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A NOVEL FIXED POINT THEOREM OF REICH-PEROV TYPE α-CONTRACTIVE MAPPINGS IN VECTOR-VALUED METRIC SPACES

SUNARSINI[(1)], MAHMUD YUNUS[(2)] AND SUBIONO[(3)]

Abstract. This article discusses a novel concept of Reich-Perov type α-contractive mappings in vector-valued metric spaces. First, we define Reich-Perov-type contractive mappings using a novel concept in vector-valued metric spaces. Later, we investigate the sufficient conditions for a Reich-Perov type contractive mapping to have a unique fixed point in the spaces. By defining an α-contractive mapping, we next show the sufficient conditions of the existence and uniqueness of a fixed point of the Reich-Perov type α-contractive mappings in vector-valued metric spaces.

1. Introduction

Since the first introduction of metric space by French mathematician Maurice Fréchet in 1906, research on metric spaces has been developed, including the fixed point theorem in the spaces. Stefan Banach, 1920, was a pioneer researcher of the fixed point theorem in complete metric spaces, also known as the Banach fixed point theorem [1]. In addition, the researchers conducted many studies on various types of contractive mapping on complete metric spaces. For instance, Reich and Kannan introduced Reich and Kannan's types of contractive mapping (see [2], [3]). Moreover, we refer to [4, 5] for further works in the form of contractive mappings. In addition, Rhoades introduced several types of contractive mapping in complete metric spaces and proved the related issues of these mappings [6].

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Meanwhile, there are various applications of the Banach fixed point theorem in the various branches of mathematics, computer science, and engineering (see [7], [8], [9], [10]). In computer science, De Bekker and De Vink [7] applied Banach fixed point theorem to abstract programming languages, both linear and branched denotations. The classical Banach fixed point theorem is the primary tool for constructing two models and the semantic operators involved. Various semantic definitions were justified through high-level transformations by characterization as a unique fixed contractive point in the complete metric space. Rousseau in [8] involved the Banach fixed point theorem on image compression. Meanwhile, Ege and Karaca used Banach fixed point theorem for digital images [9].

Existing research development focuses not only on the type of contractive mapping but also on its space. One of the research results is vector-valued metric spaces in $\mathbb{R}^n$ introduced by a Russian mathematician A.I. Perov [11]. Perov provided a new concept of contractive mapping in the spaces known as the Perov contractive type. Some authors, e.g., Altun et al. (see[12], [13]), developed a Perov type fixed point theorem in vector-valued metric spaces. They also looked into how the Perov type fixed point theorem could be applied to vector-valued metric spaces. Also, they discussed its application to semilinear operator systems. Meanwhile, Vetro and Radenović [10] introduced the Perov type contractive mapping and discussed the Perov type fixed point theorem in rectangular cone metric spaces. They are analogous to the Perov type contractive mapping in a vector-valued metric space.

In the present paper, we show the sufficient conditions of the existence and uniqueness of the fixed point of Reich-Perov α-contractive mapping in vector-valued metric spaces. The novel results of this paper are the development and fusion of the ideas of Vetro [10] and Altun [12].

We now briefly describe the content of the paper. In Section 2, we give some notations and recall the primary results used in this paper. The main results of this research are discussed in Section 3. First, we show sufficient conditions for Reich-Perov type contractive mapping to have a fixed point in vector-valued metric spaces. Then, we discuss the novel idea of Reich–Perov type α-contractive mappings in vector-valued metric spaces. We present sufficient conditions for the mapping such that the fixed point exists. Furthermore, we offer a particular hypothesis to get the uniqueness of the fixed point from the mapping. Finally, we construct an example to illustrate the main result.

2. Preliminaries

In this section, we describe the concept of vector-valued metric spaces introduced by Perov in 1964 [11]. Then, we define convergent sequences, Cauchy sequences, completeness, and continuous mapping in vector-valued metric spaces. We also describe the Perov contractive mapping in vector-valued metric spaces, which we used in the discussion.

2.1. **Vector-Valued Metric Spaces.**

Definition 2.1. ([12], [13]). Let X be a non-empty set. A function $d_v : X \times X \to \mathbb{R}^n$, is called a vector-valued metric if for every $s, t, r \in X$, satisfies:

(VM1) $\mathbf{0} \preceq d_v(s,t)$, $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$,

(VM2) $d_v(s,t) = \mathbf{0} \Leftrightarrow s = t$,

(VM3) $d_v(s,t) = d_v(t,s)$,

(VM4) $d_v(s,t) \preceq d_v(s,r) + d_v(r,t)$.

