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GENERALIZATION OF OSTROWSKI'S TYPE INEQUALITY VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL AND APPLICATIONS IN NUMERICAL INTEGRATION, PROBABILITY THEORY AND SPECIAL MEANS

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Abstract. We apply Riemann-Liouville fractional integral to get a new generalization of Ostrowski's type integral inequality. We may prove new estimates for the remainder term of the midpoint's, trapezoid's, & Simpson's formulae as a result of the generalization. Our estimates are generalized and recaptured some previously obtained estimates. Applications are also deduced for numerical integration, probability theory and special means.

1. Introduction

In the development of mathematics, inequalities are one of the most powerful tools. From two decades back, scholars researched on fractional calculus because of its importance in inequalities.

We quote from [3],

"The subject of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers in. In recent years, the fractional calculus has played a significant role in many areas of science and engineering."

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Due to worth of fractional integral inequalities, many scholars have mentioned certain generalizations of fractional integral inequalities (see [2, 21, 22, 23]).

In 1938, A. M. Ostrowski was a Ukrainian mathematician, who had presented an inequality in his article [19]. Since then this inequality is called an Ostrowski inequality and this result had obtained by applying the Montgomery identity. A number of researchers have written their articles [1, 7, 8, 15, 17] about generalizations of Ostrowski's inequality in the past some decades. Ostrowski's inequality has been proved to be a huge and remarkable tool for the enlargement of various fields of mathematics. Inequalities including integral which create bounds in the physical quantities, are of great significant in the sense that these types of inequalities are not only used in integral approximation theory, operator theory, nonlinear analysis, numerical integration, stochastic analysis, information theory, statistics and probability theory but we may also see its applications in the several branches of physics, engineering and biological sciences. We refer to the readers [4, 11, 12, 13, 16, 18] for some recent contributions to the study of Ostrowski's inequality.

S. S. Dragomir et. al. derived the generalization of Ostrowski's type inequality in [9] which is as follow:

Proposition 1.1. *Let $g : [j,k] \to \mathbb{R}$ be continuous on $[j,k]$ and differentiable on (j,k) & whose derivative $g' : (j,k) \to \mathbb{R}$ is bounded on (j,k), where $\|g'\|_\infty = sup_{\tau \in [j,k]} |g'(\tau)| < \infty$. Then,*

$$\left| \int_j^k g(\tau)d\tau - \left[g(\theta)(1-\lambda) + \lambda \frac{g(j)+g(k)}{2} \right] (k-j) \right|$$

$$\leq \left[\frac{1}{4}(k-j)^2(\lambda^2 + (1-\lambda)^2) + \left(\theta - \frac{j+k}{2} \right)^2 \right] \|g'\|_\infty, \tag{1.1}$$

for all $\lambda \in [0,1]$ and $j + \lambda(\frac{k-j}{2}) \leq \theta \leq k - \lambda(\frac{k-j}{2})$.

By applying (1.1), the scholars obtained estimates for the remainder term of the midpoint's, trapezoid's, & Simpson's formulae. They also gave applications in special means and numerical integration.

We need here to define Riemann-Liouville fractional integral(RLFI) (see[10]) for proving our next main result in the second section.

Definition 1.2. The RLFI operator of order $\gamma > 0$ is stated as

$$J_j^{\gamma} g(\theta) = \frac{1}{\Gamma(\gamma)} \int_j^{\theta} (\theta - \tau)^{\gamma-1} g(\tau) d\tau,$$
$$J_j^0 g(\theta) = g(\theta),$$

and gamma function $\Gamma(\gamma)$ is stated as

$$\Gamma(\gamma) = \int_0^{\infty} \theta^{\gamma-1} e^{-\theta} d\theta.$$

In the given article, we would prove a perturbation via RLFI of (1.1). Using obtained inequality, we would get some tighter error bounds for the midpoint's, trapezoid's and Simpson's quadrature formulae and for probability theory. Similar perturbed inequalities without RLFI are also considered in [5, 6]. In section 3, 4 and 5, we would give applications in numerical integration, probability theory and special means respectively.

