $\Longrightarrow$ (4). Let $ab \in Z_r(R)$ for two elements a, b in R. Then by the assumption, $r\alpha^n(b)a \subseteq Z_r(R)$ and $rb\alpha^n(a) \subseteq Z_r(R)$ for any $n \geq 0$ and for any $r \in R$. Since R is right Z-symmetric, $ar\alpha^n(b) \subseteq Z_r(R)$ and $\alpha^n(a)rb \subseteq Z_r(R)$ for any $n \geq 0$ and for any $r \in R$. This implies that $aR\alpha^n(b) \subseteq Z_r(R)$ and $\alpha^n(a)Rb \subseteq Z_r(R)$ for any $n \geq 0$.

(4) $\Longrightarrow$ (1). Let $ab \in Z_r(R)$ for two elements a, b in R. Then, $ba \in Z_r(R)$ as R is right Z-symmetric. It follows by the assumption that $b\alpha(a) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. □

With the similar argument as above, we have the following conclusion:

Proposition 2.4. If a ring R is left Z-symmetric, then the following are equivalent for an endomorphism α of R:

(1) R is left α-skew Z-symmetric ring;
(2) For any two elements a and b in R, if $ab \in Z_l(R)$, then $b\alpha(a) \in Z_l(R)$;
(3) For any two elements a and b in R, if $ab \in Z_l(R)$, then $\alpha^n(b)aR \subseteq Z_l(R)$ and $b\alpha^n(a)R \subseteq Z_l(R)$ for any $n \geq 0$;
(4) For any two elements a and b in R, if $ab \in Z_l(R)$, then $aR\alpha^n(b) \subseteq Z_l(R)$ and $\alpha^n(a)Rb \subseteq Z_l(R)$ for any $n \geq 0$.

Remark 1. It's not necessary for a right α-skew Z-symmetric ring to be α-skew reversible. For instance, $R = T_2(\mathbb{Z})$ is α-skew Z-symmetric for $\alpha = I$, the identity map on R, but it is neither left nor right α-skew reversible. Whether a right α-skew reversible ring is always a right α-skew Z-symmetric or not is an open problem. However, in the following, we give a partial answer to that.

Proposition 2.5. Let $\alpha \in End(R)$ be injective. If R is right (resp., left) α-skew reversible, then R is right (resp., left) α-skew Z-symmetric.

Proof. Consider two elements b and c in a right α-skew reversible R. Let $bc \in Z_r(R)$. Then, we have $abc = 0$ for some nonzero element a in R. This implies that $abc\alpha(b) = 0$. Let $ab \neq 0$. Then $c\alpha(b) \in Z_r(R)$. Assume that $ab = 0$. Then, we get $\alpha(a)\alpha(b) = 0$ and $\alpha(a) \neq 0$ as α is a monomorphism. Due to the fact that α is an injective endomorphism of a right α-skew reversible ring R, we get $\alpha(a)c\alpha(b) = 0$ by [3, Proposition 2.5(ii)]. This implies that $c\alpha(b) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. □

As per Remark 1, the converse of Proposition 2.5 is untrue. Now, we give a characterization of α-skew Z-symmetric rings as follows:

Proposition 2.6. For any ring R and an endomorphism α, the following are equivalent:

(1) R is right (resp., left) α-skew Z-symmetric;

(2) At least, one of the two criteria given below is true for any a, b, c in R with $abc = 0$:

  (a) $ac\alpha(b) = 0$ (resp., $\alpha(b)ac = 0$);

  (b) $a'c\alpha(b) = 0$ for some $0 \neq a' \in R$ (resp., $\alpha(b)ac' = 0$ for some $0 \neq c' \in R$).

Proof. (1) $\Longrightarrow$ (2) Consider three elements a, b, c in a right α-skew Z-symmetric ring R. Let $abc = 0$. If $a = 0$, then clearly we have $ac\alpha(b) = 0$. If $a \neq 0$, then $abc = 0$ implies that $bc \in Z_r(R)$ and so $c\alpha(b) \in Z_r(R)$. Hence, $a'c\alpha(b) = 0$ for some $0 \neq a' \in R$.