The pair (X, d_v) is called a vector-valued metric space.

Remark 1. *The symbol "$\preceq$" denotes coordinate-wise ordering on $\mathbb{R}^n$, i.e., $k = (k_1, k_2, \ldots, k_n), l = (l_1, l_2, \ldots, l_n) \in \mathbb{R}^n, k \preceq l \Leftrightarrow k_j \leq l_j, \forall j = 1, 2, ..., n$. Also $k \prec l \Leftrightarrow k_j < l_j, \forall j = 1, 2, ..., n$.*

Example 2.1. Let $X = \left\{ s_n = \dfrac{1}{2n} : n \in \{1, 2, 3, \ldots\} \right\} \cup \{0\}$ be a set equipped with a function $d_v : X \times X \to \mathbb{R}^2$ defined by $d_v(s,t) = (\omega|s-t|, \omega|s-t|)$ for every $s, t \in X$, and some $\omega > 0$. Then (X, d_v) is a vector-valued metric space.

Definition 2.2. Let (X, d_v) be a vector-valued metric space.

(i) A sequence (s_n) in (X, d_v) is said to converge to $s \in X$, denoted $s_n \to s$ as $s \to \infty$ or $\lim_{n\to\infty} s_n = s$, if $d_v(s_n, s) \to \mathbf{0}$ as $n \to \infty$; in other words, for every ε with $\mathbf{0} \prec \varepsilon$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that $d_v(s_n, s) \prec \varepsilon$, for every $n \geq N$.

(ii) A sequence (s_n) in (X, d_v) is said to be a Cauchy sequence if for every ε with $\mathbf{0} \prec \varepsilon$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^n$, there exists $K \in \mathbb{N}$ such that $d_v(s_m, s_n) \prec \varepsilon$, for every $m, n \geq K$.

(iii) Let $E \subseteq X$. If every Cauchy sequence in E has a limit at E, then E is said to be complete. If X is complete, then the pair (X, d_v) is called a complete vector-valued metric space.

Definition 2.3. Let (X, d_v) be a vector-valued metric space. A mapping $T : X \to X$ is said to be continuous at $s \in X$ if whenever a sequence (s_n) in X converges to $s \in X$, the sequence (Ts_n) in X converges to $Ts \in X$.

2.2. **Perov Type Contractive Mapping.**

We discuss the Perov type fixed point theorem in vector-valued metric spaces. However, we first provide the following notations:

(i) Θ is a matrix of size $n \times n$ with zero entries.

(ii) A is a matrix of size $n \times n$ with real number entries. We denote it briefly by $A = [a_{ij}]$, $1 \leq i, j \leq n$.

(iii) I is an identity matrix of size $n \times n$

(iv) A^2 is a multiplication $A \cdot A$.
In general, we get $A^m = \underbrace{A \cdot A \cdots A}_{m\text{-times}}, m \in \mathbb{N}$.

(v) $\mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ is the set of all matrices of size $n \times n$ with non-negative real numbers entries.

(vi) $\mathcal{M}_{n\times n}(\mathbb{R})$ is the set of all matrices of size $n \times n$ with real numbers entries.

We now discuss notions of matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ that converges to zero, a convergence theorem of matrices, and Perov type contractive mappings in vector-valued metric spaces.

Definition 2.4. ([15]). Let $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ with eigenvalues $\lambda_i, 1 \leq i \leq n$. The spectral radius of matrix A, $\rho(A)$, is defined by $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$.

Remark 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. In particular, there exists a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ such that $f(x) = Ax$ for every $x \in \mathbb{R}^n$. If the function f satisfies the*

Lipschitz condition on $\mathbb{R}^n$*, then we can choose a positive real constant* K *such that* $\|Ax - Ay\| \leq K\|x - y\|$ *for any* $x, y \in \mathbb{R}^n$*.*

From Definition 2.4 and Remark 2, we can compare the spectral radius and the Lipschitz constant. Geometrically, all eigenvalues of matrix A are plotted in the complex z-plane, then $\rho(A)$ is the radius of the smallest disc $|z| \leq R$ with center at the origin, which includes all the eigenvalues of matrix A. The spectral radius of A is finite for the set of finite matrices $A = [a_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ and does not depend on the norm $\|\cdot\|$. However, the Lipschitz constant depends on the norm $\|\cdot\|$.