2. Generalization of Ostrowski's Type Inequality Via RLFI

Under present section we would give our results about Ostrowski's type inequality via RLFI which are as follow:

Theorem 2.1. *Let $J \subset \mathbb{R}$ and $j, k \in J, j < k$. If $g : J \to \mathbb{R}$ is differentiable function such that $\mathfrak{M} \leq g'(\tau) \leq \mathfrak{N}$, $\forall\ \tau \in [j, k]$, for some constants $\mathfrak{M}, \mathfrak{N} \in \mathbb{R}$, then*

$$\Bigg|(1-\lambda)g(\theta) - \frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) + \frac{\lambda(k-\theta)^{1-\gamma}}{2(k-j)^{1-\gamma}}J_j^0 g(j) + J_j^{\gamma-1}(P(\theta,k)g(k))$$
$$- \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)\left(\theta - \frac{j+k}{2}\right)(k-\theta)^{1-\gamma}\Bigg| \leq (k-\theta)^{1-\gamma}\Bigg((k-j)^{\gamma-1}\mathfrak{N}$$
$$- \frac{\mathfrak{N}+\mathfrak{M}}{2}\Bigg)\left(\frac{(k-j)}{4}[\lambda^2 + (1-\lambda)^2] + \frac{1}{(k-j)}\left(\theta - \frac{j+k}{2}\right)^2\right), \tag{2.1}$$

holds. Where $j + \lambda(\frac{k-j}{2}) \leq \theta \leq k - \lambda(\frac{k-j}{2})$ and $\lambda \in [0, 1]$.

Proof. For the sake of proof we state the fractional Peano kernel as;

$$P(\theta,\tau)=\frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)\begin{cases}\tau-\left(j+\lambda\frac{k-j}{2}\right), & \text{if} \quad \tau\in[j,\theta),\\ \\ \tau-\left(k-\lambda\frac{k-j}{2}\right), & \text{if} \quad \tau\in[\theta,k].\end{cases} \tag{2.2}$$

Applying RLFI operator and by parts formula of integration, obtain

$$\begin{aligned} J_j^{\gamma}(P(\theta,k)g(k)) &= \frac{1}{\Gamma(\gamma)}\int_j^k (k-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau \\ &= (1-\lambda)g(\theta)-\frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) \\ &+ \frac{\lambda(k-\theta)^{1-\gamma}}{2(k-j)^{1-\gamma}}J_j^0 g(j)+J_j^{\gamma-1}(P(\theta,k)g(k)). \end{aligned} \tag{2.3}$$

It is clear that

$$\frac{1}{\Gamma(\gamma)}\int_j^k P(\theta,\tau)d\tau=(1-\lambda)\left(\theta-\frac{j+k}{2}\right)(k-\theta)^{1-\gamma}. \tag{2.4}$$

Suppose $C=\frac{\mathfrak{N}+\mathfrak{M}}{2}$. From (2.3) and (2.4), it follows that

$$\begin{aligned} &\frac{1}{\Gamma(\gamma)}\int_j^k P(\theta,\tau)[(k-\tau)^{\gamma-1}g'(\tau)-C]d\tau \\ &= (1-\lambda)g(\theta)-\frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) \\ &+\frac{\lambda(k-\theta)^{1-\gamma}}{2(k-j)^{1-\gamma}}J_j^0 g(j)+J_j^{\gamma-1}(P(\theta,k)g(k)) \\ &-C(1-\lambda)\left(\theta-\frac{j+k}{2}\right)(k-\theta)^{1-\gamma}. \end{aligned} \tag{2.5}$$

Another way we have

$$\begin{aligned} &\left|\frac{1}{\Gamma(\gamma)}\int_j^k P(\theta,\tau)[(k-\tau)^{\gamma-1}g'(\tau)-C]d\tau\right| \\ &\le \frac{1}{\Gamma(\gamma)}\max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}g'(\tau)-C|\cdot\int_j^k |P(\theta,\tau)|d\tau. \end{aligned} \tag{2.6}$$

Since

$$\max_{\tau\in[j,k]}|(k-\tau)^{\gamma-1}g'(\tau)-C|\le(k-j)^{\gamma-1}\mathfrak{N}-C, \tag{2.7}$$

further we have

$$\int_j^k |P(\theta,\tau)|d\tau = (k-\theta)^{1-\gamma}\Gamma(\gamma)$$

$$\times \left(\frac{(k-j)}{4}[\lambda^2+(1-\lambda)^2]+\frac{1}{(k-j)}\left(\theta-\frac{j+k}{2}\right)^2\right) \tag{2.8}$$