(2) $\Longrightarrow$ (1) Consider two elements b and c in R. Let $bc \in Z_r(R)$. Then $abc = 0$ for some $0 \neq a \in R$. So, by (2), we have $ac\alpha(b) = 0$ or $a'c\alpha(b) = 0$ for some $0 \neq a' \in R$. If $ac\alpha(b) = 0$, then $c\alpha(b) \in Z_r(R)$ as $a \neq 0$. Similarly, if $a'c\alpha(b) = 0$ for some $0 \neq a' \in R$, then $c\alpha(b) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. ☐

Corollary 2.1. [6, Proposition 3.2] The following are equivalent for a ring R:

(1) R is right Z-symmetric;

(2) At least, one of the two criteria given below is true for any $a, b, c \in R$ with $abc = 0$:

  (a) $acb = 0$;

  (b) $a'cb = 0$ for some $0 \neq a' \in R$.

Recall from [2] that if $ab = 0$ gives $aR\alpha(b) = 0$ for any $a, b \in R$, then the endomorphism α of R is called *semicommutative.* If α is a semicommutative endomorphism of R, then R is referred to as being *α-semicommutative.*

Proposition 2.7. Let $\alpha \in End(R)$. If R is α-semicommutative, then R is right α-skew Z-symmetric.

Proof. Consider two elements b and c in R. Assume $bc \in Z_r(R)$. Then, $abc = 0$ for some $0 \neq a \in R$ and so $abc\alpha(b) = 0$. If $ab \neq 0$, clearly then $c\alpha(b) \in Z_r(R)$. Suppose that $ab = 0$. Then, $ac\alpha(b) = 0$ because R is α-semicommutative. Hence, $c\alpha(b) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. □

A right α-skew Z-symmetric ring need not be α-semicommutative. For example, let α be an endomorphism of $R = \mathbb{Z} \times \mathbb{Z}$ given by $\alpha(x, y) = (y, x)$. Let (x, y) and (u, v) be two nonzero elements of R for which $(x, y)(u, v) \in Z_r(R)$. Then, $(c, d)(x, y)(u, v) = (0, 0)$ for some $(0, 0) \neq (c, d) \in R$. Since all of $(x, y), (u, v)$ and (c, d) are nonzero, at least one of x, y, u, v must be zero. If $x = 0$ or $v = 0$, then $(0, 1)(u, v)\alpha(x, y) = (0, 0)$; and if $y = 0$ or $u = 0$, then $(1, 0)(u, v)\alpha(x, y) = (0, 0)$. This implies that, $(u, v)\alpha(x, y) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. But, R is not α-semicommutative as $(1, 0)(0, 1) = (0, 0)$ but $(1, 0)(1, 1)\alpha(0, 1) = (1, 0) \neq (0, 0)$.

Corollary 2.2. All semicommutative rings are right Z-symmetric rings.

Homomorphic images do not preserve the idea of left and right α-skew Z-symmetric rings. For illustration, if we consider R as the free algebra $F < x, y >$ in two noncommuting inderminates x and y over a field F, $\alpha = I_R$ as the identity homomorphism of R and J as the ideal $< x^2 >$, then R is Z-symmetric but R/J is not so by [6, Remark 2.10]. Thus, R is α-skew Z-symmetric but R/J is not $\bar{\alpha}$-skew Z-symmetric, where $\bar{\alpha}(f + I) = \alpha(f) + I$. In the following, we find that it is preserved under the isomorphism:

Proposition 2.8. Let $\sigma : R \to S$ be an isomorphism of two rings R and S. Then, for any $\alpha \in End(R)$, R is right (resp., left) α-skew Z-symmetric if and only if S is right (resp., left) $\bar{\alpha}$-skew Z-symmetric, where $\bar{\alpha} = \sigma\alpha\sigma^{-1}$.