Theorem 2.1. ([15]). *$A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ converges to zero if and only if $\rho(A) < 1$.*

Definition 2.5. ([15]). Let $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. A matrix A is said to be convergent (to zero) if the sequence of matrices $(A, A^2, A^3, A^4, \dots)$ converges to Θ, denoted $A^m \to \Theta$ as $m \to \infty$.

By Definition 2.5, the matrix $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ converges to zero. We can easily show that $A^m = \begin{bmatrix} (\frac{1}{4})^m & \frac{m}{2^{2m-1}} \\ 0 & (\frac{1}{4})^m \end{bmatrix} \to \Theta$ as $m \to \infty$.

Theorem 2.2. ([15]). *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. The following statements are equivalent:*

(i) *A converges to zero.*

(ii) *The eigenvalues of A are in the open unit disc.*

(iii) *The matrix $I - A$ is a non-singular matrix and $(I - A)^{-1} = I + A + A^2 + A^3 + \cdots + A^m + \cdots$ is a matrix with non-negative real numbers entries.*

Example 2.2. If c, d, and e are non-negative real numbers with $\max\{c, d, e\} < 1$, then the matrix

$$A = \begin{bmatrix} c & 0 & 0 \\ d & d & d \\ 0 & 0 & e \end{bmatrix}$$

converges to zero.

Example 2.3. If p, q are non-negative real numbers with $\min\{p, q\} \geq 1$, then the matrix

$$A = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

does not converge to zero.

Definition 2.6. ([12],[13]). Let (X, d_v) be a vector-valued metric space and the mapping $T : X \to X$. If there exists a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ that converges to zero, and for every $s, t \in X$ satisfies

$$d_v(Ts, Tt) \preceq A d_v(s, t), \tag{2.1}$$

then the mapping T is said to be a Perov type contractive in X.

Example 2.4. Let $X = \left\{ s_n = \dfrac{1}{2n} : n \in \mathbb{N} \right\} \cup \{0\}$ be a set equipped with a function $d_v : X \times X \to \mathbb{R}^2$ defined by $d_v(s, t) = (\omega|s - t|, \omega|s - t|)$ for each $s, t \in X$ and some $\omega > 0$. Thus (X, d_v) is a vector-valued metric space. Furthermore, if $T : X \to X$ is defined by

$$Ts = \begin{cases} 0, & s = 0 \\ \omega s_{n+1}, & s = s_n, \end{cases}$$

then T is not a Perov type contractive mapping in X. It can be shown via a contradiction argument. Assume that T is a Perov type contractive mapping. So there exists a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}_0^+)$ such that for every $s, t \in X$ satisfies (2.1). Now, taking $s = s_n$ and $t = 0$, we obtain

$$d_v(Ts, Tt) = d_v(Ts_n, T0) = d_v(s_{n+1}, 0) = (\omega s_{n+1}, \omega s_{n+1}).$$

$$A d_v(s, t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} d_v(s_n, 0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\omega s_{n+1}, \omega s_{n+1}).$$

By (2.1), it follows that

$$s_{n+1} \leq (a + b)s_n$$

and

$$s_{n+1} \leq (c + d)s_n.$$

Because $\lim\limits_{n\to\infty} \dfrac{s_{n+1}}{s_n} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{2(n+1)}}{\frac{1}{2n}} = 1$, we get $1 \leq a + b$ and $1 \leq c + d$. Contradicting the hypothesis that matrix A converges to zero.

Example 2.5. Let $X = \left\{ s_n = \frac{1}{2^n} : n \in \mathbb{N} \right\} \cup \{0\}$ be a set equipped with a function defined by $d_v : X \times X \to \mathbb{R}^2$ with $d_v(s,t) = (|s-t|, |s-t|)$ for each $s, t \in X$. We can observe that (X, d_v) is a vector-valued metric space. Furthermore, suppose that $T : X \to X$ is defined by

$$Ts = \begin{cases} 0, & s = 0 \\ s_{n+1}, & s = s_n. \end{cases}$$

Next, we prove that T is a Perov type contractive mapping in X. In particular, we construct a matrix $A \in \mathcal{M}_{2\times 2}(\mathbb{R}_0^+)$, which converges to zero such that (2.1) is satisfied. Firstly, we consider sufficient conditions for non-negative real numbers a, b, c, and d. Here, we observe for all possible values of $s, t \in X$: (i) $s = s_n, n = 1, 2, \cdots$ and $t = 0$, (ii) $s \neq t \neq 0$, (iii) $s = t$.