Using (2.6) to (2.8) we may written as

$$\left|\frac{1}{\Gamma(\gamma)}\int_j^k P(\theta,\tau)\left[(k-\tau)^{\gamma-1}g'(\tau)-\frac{\mathfrak{N}+\mathfrak{M}}{2}\right]d\tau\right|$$

$$\leq (k-\theta)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N}-\frac{\mathfrak{N}+\mathfrak{M}}{2}\right)$$

$$\times\left(\frac{(k-j)}{4}[\lambda^2+(1-\lambda)^2]+\frac{1}{(k-j)}\left(\theta-\frac{j+k}{2}\right)^2\right). \tag{2.9}$$

Using (2.5) to (2.9) we easily obtain our required result (2.1). □

Remark 2.2. If put $\gamma=1$ in Theorem 2.1, then we recapture the Theorem 2 of [24].

Remark 2.3. If put $\lambda=0$ in (2.3) of Theorem 2.1 and after some rearrangements, then we recapture the Montgomery fractional identity (see Lemma 3.1 of [2]).

Remark 2.4. If put $\gamma=1$ in (2.3) of Theorem 2.1, then we recapture the Montgomery identity with parameter (see equation (2.2) of Theorem 2 of [9]).

Corollary 2.5. *Let the supposition of Theorem* 2.1 *be true, then*

$$\left|g(\theta)-\frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k)+J_j^{\gamma-1}(P(\theta,k)g(k))\right.$$

$$\left.-\frac{(\mathfrak{N}+\mathfrak{M})}{2}\left(\theta-\frac{j+k}{2}\right)(k-\theta)^{1-\gamma}\right|\leq (k-\theta)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N}-\frac{\mathfrak{N}+\mathfrak{M}}{2}\right)$$

$$\times\left(\frac{(k-j)}{4}+\frac{1}{(k-j)}\left(\theta-\frac{j+k}{2}\right)^2\right), \tag{2.10}$$

holds and specially

$$\left|g(\frac{j+k}{2})-\frac{(\frac{k-j}{2})^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k)+J_j^{\gamma-1}(P(\frac{j+k}{2},k)g(k))\right| \tag{2.11}$$

$$\leq\frac{(k-j)}{4}\left(\frac{k-j}{2}\right)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N}-\frac{\mathfrak{N}+\mathfrak{M}}{2}\right).$$

Proof. By putting $\lambda = 0$ in (2.1) we obtain (2.10) and $\theta = \frac{j+k}{2}$ in (2.10) we obtain (2.11). □

Remark 2.6. If put $\gamma = 1$ in (2.10) of Corollary 2.5, then we recapture the result (2.10) of Corollary 1 of [24].

Remark 2.7. If put $\gamma = 1$ in (2.11) of Corollary 2.5, then we recapture the result (2.11) of Corollary 1 of [24].

Corollary 2.8. *Let the supposition of Theorem* 2.1 *be true, then*

$$\left| -\frac{(\frac{k-j}{2})^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) + \frac{(\frac{k-j}{2})^{1-\gamma}}{2(k-j)^{1-\gamma}}J_j^0 g(j) + J_j^{\gamma-1}(P(\frac{j+k}{2},k)g(k))\right| \leq \frac{(k-j)}{4}\left(\frac{k-j}{2}\right)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right). \tag{2.12}$$

Proof. By putting $\lambda = 1$ and $\theta = \frac{j+k}{2}$ in (2.1) we obtain (2.12). □

Remark 2.9. If put $\gamma = 1$ in (2.12) of Corollary 2.8, then we recapture the result (2.12) of Corollary 2 of [24].

Corollary 2.10. *Let the supposition of Theorem* 2.1 *be true, then*

$$\left|\frac{g(\theta)}{2} - \frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) + \frac{(k-\theta)^{1-\gamma}}{4(k-j)^{1-\gamma}}J_j^0 g(j) + J_j^{\gamma-1}(P(\theta,k)g(k)) - \frac{(\mathfrak{N}+\mathfrak{M})}{4}\left(\theta - \frac{j+k}{2}\right)(k-\theta)^{1-\gamma}\right| \leq (k-\theta)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right)\left(\frac{(k-j)}{8} + \frac{1}{(k-j)}\left(\theta - \frac{j+k}{2}\right)^2\right), \tag{2.13}$$

holds and specially

$$\left|\frac{g(\frac{j+k}{2})}{2} - \frac{(\frac{k-j}{2})^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) + \frac{(\frac{k-j}{2})^{1-\gamma}}{4(k-j)^{1-\gamma}}J_j^0 g(j) + J_j^{\gamma-1}(P(\frac{j+k}{2},k)g(k))\right| \leq \frac{(k-j)}{8}\left(\frac{k-j}{2}\right)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right). \tag{2.14}$$

Proof. By putting $\lambda = \frac{1}{2}$ in (2.13) we obtain (2.13) and $\theta = \frac{j+k}{2}$ in (2.13) we obtain (2.14). □

Remark 2.11. If put $\gamma = 1$ in (2.10) of Corollary 2.10, then we recapture the result (2.13) of Corollary 3 of [24].