Proof. Suppose that R is right α-skew Z-symmetric. Consider two elements y and z in S. Assume $yz \in Z_r(S)$. Then, $xyz = 0$ for some nonzero element x in S. Since $\sigma : R \to S$ is an isomorphism, $\sigma(a) = x, \sigma(b) = y, \sigma(c) = z$ for some $0 \neq a, b, c \in R$. Hence, we have $\sigma(abc) = \sigma(a)\sigma(b)\sigma(c) = xyz = 0$ and so $abc = 0$ as σ is injective. Since $a \neq 0$, we have $bc \in Z_r(R)$. So, $c\alpha(b) \in Z_r(R)$ because R is a right α-skew Z-symmetric ring. Hence, $dc\alpha(b) = 0$ for some $0 \neq d \in R$. Therefore, $\sigma(d)z\bar{\alpha}(y) =$

$\sigma(d)\sigma(c)\sigma\alpha\sigma^{-1}(\sigma(b)) = \sigma(d)\sigma(c)\sigma(\alpha(b)) = \sigma(dc\alpha(b)) = 0$. Since σ is an injective and $d \neq 0$, $\sigma(d) \neq 0$. So, $z\bar{\alpha}(y) \in Z_r(S)$. Thus, S is right $\bar{\alpha}$-skew Z-symmetric.

Next, suppose that S is a right $\bar{\alpha}$-skew Z-symmetric ring. Consider two elements b and c in R. Let $bc \in Z_r(R)$. Then, it gives a nonzero element a in R for which $abc = 0$ and so $\sigma(a)\sigma(b)\sigma(c) = \sigma(abc) = 0$. As σ is injective and $a \neq 0$, $\sigma(a) \neq 0$. Hence, $\sigma(b)\sigma(c) \in Z_r(S)$ and so $\sigma(c)\bar{\alpha}((\sigma)(b)) \in Z_r(S)$ as S is right $\bar{\alpha}$-skew Z-symmetric. This implies that $x\sigma(c)\bar{\alpha}((\sigma)(b)) = 0$ for some $0 \neq x \in S$. As σ is isomorphism, $\sigma(d) = x$ for some $0 \neq d \in R$. Therefore, we have $\sigma(dc\alpha(b)) = \sigma(d)\sigma(c)\sigma(\alpha(b)) = x\sigma(c)\sigma\alpha\sigma^{-1}(\sigma(b)) = x\sigma(c)\bar{\alpha}(\sigma(b)) = 0$ and so $dc\alpha(b) = 0$ as σ is injective. So, $c\alpha(b) \in Z_r(R)$. Hence, R is right α-skew Z-symmetric. □

In the following result, we demonstrate that α-skew Z-symmetric rings are preserved under arbitrary products:

Theorem 2.1. Let $\{(R_i, \alpha_i) : i \in I\}$ be a family of rings with an endomorphism $\alpha_i : R_i \to R_i$. Then, the product of all rings R is right (resp., left) α-skew Z-symmetric if and only if each R_i is right (resp., left) α_i-skew Z-symmetric, where $\alpha := (\alpha_i)_{i\in I}$ and $R = \prod_{i\in I} R_i$.

Proof. Suppose that $R = \prod_{i\in I} R_i$ is a right α-skew Z-symmetric ring. Let x and y be two elements in R_j such that $xy \in Z_r(R_j)$. Then, $zxy = 0$ for some $0 \neq z \in R_j$. Consider elements $l = (l_i)_{i\in I}, m = (m_i)_{i\in I}, n = (n_i)_{i\in I} \in R$, where $l_i = m_i = 1, n_i = 0$, for each $i \neq j$ and $l_j = x, m_j = y, n_j = z$. Then, $n \neq 0$ and $nlm = 0$. Hence, $lm \in Z_r(R)$ and so $m\alpha(l) \in Z_r(R)$ as R is right α-skew Z-symmetric. Therefore, $dm\alpha(l) = 0$ for some nonzero $d = (d_i)_{i\in I} \in R$. This implies that $d_i = 0$, for all $i \neq j$ and $d_j y\alpha_j(x) = 0$. Since $d \neq 0$, $d_j \neq 0$. Hence, $y\alpha_j(x) \in Z_r(R_j)$. Thus, we have shown that for any $j \in I$, R_j is a right α_j-skew Z-symmetric ring.