Case (i). For $s = s_n, n = 1, 2, \cdots$ and $t = 0$, we obtain

$$d_v(Ts, Tt) = d_v(Ts_n, T0) = d_v(s_{n+1}, 0) = (s_{n+1}, s_{n+1})$$

and

$$Ad_v(s,t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} d_v(s_n, 0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (s_n, s_n).$$

Thus, we get

$$(s_{n+1}, s_{n+1}) \preceq \begin{bmatrix} a & b \\ c & d \end{bmatrix} (s_n, s_n).$$

It follows

$$s_{n+1} \leq (a+b)s_n$$

and

$$s_{n+1} \leq (c+d)s_n.$$

Because $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2}$, we have $\frac{1}{2} \leq a+b$ and $\frac{1}{2} \leq c+d$. To get a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ that converges to zero and satisfies (2.1), we can choose $\frac{1}{2} \leq a < 1,\ b = 0,\ c = 0$, and $\frac{1}{2} \leq d < 1$.

Case (ii). Analogous to Case (i).

Case (iii). Analogous to Case (i).

From Case (i)–(iii), we get the matrix $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}_0^+)$ with $\frac{1}{2} \leq a < 1$ and $\frac{1}{2} \leq d < 1$, which converges to zero and satisfies (2.1). This yields that T is a Perov type contractive mapping.

3. Main results

3.1. Reich-Perov Type Contractive Mapping in Vector-Valued Metric Spaces.

In this subsection, we discuss the fixed point theorem of Reich-Perov contractive mapping in vector-valued metric spaces. Reich-Perov type contractive mapping is a generalization of Perov contractive mapping in vector-valued metric spaces. Each of these mappings has the property that a unique fixed point can be obtained. Now, we recall the following definition of Reich contractive mapping in complete metric spaces from [2].

Definition 3.1. Let (X, d) be a complete metric space and $T : X \to X$. If there are non-negative real numbers a, b, and c where $a+b+c < 1$ such that for every $s, t \in X$ satisfies

$$d(Ts, Tt) \leq ad(s, Ts) + bd(t, Tt) + cd(s, t),$$

then T is called Reich contractive mapping in X.

Inspired by Definitions 2.6 and 3.1, we introduce the novel idea of Reich-Perov contractive mapping in vector-valued metric spaces.

Definition 3.2. Let (X, d_v) be a vector-valued metric space. A mapping $T : X \to X$ is called Reich-Perov type contractive mapping in X if there are three matrices $A_1, A_2, A_3 \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ where $\rho(A_1)+\rho(A_2)+\rho(A_3) < 1$ such that for every $s, t \in X$ satisfies

$$d_v(Ts, Tt) \leq A_1 d_v(s, Ts) + A_2 d_v(t, Tt) + A_3 d_v(s, t). \tag{3.1}$$

Next, we show the sufficient conditions for a Reich-Perov type contractive mapping to have a unique fixed point in vector-valued metric spaces. We require the following Lemma.

Lemma 3.1. *Let (X, d_v) be a vector-valued metric space. If $B \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ converges to zero and $d_v(s,t) \preceq B d_v(s,t)$ for any $s, t \in X$ then $d_v(s,t) = \mathbf{0}$.*

Proof. Based on the fact that $B \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ converges to zero, according to Theorem 2.2, $(I-B)$ is a non-singular matrix and $(I-B)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. Then, from $d_v(s,t) \preceq B d_v(s,t)$ for every $s, t \in X$, we obtain

$$\begin{aligned} (I-B)d_v(s,t) &\preceq \mathbf{0} \\ (I-B)^{-1}(I-B)d_v(s,t) &\preceq (I-B)^{-1}\mathbf{0} = \mathbf{0}. \end{aligned}$$

Since $\mathbf{0} \preceq d_v(s,t) \preceq \mathbf{0}$, it follows that $d_v(s,t) = \mathbf{0}$. □