Remark 2.12. If put $\gamma = 1$ in (2.14) of Corollary 2.10, then we recapture the result (2.14) of Corollary 3 of [24].

Corollary 2.13. *Let the supposition of Theorem* 2.1 *be true, then*

$$\Bigg|\frac{2}{3}g(\theta) - \frac{(k-\theta)^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) + \frac{(k-\theta)^{1-\gamma}}{6(k-j)^{1-\gamma}}J_j^{0}g(j) + J_j^{\gamma-1}(P(\theta,k)g(k)) - \frac{(\mathfrak{N}+\mathfrak{M})}{3}\left(\theta-\frac{j+k}{2}\right)(k-\theta)^{1-\gamma}\Bigg| \leq (k-\theta)^{1-\gamma}\Bigg((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\Bigg)\left(\frac{5(k-j)}{36} + \frac{1}{(k-j)}\left(\theta-\frac{j+k}{2}\right)^2\right), \tag{2.15}$$

holds and specially

$$\left|\frac{2g(\frac{j+k}{2})}{3} - \frac{(\frac{k-j}{2})^{1-\gamma}}{(k-j)}\Gamma(\gamma)J_j^{\gamma}g(k) + \frac{(\frac{k-j}{2})^{1-\gamma}}{6(k-j)^{1-\gamma}}J_j^{0}g(j) + J_j^{\gamma-1}(P(\frac{j+k}{2},k)g(k))\right| \leq \frac{5(k-j)}{36}\left(\frac{k-j}{2}\right)^{1-\gamma}\left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right). \tag{2.16}$$

Proof. By putting $\lambda = \frac{1}{3}$ in (2.1) we obtain (2.15) and $\theta = \frac{j+k}{2}$ in (2.15) we obtain (2.16). □

Remark 2.14. If put $\gamma = 1$ in (2.15) of Corollary 2.13, then we recapture the result (2.15) of Corollary 4 of [24].

Remark 2.15. If put $\gamma = 1$ in (2.16) of Corollary 2.13, then we recapture the result (2.16) of Corollary 4 of [24].

Remark 2.16. Inequalities (2.11), (2.12), (2.14) and (2.16) are generalized and better estimates than the corresponding estimates presented in [9] and further that above said estimates are generalization of the corresponding estimates which are obtained in [24]. For example, consider the following inequality, obtained in [9]:

$$\left|\frac{k-j}{6}\left[g(j) + 4g\left(\frac{j+k}{2}\right) + g(k)\right] - \int_j^k g(\tau)d\tau\right| \leq \frac{5}{36}(k-j)^2\|g'\|_{\infty}. \tag{2.17}$$

If choose

$$\mathfrak{N} = \sup_{\tau\in[j,k]} g'(\tau), \qquad \text{and} \qquad \mathfrak{M} = \inf_{\tau\in[j,k]} g'(\tau)$$

then $\frac{(\mathfrak{N}-\mathfrak{M})}{2} \leq \|g'\|_\infty$ when $\gamma = 1$. Therefore, (2.16) is generalized and better than (2.17). In fact, if $sgn\,\mathfrak{M} = sgn\,\mathfrak{N}$ and $\mathfrak{M} \approx \mathfrak{N}$, then (2.16) may be much better than (2.17). It is also generalized and better than a corresponding inequality got in [20].

Remark 2.17. Note that the simple three-point quadrature rule that is given in (2.14) has a better estimate of error than the well known three-point Simpson's quadrature rule is given in (2.16).

3. Applications to Numerical Integration

We restrict further considerations to the trapezoidal quadrature rule. We also emphasize that similar considerations may be done for all quadrature rules considered in the previous section.