Conversely, assume that all R_i's are right α_i-skew Z-symmetric rings. Let $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$ be two elements in R for which $xy \in Z_r(R)$. Then, $zxy = (z_i x_i y_i)_{i\in I} = 0$ for some nonzero $z = (z_i)_{i\in I} \in R$. As z is nonzero, z_j is nonzero for some $j \in I$. Hence, $x_j y_j \in Z_r(R_j)$ and so $y_j\alpha_j(x_j) \in Z_r(R_j)$. Therefore, $uy_j\alpha_j(x_j) = 0$ for some nonzero $u \in R_j$. Consider the element $v = (v_i)_{i\in I} \in R$ in which $v_i = 0$, for each $i \neq j$ and $v_j = u$. Then, clearly $v \neq 0$ and $vy\alpha(x) = (v_i y_i \alpha_i(x_i))_{i \in I} = 0$. Hence, $y\alpha(x) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. □

Corollary 2.3. [6, Theorem 3.3] An arbitrary product of rings is right (resp., left) Z-symmetric if and only if each ring of the product is right (resp., left) Z-symmetric.

Corollary 2.4. Let $\alpha \in End(R)$ and $e \in C(R) \cap I(R)$ such that $\alpha(e) = e$ and $\alpha(1) = 1$. Then, R is right (resp., left) α-skew Z-symmetric if and only if $(1-e)R$ and eR are right (resp., left) α-skew Z-symmetric.

3. Some extensions

Recall from [4], for any algebra R over S where S is any commutative ring, abelian group $R \oplus S$ forms a ring with respect to the multiplication given by $(x_1, y_1)(x_2, y_2) = (x_1x_2 + y_1x_2 + y_2x_1, y_1y_2)$, where $x_i \in R$ and $y_i \in S$. The term Dorroh extension of the ring R by S is used for this ring because for the first time, such extension ring was constructed by J. L. Dorroh [8] for ring of integers. The next outcome is analogous to [4, Proposition 2.26].

Theorem 3.1. Let D be Dorroh extension of the unitary algebra R over commutative ring S. Then for any S-endomorphism α of R satisfying $\alpha(1) = 1$, R is right (resp., left) α-skew Z-symmetric if and only if D is right (resp., left) $\bar{\alpha}$-skew Z-symmetric, where $\bar{\alpha}$ is the S-algebra endomorphism of D given by $\bar{\alpha}(x, y) = (\alpha(x), y)$.

Proof. Since any $s \in S$ can be written as $s = s1 \in R$, so $R = \{r + s : (r, s) \in D\}$. Suppose that R is right α-skew Z-symmetric. Let (x_2, y_2) and (x_3, y_3) be two elements in D for which $(x_2, y_2)(x_3, y_3) \in Z_r(D)$. Then, $(x_1, y_1)(x_2, y_2)(x_3, y_3) = (0, 0)$ for some nonzero (x_1, y_1) in D. So, we have an equivalent expression $(x_1 + y_1 1)(x_2 + y_2 1)(x_3 + y_3 1) = 0$ together with $y_1y_2y_3 = 0$. There are two cases.

<u>Case-I:- $x_1 + y_1 1 = 0$:</u> In this case, $(x_1, y_1) = (-y_1 1, y_1)$. So, $(x_1, y_1)(x, y) = 0 = (x, y)(x_1, y_1) \iff yy_1 = 0$ for any $(x, y) \in D$. Since S is commutative, we have $y_1y_3y_2 = 0$. Hence, we get

$$(x_1, y_1)(x_3, y_3)\bar{\alpha}(x_2, y_2) = (x_1, y_1)(x_3, y_3)(\alpha(x_2), y_2) = (0, 0)$$

which implies that $(x_3, y_3)\bar{\alpha}(x_2, y_2) \in Z_r(D)$.