Theorem 3.1. *Let (X, d_v) be a complete vector-valued metric space. If $T : X \to X$ is a Reich-Perov type contractive mapping in X, it means there are three matrices $A_1, A_2, A_3 \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ where $\rho(A_1)+\rho(A_2)+\rho(A_3) < 1$ such that for every $s, t \in X$ satisfies* (3.1) *and $A \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ converges to zero where $A = (I-A_2)^{-1}(A_1 + A_3)$, then T has a unique fixed point.*

Proof. Our proof begins by forming a Picard sequence with an initial value $s_0 \in X$. We take any $s_0 \in X$ and let $s_1 = Ts_0, s_2 = Ts_1 = TTs_0 = T^2 s_0, \ldots, s_n = T^n s_0, \ldots$. The process is continued; we get an iterative sequence

$$s_n = T^n s_0, \text{ for } n = 1, 2, \ldots.$$

If $s_{k-1} = s_k$ for some $k \in \mathbb{N}$, then $s_k = Ts_{k-1} = Ts_k$. This implies that s_k is a fixed point of T. We now turn to the case $s_{n-1} \neq s_n$ for every $n \in \mathbb{N}$. By using (3.1) with $s = s_{n-1}$ and $t = s_n$, we obtain

$$\begin{aligned} d_v(s_n, s_{n+1}) &= d_v(Ts_{n-1}, Ts_n) \\ &\preceq A_1 d_v(s_{n-1}, Ts_{n-1}) + A_2 d_v(s_n, Ts_n) + A_3 d_v(s_{n-1}, s_n) \\ &= A_1 d_v(s_{n-1}, s_n) + A_2 d_v(s_n, s_{n-1}) + A_3 d_v(s_{n-1}, s_n) \end{aligned}$$

It follows that

$$(I-A_2)d_v(s_n, s_{n+1}) \preceq (A_1 + A_3)d_v(s_{n-1}, s_n). \tag{3.2}$$

Since $\rho(A_2) < 1$, we get A_2 converges to zero based on Theorem 2.1. Furthermore, according to Theorem 2.2, we conclude that $(I-A_2)$ is a non-singular matrix and

$(I-A_2)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. Then, we can write

$$\begin{aligned} d_v(s_n, s_{n+1}) &\preceq (I-A_2)^{-1}(A_1+A_3)d_v(s_{n-1}, s_n) \\ &\preceq Ad_v(s_{n-1}, s_n) \end{aligned} \tag{3.3}$$

where $A = (I-A_2)^{-1}(A_1+A_3)$. Moreover, by (3.3), we get

$$\begin{aligned} d_v(s_n, s_{n+1}) &\preceq Ad_v(s_{n-1}, s_n) \\ &\preceq A^2 d_v(s_{n-2}, s_{n-1}) \\ &\preceq A^3 d_v(s_{n-3}, s_{n-2}) \preceq \cdots \preceq A^n d_v(s_0, s_1). \end{aligned}$$

Thus we obtain

$$d_v(s_n, s_{n+1}) \preceq A^n d_v(s_0, s_1). \tag{3.4}$$

Then, we show (s_n) is a Cauchy sequence. By using Definition 2.1(VM4) and (3.4), for every $p \in \mathbb{N}$, it follows that

$$\begin{aligned} d_v(s_n, s_{n+p}) &\preceq d_v(s_n, s_{n+1}) + d_v(s_{n+1}, s_{n+2}) + \cdots + d_v(s_{n+p-1}, s_{n+p}) \\ &\preceq A^n d_v(s_0, s_1) + A^{n+1} d_v(s_0, s_1) + \cdots + A^{n+p-1} d_v(s_0, s_1) \\ &= A^n(I + A + A^2 + A^3 + \cdots + A^{p-1}) d_v(s_0, s_1) \\ &\preceq A^n (I-A)^{-1} d_v(s_0, s_1). \end{aligned}$$

Since it is known that A converges to zero, from Theorem 2.2, it follows that $(I-A)$ is a non-singular matrix and $(I-A)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. We conclude that

$$d_v(s_n, s_{n+p}) \preceq A^n (I-A)^{-1} d_v(s_0, s_1) \to \mathbf{0} \text{ as } n \to \infty.$$