Suppose $I_m = \{j = \theta_0 < \theta_1 < ... < \theta_{m-1} = k\}$ be given sub-division of the $[j,k]$, where $\theta_{i+1} - \theta_i = h$. If we apply Theorem 2.1 to the $[\theta_i, \theta_{i+1}]$ with $\lambda = 1$ & sum over i from 0 to $m-1$, then we acquire the composite trapezoidal rule

$$\left(\frac{h}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^\gamma g(k) = Q_T(g, I_m) + E_T(g), \tag{3.1}$$

where

$$Q_T(g, I_m) = \sum_{i=0}^{m-1} h \left[\frac{\left(\frac{h}{2}\right)^{1-\gamma}}{2(h)^{1-\gamma}} J^0_{\theta_i} g(\theta_i) + J^{\gamma-1}_{\theta_i}(P(\frac{\theta_i + \theta_{i+1}}{2}, \theta_{i+1}) g(\theta_{i+1})) \right].$$

Using triangle inequality then get

$$\begin{aligned} |E_T(g)| &\leq \sum_{i=0}^{m-1} \frac{h^2}{4} \left(\frac{h}{2}\right)^{1-\gamma} \left(h^{\gamma-1}\mathfrak{N}_i - \frac{\mathfrak{N}_i + \mathfrak{M}_i}{2} \right) \\ &\leq \frac{(k-j)^2}{4m} \left(\frac{k-j}{2}\right)^{1-\gamma} \left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N} + \mathfrak{M}}{2} \right), \end{aligned} \tag{3.2}$$

where we choose $\mathfrak{N}_i = \max_{\tau\in[\theta_i,\theta_{i+1}]} g'(\tau)$, $\mathfrak{M}_i = \min_{\tau\in[\theta_i,\theta_{i+1}]} g'(\tau)$, $\mathfrak{M} \leq g'(\tau) \leq \mathfrak{N}$, $\forall\ \tau \in [j,k]$. The following classical estimation is given by [24]

$$|E_T(g)| \leq \frac{\|g''\|_\infty}{12m^2}(k-j)^3. \tag{3.3}$$

The following inequality is the fractional form of (3.3) .

$$|E_T(g)| \leq \frac{\|g''\|_\infty (k-j)^2}{6m^2} \left(\frac{k-j}{2}\right)^{1-\gamma} \left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right) \tag{3.4}$$

Note that we can apply (3.4) only if $g \in C^2[j,k]$, while we may apply (3.2) if $g \in C^1[j,k]$. Hence, the above obtained result enlarges the applicability of the trapezoidal rule.

Moreover, we may find a consequence of Theorem 5.3.2 of [14] which for the trapezoidal formula gives

$$\begin{aligned} \lim_{m\to\infty} m^2 \left[Q_T(g, I_m) - \left(\frac{h}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^\gamma g(k)\right] \\ = \frac{(k-j)}{6m^2} \left(\frac{k-j}{2}\right)^{1-\gamma} \left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right)[g'(k) - g'(j)], \end{aligned} \tag{3.5}$$

if $g \in C^2[j,k]$. This consequence characterizes the order of convergence of the composite trapezoidal rule. From (3.5), it follows

$$\lim_{m\to\infty} m \left[Q_T(g, I_m) - \left(\frac{h}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^\gamma g(k)\right] = 0, \tag{3.6}$$

if $g \in C^2[j,k]$. Using inequality (3.2), we may acquire a stronger consequence. Namely, we may derive that (3.6) holds for functions which belong to $C^1[j,k]$.

Theorem 3.1. *Let $g \in C^1[j,k]$. Then* (3.6) *holds.*

Proof. Let $\mathfrak{N}_i = \max_{\tau\in[\theta_i,\theta_{i+1}]} g'(\tau)$, $\mathfrak{M}_i = \min_{\tau\in[\theta_i,\theta_{i+1}]} g'(\tau)$. Since $g \in C^1[j,k]$, it follows that there exist $\zeta_i, \varsigma_i \in [\theta_i, \theta_{i+1}]$ such that $\mathfrak{N}_i = g'(\zeta_i)$ and $\mathfrak{M}_i = g'(\varsigma_i)$. From (3.1) and (3.2) we get

$$\begin{aligned} m \left[Q_T(g, I_m) - \left(\frac{h}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^\gamma g(k)\right] \\ \leq \frac{1}{4}\left(\frac{k-j}{2}\right)^{1-\gamma} \left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right) \sum_{i=0}^{m-1} \frac{(k-j)}{m}[g'(\zeta_i) - g'(\varsigma_i)]. \end{aligned} \tag{3.7}$$