Case-II:- $x_1 + y_1 1 \neq 0$: In this case, $(x_2 + y_2 1)(x_3 + y_3 1) \in Z_r(R)$. Hence, $(x_3 + y_3 1)(\alpha(x_2) + y_2 1) = (x_3 + y_3 1)\alpha(x_2 + y_2 1) \in Z_r(R)$ because R is right α-skew Z-symmetric and $\alpha(1) = 1$. So, $((v - y_1 1) + y_1 1)(x_3 + y_3 1)(\alpha(x_2) + y_2 1) = v(x_3 + y_3 1)(\alpha(x_2) + y_2 1) = 0$ for some $0 \neq v \in R$. Since $y_2 y_3 y_1 = 0$ due to commutativity of S, we have $((v - y_1 1), y_1)(x_3, y_3)(\alpha(x_2), y_2) = (0, 0)$. Since $v \neq 0$, $(v - y_1 1), y_1)$ cannot be zero. Thus, $(x_3, y_3)(\alpha(x_2), y_2) = (x_3, y_3)\bar{\alpha}(x_2, y_2) \in Z_r(D)$. Hence, D is right $\bar{\alpha}$-skew Z-symmetric.

On the other hand, suppose that D is a right $\bar{\alpha}$-skew Z-symmetric ring. Let b and c be two elements in R for which $bc \in Z_r(R)$. Then, $abc = 0$ for some $0 \neq a \in R$. Then $(a, 0), (b - 1, 1_S), (c - 1, 1_S) \in D$, $(a, 0)$ is nonzero and $(a, 0)(b - 1, 1_S)(c - 1, 1_S) = (abc, 0) = (0, 0)$ which implies $(b - 1, 1_S)(c - 1, 1_S) \in Z_r(D)$. Hence, $(c - 1, 1_S)(\alpha(b) - 1, 1_S) \in Z_r(D)$ as D is right $\bar{\alpha}$-skew Z-symmetric. So, $(r, s)((c - 1, 1_S)(\alpha(b) - 1, 1_S)) = ((r + s1)c\alpha(b), 0) = (rc\alpha(b) + s(c\alpha(b) - 1), s)(0, 0)$ for some $(0, 0) \neq (r, s) \in D$. This gives $s = 0$ and $rc\alpha(b) = 0$. Since $(r, s) \neq 0$ and $s = 0$, r must be nonzero. Hence, $c\alpha(b) \in Z_r(R)$. Thus, R is right α-skew Z-symmetric. □

Corollary 3.1. [6, Proposition 3.8] Let D be the Dorroh extension of a unitary algebra R over a commutative ring S. Then, R is right (resp., left) Z-symmetric if and only if D is so.

Recall from [4] that if S is a subring of the ring R and n is a positive integer, then the set, denoted and defined by,

$$[R, S]_n = \{(u_1, u_2, \cdots, u_n, v, v, \cdots) : v \in S, u_i \in R, 1 \leq i \leq n\}$$

is a subring of the product ring of R by S. For any $\alpha \in End(R)$ satisfying $\alpha(S) \subseteq S$, the map $\bar{\alpha}$, given by $\bar{\alpha}(u_1, u_2, \cdots, u_n, v, v, \cdots) = (\alpha(u_1), \alpha(u_2), \cdots, \alpha(u_n), \alpha(v), \alpha(v), \cdots)$, lies in $End([R, S]_n)$.

Proposition 3.1. For any $\alpha \in End(R)$ satisfying condition $\alpha(1) = 1$ and any subring S of R having the same identity with the condition that $\alpha(S) \subseteq S$, the ring $[R, S]_n$ is right (resp., left) $\bar{\alpha}$-skew Z-symmetric if and only if R and S are right (resp., left) α-skew Z-symmetric.

Proof. Suppose that R and S are right α-skew Z-symmetric and let $Y = (b_1, b_2, \cdots, b_n, s_2, s_2, \cdots), Z = (c_1, c_2, \cdots, c_n, s_3, s_3, \cdots) \in [R, S]_n$ for which $YZ \in Z_r([R, S]_n)$. Then, $XYZ = 0$ for some nonzero $\neq X = (a_1, a_2, \cdots, a_n, s_1, s_1, \cdots) \in [R, S]_n$. Hence, it follows that $a_i b_i c_i = 0$ for each i and $s_1 s_2 s_3 = 0$. There are two cases:

Case-I: $s_1 \neq 0$:- Then $s_2 s_3 \in Z_r(S)$ and so $s_3 \alpha(s_2) \in Z_r(S)$ as S is right α-skew Z-symmetric. So, $s s_3 \alpha(s_2) = 0$ for some $0 \neq s \in S$. Hence, for $A = (0, 0, \cdots, 0, s, s, \cdots) \in [R, S]_n$, $A \neq 0$ and $AZ\bar{\alpha}(Y) = 0$. Thus, $Z\bar{\alpha}(Y) \in Z_r([R, S]_n)$.