Thus (s_n) is a Cauchy sequence in complete vector-valued metric space (X, d_v). This implies that (s_n) converges, say to $s^\star \in X$, it means $d_v(s_n, s^\star) \to \mathbf{0}$ as $n \to \infty$. Further, we show that $s^\star$ is a fixed point of T. By Definition 2.1(VM4) and (3.1), we obtain

$$\begin{aligned} d_v(s^\star, Ts^\star) &\preceq d_v(s^\star, s_n) + d_v(s_n, Ts^\star) \\ &= d_v(s^\star, s_n) + d_v(Ts_{n-1}, Ts^\star) \\ &\preceq d_v(s^\star, s_n) + A_1 d_v(s_{n-1}, Ts_{n-1}) + A_2 d_v(s^\star, Ts^\star) + A_3 d_v(s_{n-1}, s^\star). \end{aligned}$$

Taking $n \to \infty$ and Lemma 3.1, we conclude that $d_v(s^\star, Ts^\star) = \mathbf{0}$. The consequence is $s^\star = Ts^\star$. This yields that $s^\star$ is a fixed point of T.

Now, we need to show uniqueness. Suppose that $s^{\star\star} \in X$ is another fixed point of T such that $s^{\star\star} = Ts^{\star\star}$. Then, we can write

$$\begin{aligned} d_v(s^\star, s^{\star\star}) = d_v(Ts^\star, Ts^{\star\star}) &\preceq A_1 d_v(s^\star, Ts^\star) + A_2 d_2(s^{\star\star}, Ts^{\star\star}) + A_3 d_v(s^\star, s^{\star\star}) \\ &= A_1 d_v(s^\star, s^\star) + A_2 d_2(s^{\star\star}, s^{\star\star}) + A_3 d_v(s^\star, s^{\star\star}) \\ &= A_3 d_v(s^\star, s^{\star\star}). \end{aligned}$$

Hence, by Lemma 3.1, we conclude that $d_v(s^\star, s^{\star\star}) = \mathbf{0}$. It yields that $s^\star = s^{\star\star}$. □

Inspired by the definition of Kannan contractive mapping in complete metric space [2] and Definition 2.6, we introduced a novel notion of Kannan-Perov type contractive mapping in vector-valued metric spaces.

Definition 3.3. Let (X, d_v) be a vector-valued metric space. A mapping $T : X \to X$ is said to be Kannan-Perov type contractive mapping in X if there are two matrices $A_1, A_2 \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ where $\rho(A_1) + \rho(A_2) < 1$ such that for every $s, t \in X$ satisfies

$$d_v(Ts, Tt) \leq A_1 d_v(s, Ts) + A_2 d_v(t, Tt). \tag{3.5}$$

By Definition 3.3, we get the following corollary whose proof is analogous to Theorem 3.1.

Corollary 3.1. *Let (X, d_v) be a complete vector-valued space. If $T : X \to X$ is a Kannan-Perov type contractive mapping in (X, d_v), it means there are two matrices $A_1, A_2 \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ where $\rho(A_1) + \rho(A_2) < 1$ satisfies* (3.5)*, and $C \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ converges to zero where $C = (I - A_2)^{-1} A_1$, then T has a unique fixed point.*

3.2. **Reich–Perov Type Contractive Mapping in Vector-Valued Metric Spaces.**

The Reich–Perov α-contractive mapping is a generalization of the Perov and Reich–Perov contractive mapping. In this subsection, we discuss the novel idea of Reich–Perov type α-contractive mapping in vector-valued metric spaces. We show sufficient conditions to derive the existence and uniqueness of a fixed point in the spaces.

Definition 3.4. ([14]). Let X be a non-empty set. Consider $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. A function T is said to be α-admissible if for every $s, t \in X$, $\alpha(s, t) \geq 1$, then $\alpha(Ts, Tt) \geq 1$.

Example 3.1. Suppose that $X = (0, \infty)$. Consider $T : X \to X$ with $Ts = s^2$, $\forall s \in X$ and $\alpha : X \times X \to [0, \infty)$ is defined by

$$\alpha(s,t) = \begin{cases} 2, & s \geq t \\ \frac{1}{2}, & s < t \end{cases}$$

for every $s, t \in X$. Since $Ts = s^2$, $\forall s \in X$ is an increasing function, and it follows that $s \geq t$ implies $Ts \geq Tt$. By Definition 3.4, it is clear that for $s \geq t$, $\alpha(s,t) \geq 1$ the result is $\alpha(Ts, Tt) \geq 1$. This yields that T is an α-admissible function.