Since $((k-j)/m)\sum_{i=0}^{m-1} g'(\zeta_i)$ and $((k-j)/m)\sum_{i=0}^{m-1} g'(\varsigma_i)$ are Riemann sums, we have

$$\lim_{m\to\infty} \frac{(k-j)}{m} \sum_{i=0}^{m-1} g'(\zeta_i) = \lim_{m\to\infty} \frac{(k-j)}{m} \sum_{i=0}^{m-1} g'(\varsigma_i) = \left(\frac{h}{2}\right)^{1-\gamma} \Gamma(\gamma) J_j^\gamma g'(k). \tag{3.8}$$

Thus, (3.6) holds. □

The consequence obtained in the above theorem characterizes the order of convergence of the composite trapezoidal quadrature formula for functions which belong to $C^1[j,k]$.

Remark 3.2. If put $\gamma = 1$ in Theorem 3.1, then we recapture the result of Theorem 3 of [24].

4. Applications to Probability Theory

Suppose random variable 'Z' be continuous with probability density function $g : [j,k] \to [0,1]$ & cumulative distribution function Φ is introduced and defined by us, i.e,

$$\Phi(\theta) = \Gamma(\gamma) J_j^{\gamma} g(\theta) = \int_j^{\theta} (\theta-\tau)^{\gamma-1} g(\tau) d\tau, \qquad j+\lambda\frac{k-j}{2} \le \theta \le k-\lambda\frac{k-j}{2},$$

and

$$\begin{aligned} E_{f1}(Z) &= \Gamma(\gamma) J_j^{\gamma-1}(kg(k)) = \int_j^k \tau(k-\tau)^{\gamma-2} g(\tau) d\tau, \\ E_{f2}(Z) &= \Gamma(\gamma) J_j^{\gamma}(kg'(k)) = \int_j^k \tau(k-\tau)^{\gamma-1} g'(\tau) d\tau, \\ \text{if } \gamma = 1, \text{ then } E(Z) &= \int_j^k \tau g(\tau) d\tau \end{aligned}$$

are the fractional expectation of random variable 'Z' in interval $[j,k]$. Then we can write the following theorem as:

Theorem 4.1. *Let the suppositions of Theorem* 2.1 *be true. Then get the following*

$$\Bigg|(1-\lambda)\Phi(\theta) - \frac{(k-\theta)^{1-\gamma}}{(k-j)}\Big((\gamma-1)E_{f1}(Z) - E_{f2}(Z)\Big) + \frac{\lambda(k-\theta)^{1-\gamma}}{2(k-j)^{1-\gamma}} J_j^0 \Phi(j)$$
$$+ J_j^{\gamma-1}(P(\theta,k)\Phi(k)) - \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)\left(\theta - \frac{j+k}{2}\right)(k-\theta)^{1-\gamma}\Bigg| \le (k-\theta)^{1-\gamma}$$

(4.1)

$$\times\left((k-j)^{\gamma-1}\mathfrak{N} - \frac{\mathfrak{N}+\mathfrak{M}}{2}\right)\left(\frac{(k-j)}{4}[\lambda^2+(1-\lambda)^2] + \frac{1}{(k-j)}\left(\theta - \frac{j+k}{2}\right)^2\right),$$

where $j+\lambda(\frac{k-j}{2}) \le \theta \le k-\lambda(\frac{k-j}{2})$ *and* $\lambda \in [0,1]$.

Proof. Put $g = \Phi$ we obtain (4.1), by applying the identity

$$\Gamma(\gamma) J_j^{\gamma} g(k) = \int_j^k (k-\tau)^{\gamma-1} g(\tau) d\tau = (\gamma-1) E_{f1}(Z) - E_{f2}(Z)$$

and $\Phi(j) = 0, \ \Phi(k) = 1.$

□

Corollary 4.2. *Select $\gamma = 1$ in Theorem 4.1. Then get the following*

$$\left| (k-j) \left[\frac{\lambda}{2} + (1-\lambda)\Phi(\theta) - \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)\left(\theta - \frac{j+k}{2}\right)\right] - k + E(Z) \right|$$

$$\leq \frac{\mathfrak{N}-\mathfrak{M}}{2}\left(\frac{(k-j)^2}{4}[\lambda^2 + (1-\lambda)^2] + \left(\theta - \frac{j+k}{2}\right)^2\right), \tag{4.2}$$

where $j + \lambda(\frac{k-j}{2}) \leq \theta \leq k - \lambda(\frac{k-j}{2})$ *and* $\lambda \in [0,1]$.