Case-II: $s_1 = 0$: Since $X \neq 0$, $a_i \neq 0$ for some i. This implies that $b_i c_i \in Z_r(R)$. Hence, we have $c_i \alpha(b_i) \in Z_r(R)$. So, $d_i c_i \alpha(b_i) = 0$ for some nonzero $d_i \in R$. Hence, for $B = (0, 0, \cdots, d_i, \cdots, 0, 0, 0 \cdots) \in [R, S]_n$, $B \neq 0$ and $BZ\bar{\alpha}(Y) = 0$. Thus, $Z\bar{\alpha}(Y) \in Z_r([R, S]_n)$.

Therefore, $Z\bar{\alpha}(Y) \in Z_r([R, S]_n)$ in each case and so $[R, S]_n$ is a right $\bar{\alpha}$-skew Z-symmetric ring.

Conversely, suppose that the ring $[R, S]_n$ is right $\bar{\alpha}$-skew Z-symmetric. Let $b, c \in R$ and $s_2, s_3 \in S$ be such that $bc \in Z_r(R)$ and $s_2 s_3 \in Z_r(S)$. Then, there exist $0 \neq a \in R, 0 \neq s_1 \in S$ such that $abc = 0$ and $s_1 s_2 s_3 = 0$. If we take $A = (a, 0, \cdots 0, 0, 0, \cdots), B = (b, 1, \cdots, 1, 1, 1, \cdots), C = (c, 1, \cdots, 1, 1, 1, \cdots), X = (0, 0, \cdots, 0, s_1, s_1, \cdots,), Y = (1, 1, \cdots, 1, s_2, s_2, \cdots,), Z = (1, 1, \cdots, 1, s_3, s_3, \cdots,) \in [R, S]_n$, then we have $A \neq 0, B \neq 0$ and $ABC = 0 = XYZ$ which implies that $BC, YZ \in Z_r([R, S]_n)$; and so $C\bar{\alpha}(B), Z\bar{\alpha}(Y) \in Z_r([R, S]_n)$. Hence, there exist nonzero elements $U = (d_1, d_2, \cdots, d_n, s, s, \cdots), V = (e_1, e_2, \cdots, e_n, s', s', \cdots) \in [R, S]_n$ such that $UC\bar{\alpha}(B) = 0 = VZ\bar{\alpha}(Y)$. Since $UC\bar{\alpha}(B) = 0$ and $U \neq 0$, we must have $d_1 \neq 0$ and $d_1 c \alpha(b) = 0$. This implies that $c\alpha(b) \in Z_r(R)$. Similarly, $s_3 \alpha(s_2) \in Z_r(S)$. Thus, the rings R and S are right α-skew Z-symmetric. □

Corollary 3.2. If a ring R and its a subring S share the same identity, then $[R, S]_n$ is right Z-symmetric if and only if R and S are so.

The following proposition is a generalization of [6, Proposition 3.10] and it has an analogous proof.

Proposition 3.2. The following holds for any endomorphism α of a ring R:

(1) R is right α-skew Z-symmetric if and only if $T_n(R)$ is right $\bar{\alpha}$-skew Z-symmetric, where $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$.

(2) R is right α-skew Z-symmetric if $M_n(R)$ is right $\bar{\alpha}$-skew Z-symmetric, where $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$.

Corollary 3.3. [6, Proposition 3.10] The following holds for any ring R:

(1) R is right (resp., left) Z-symmetric if and only if $T_n(R)$ is right (resp., left) Z-symmetric.

(2) R is right (resp., left) Z-symmetric if $M_n(R)$ is right (resp., left) Z-symmetric.

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