Altun et al. [13] generalize the concept of the α-admissible function by replacing the codomain of the function with $\mathcal{M}_{n\times n}(\mathbb{R})$.

Definition 3.5. ([13]). Let X be a non-empty set, $\alpha : X \times X \to \mathcal{M}_{n\times n}(\mathbb{R})$ and $T : X \to X$. If every $s, t \in X$, $\alpha(s,t) \geq I$ result in $\alpha(Ts, Tt) \geq I$, then a function T is called α-admissible.

Remark 3. *Assume that $U = [u_{ij}], V = [v_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{R})$. Then $U \geq V$ means $u_{ij} \geq v_{ij}$, for every $i, j \in \{1, 2, 3, \ldots, n\}$.*

Example 3.2. Suppose that $X = (1, \infty)$, $\alpha : X \times X \to \mathcal{M}_{n\times n}(\mathbb{R})$ is defined by

$$\alpha(s,t) = \begin{cases} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}, & s \geq t \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & s < t \end{cases}$$

$\forall s, t \in X$ and $T : X \to X$ with $Ts = 2s$. It is easy to check that T is an increasing function. By Definition 3.5, for $s \geq t$, $\alpha(s,t) \geq I$ result in

$$\alpha(Ts,Tt) = \begin{cases} \begin{bmatrix} 2s & 0 \\ 0 & 2t \end{bmatrix}, & Ts \geq Tt \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & Ts < Tt. \end{cases}$$

It follows $\alpha(Ts, Tt) \geq I$. This implies that T is an α-admissible function.

By Definitions 3.2 and 3.5, we introduce a novel idea of the Reich-Perov α-contractive mapping in vector-valued metric spaces.

Definition 3.6. Let (X, d_v) be a vector-valued metric space and $T : X \to X$. If there exists a function $\alpha : X \times X \to \mathcal{M}_{n\times n}(\mathbb{R})$ defined by $\alpha(s,t) \geq I$ and three matrices $A_1, A_2, A_3 \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$ where $\rho(A_1) + \rho(A_2) + \rho(A_3) < 1$, such that for every $s, t \in X$ satisfies

$$d_v(Ts, Tt) \preceq A_1 d_v(s, Ts) + A_2 d_v(t, Tt) + A_3 d_v(s, t), \tag{3.6}$$

then T is called a Reich-Perov α-contractive mapping in X.

The following theorem is the extension of two references [10, 12], which we will later call the fixed point theorem of Reich-Perov type α-contractive mapping in vector-valued metric spaces. In these novel theorems, we show sufficient conditions to derive the existence and uniqueness of a fixed point of the mapping.

Theorem 3.2. *Let (X, d_v) be a vector-valued metric space and $T : X \to X$ be a Reich-Perov α-contractive mapping with respect to function $\alpha : X \times X \to \mathcal{M}_{n\times n}(\mathbb{R})$. Assume that* (i)-(iii) *hold.*

(i) *T is continuous,*

(ii) *T is α-admissible,*

(iii) *There exists $s_0 \in X$ such that $\alpha(s_0, Ts_0) \geq I$.*

Then T has a fixed point.

Proof. From (iii), it is guaranteed that there exists $s_0 \in X$ such that $\alpha(s_0, Ts_0) \geq I$. We assume $s_0 \in X$ as the initial point. Next, we construct the Picard sequence with initial point $s_0 \in X$. We can write

$$s_1 = Ts_0,\ s_2 = Ts_1 = TTs_0 = T^2 s_0, \ldots, s_n = T^n s_0, \ldots$$

The process is continued until we get the iterative sequence

$$s_n = T^n s_0, n = 1, 2, 3, \ldots.$$

If $s_{k-1} = s_k$ for some $k \in \mathbb{N}$, then $s_k = Ts_{k-1} = Ts_k$. So, s_k is a fixed point of T. Now, we assume $s_{n-1} \neq s_n$ for every $n \in \mathbb{N}$. Since T is an α-admissible, it follows that

$$\alpha(s_0, s_1) = \alpha(s_0, Ts_0) \geq I.$$

Consequently, we have $\alpha(s_1, s_2) = \alpha(s_1, Ts_1) \geq I$. The process is terminated, and we get