Before application to special means, we would present some special means and these means will apply in the 5th section.

Special Means: These means can be found in [25].

(a) The Arithmetic Mean

$$A = \frac{j+k}{2}; \quad j,k \geq 0.$$

(b) The Geometric Mean

$$G = G(j,k) = \sqrt{jk}; \quad j,k \geq 0.$$

(c) The Harmonic Mean

$$H = H(j,k) = \frac{2}{\frac{1}{j} + \frac{1}{k}}; \quad j,k > 0.$$

(d) The Logarithmic Mean

$$L = L(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \dfrac{k-j}{\ln k - \ln j}, & \text{if} \quad j \neq k; \end{cases} \quad j,k > 0.$$

(e) Identric Mean

$$I = I(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \ln\left(\dfrac{\left(\frac{k^k}{j^j}\right)^{\frac{1}{k-j}}}{e}\right), & \text{if} \quad j \neq k; \end{cases} \quad j,k > 0.$$

(f) $p-$Logarithmic Mean

$$L_p = L_p(j,k) = \begin{cases} j, & \text{if} \quad j = k \\ \left(\dfrac{k^{p+1} - j^{p+1}}{(p+1)(k-j)}\right)^{\frac{1}{p}}, & \text{if} \quad j \neq k, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}, j,k > 0$. It is known that L_p monotonically increasing over $p \in \mathbb{R}$, $L_0 = I$ and $L_{-1} = L$.

5. Application to Special Means

Example no. 1: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= \theta^p,\ p \in \mathbb{R}\backslash\{-1,0\},\ then\, for\ j < k, \\ then \qquad \frac{1}{(k-j)}\int_j^k g(\tau)d\tau &= L_p^p(j,k), \\ \frac{g(j)+g(k)}{2} &= A(j^p,k^p), \\ and \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j + \lambda(\frac{k-j}{2}), k - \lambda(\frac{k-j}{2})]$.

Therefore, (2.1) becomes

$$\left|(k-j)\left[\lambda A(j^p,k^p) + (1-\lambda)\theta^p - \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)(\theta - A) - L_p^p(j,k)\right]\right|$$

$$\leq \frac{\mathfrak{N}-\mathfrak{M}}{2}\left(\frac{(k-j)^2}{4}[\lambda^2 + (1-\lambda)^2] + (\theta - A)^2\right). \tag{5.1}$$

Choose $\theta = A$ in (5.1), get

$$\left|(k-j)\left[\lambda A(j^p,k^p) + (1-\lambda)A^p - L_p^p(j,k)\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2[\lambda^2 + (1-\lambda)^2].$$

Moreover for $\lambda = 1$

$$\left|(k-j)\left[A(j^p,k^p)-L_p^p(j,k)\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2.$$

Example no. 2: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= \frac{1}{\theta}, \quad \theta \neq 0 \\ \textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= L^{-1}(j,k), \\ \frac{g(j)+g(k)}{2} &= \frac{A}{G^2}, \\ \textit{and} \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j+\lambda(\frac{k-j}{2}), k-\lambda(\frac{k-j}{2})] \subset (0,\infty)$.

Therefore, (2.1) becomes

$$\left|(k-j)\left[\lambda\frac{A}{G^2}+(1-\lambda)\frac{1}{\theta}-\frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)(\theta-A)-L^{-1}(j,k)\right]\right|$$

$$\leq \frac{\mathfrak{N}-\mathfrak{M}}{2}\left(\frac{(k-j)^2}{4}[\lambda^2+(1-\lambda)^2]+(\theta-A)^2\right). \tag{5.2}$$

If we choose $\theta = A$ in (5.2), we get

$$\left|(k-j)\left[\lambda\frac{A}{G^2}+(1-\lambda)\frac{1}{A}-L^{-1}(j,k)\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2[\lambda^2+(1-\lambda)^2].$$

For $\lambda = 1$

$$\left|(k-j)\left[\frac{A}{G^2}-L^{-1}(j,k)\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2.$$

Example no. 3: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= ln\theta, \quad \theta \in (0,\infty) \\ \textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= ln(I(j,k)), \\ \frac{g(j)+g(k)}{2} &= lnG, \\ \textit{and} \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j+\lambda(\frac{k-j}{2}), k-\lambda(\frac{k-j}{2})] \subset (0,\infty)$.

Therefore, (2.1) becomes

$$\left|(k-j)\left[ln\frac{G^{\lambda}\theta^{(1-\lambda)}}{I(j,k)} - \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)(\theta-A)\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{2}\left(\frac{(k-j)^2}{4}[\lambda^2+(1-\lambda)^2]+(\theta-A)^2\right). \tag{5.3}$$

If we choose $\theta = A$ in (5.3), we get

$$\left|ln\left[\frac{G^{\lambda}A^{(1-\lambda)}}{I(j,k)}\right]^{(k-j)}\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2[\lambda^2+(1-\lambda)^2].$$

For $\lambda = 1$

$$\left|ln\left[\frac{G}{I(j,k)}\right]^{(k-j)}\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2.$$

Example no. 4: Consider

$$\begin{aligned} \gamma &= 1, \\ g(\theta) &= e^{\theta}, \quad \theta \in (-\infty,\infty) \\ \textit{then} \qquad \frac{1}{k-j}\int_j^k g(\tau)d\tau &= \frac{e^k-e^j}{k-j}, \\ \frac{g(j)+g(k)}{2} &= A(e^j,e^k), \\ \textit{and} \qquad \frac{j+k}{2} &= A, \end{aligned}$$

where $\theta \in [j+\lambda(\frac{k-j}{2}), k-\lambda(\frac{k-j}{2})]$.

Therefore, (2.1) becomes

$$\left|(k-j)\left[\lambda A(e^j,e^k)+(1-\lambda)e^{\theta} - \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)(\theta-A) - \frac{e^k-e^j}{k-j}\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{2}\left(\frac{(k-j)^2}{4}[\lambda^2+(1-\lambda)^2]+(\theta-A)^2\right). \tag{5.4}$$

If we choose $\theta = A$ in (5.4), we get

$$\left|(k-j)\left[\lambda A(e^j,e^k)+(1-\lambda)e^{A} - \frac{e^k-e^j}{k-j}\right]\right| \leq \frac{\mathfrak{N}-\mathfrak{M}}{8}(k-j)^2[\lambda^2+(1-\lambda)^2].$$

For $\lambda = 1$

$$\left|(k-j)A(e^j, e^k) - (e^k - e^j)\right| \leq \frac{\mathfrak{N} - \mathfrak{M}}{8}(k-j)^2.$$

Example no. 5: Consider

$$\begin{array}{lrcl} & \gamma &=& 1, \\ & g(\theta) &=& \tan\theta, \quad \theta \neq \frac{\pi}{2} \pm n\pi \\ \textit{then} & \frac{1}{k-j}\int_j^k g(\tau)d\tau &=& \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}, \\ & \frac{g(j)+g(k)}{2} &=& A(\tan j, \tan k), \\ \textit{and} & \frac{j+k}{2} &=& A, \end{array}$$

where $\theta \in [j + \lambda(\frac{k-j}{2}), k - \lambda(\frac{k-j}{2})]$.
Therefore, (2.1) becomes

$$\Bigg|(k-j)\Bigg[\lambda A(\tan j, \tan k) + (1-\lambda)\tan\theta - \frac{(\mathfrak{N}+\mathfrak{M})}{2}(1-\lambda)(\theta - A)$$

$$-\ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\Bigg]\Bigg| \leq \frac{\mathfrak{N} - \mathfrak{M}}{2}\left(\frac{(k-j)^2}{4}[\lambda^2 + (1-\lambda)^2] + (\theta - A)^2\right). \tag{5.5}$$

If we choose $\theta = A$ in (5.5), we get

$$\left|(k-j)\left[\lambda A(\tan j, \tan k) + (1-\lambda)\tan A - \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\right]\right|$$
$$\leq \frac{\mathfrak{N} - \mathfrak{M}}{8}(k-j)^2[\lambda^2 + (1-\lambda)^2].$$

For $\lambda = 1$

$$\left|(k-j)\left[A(\tan j, \tan k) - \ln\left[\frac{\sec k}{\sec j}\right]^{k-j}\right]\right| \leq \frac{\mathfrak{N} - \mathfrak{M}}{8}(k-j)^2.$$

6. Conclusion

We proved new Ostrowski's type estimates for the remainder term of the midpoint's, trapezoid's, & Simpson's formulae as consequences of the generalized fractional integral. Our estimates are generalized and recaptured the results of articles [2], [9] and [24] that are previously obtained estimates. Moreover, we have given applications to numerical integration, probability theory and special means.

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