$$\alpha(s_{n-1}, s_n) = \alpha(s_{n-1}, Ts_{n-1}) \geq I$$

for every $n \in \mathbb{N}$. By (3.6) with $s = s_{n-1}$ and $t = s_n$, we obtain

$$\begin{aligned} d_v(s_n, s_{n+1}) &= d_v(Ts_{n-1}, Ts_n) \\ &\preceq A_1 d_v(s_{n-1}, Ts_{n-1}) + A_2 d_v(s_n, Ts_n) + A_3 d_v(s_n, s_{n-1}) \\ &= A_1 d_v(s_{n-1}, s_n) + A_2 d_v(s_n, s_{n+1}) + A_3 d_v(s_n, s_{n-1}) \\ &= (A_1 + A_3) d_v(s_{n-1}, s_n) + A_2 d_v(s_n, s_{n+1}) \\ (I - A_2) d_v(s_n, s_{n+1}) &\preceq (A_1 + A_3) d_v(s_{n-1}, s_n). \end{aligned}$$

Since $\rho(A_2) < 1$, it follows that A_2 converges to zero. From Theorem 2.2, we obtain $(I - A_2)$ is a non-singular matrix and $(I - A_2)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. Therefore, we get

$$\begin{aligned} d_v(s_n, s_{n+1}) &\preceq (I - A_2)^{-1}(A_1 + A_3) d_v(s_{n-1}, s_n) \\ &\preceq A d_v(s_{n-1}, s_n) \end{aligned} \tag{3.7}$$

where $A = (I - A_2)^{-1}(A_1 + A_3)$.

Hence, by (3.7) we can write

$$\begin{aligned} d_v(s_n, s_{n+1}) &\preceq A d_v(s_{n-1}, s_n) \\ &\preceq A^2 d_v(s_{n-2}, s_{n-1}) \prec A^3 d_v(s_{n-3}, s_{n-2}) \preceq \cdots \preceq A^n d_v(s_0, s_1) \\ &\preceq A^n d_v(s_0, s_1). \end{aligned} \tag{3.8}$$

To show that (s_n) is a Cauchy sequence, we use Definition 2.1(VM4) and (3.8). For every $p \in \mathbb{N}$, we have

$$\begin{aligned} d_v(s_n, s_{n+p}) &\preceq d_v(s_n, s_{n+1}) + d_v(s_{n+1}, s_{n+2}) + \cdots + d_v(s_{n+p-1}, s_{n+p}) \\ &\preceq A^n d_v(s_0, s_1) + A^{n+1} d_v(s_0, s_1) + \cdots + A^{n+p-1} d_v(s_0, s_1) \\ &= (A^n + A^{n+1} + A^{n+2} + \cdots + A^{n+p-1}) d_v(s_0, s_1) \\ &= A^n (I + A + A^2 + A^3 + \cdots + A^{p-1}) d_v(s_0, s_1) \\ &\preceq A^n (I - A)^{-1} d_v(s_0, s_1). \end{aligned}$$

According to Theorem 2.2, it follows that the matrix $(I - A)$ is a non-singular and $(I - A)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_0^+)$. We conclude that

$$d_v(s_n, s_{n+p}) \preceq A^n(I - A)^{-1}d_v(s_0, s_1) \to \mathbf{0} \text{ as } n \to \infty.$$

Thus (s_n) is a Cauchy sequence in the complete vector-valued metric space (X, d_v). Hence, the sequence (s_n) converges, say to $s^\star \in (X, d_v)$, which means $d_v(s_n, s^\star) \to \mathbf{0}$ as $n \to \infty$. Since T is continuous, according to Definition 2.3, any sequence (s_n) in X converges to $s^\star$ in X implies (Ts_n) in X converges to $Ts^\star$ in X. In other words, sequence (s_{n+1}) converges to $Ts^\star$. So, we get $Ts^\star = s^\star$. This yields that $s^\star$ is a fixed point of T. □

Next, to prove the uniqueness of the fixed point, we need the following hypothesis.

(K) : Let (X, d_v) be a vector-valued metric space and $\mathrm{Fix}(X) := \{z \in X : Tz = z\}$. For all $v, w \in \mathrm{Fix}(X)$, we get $\alpha(v, w) \geq I$.

Theorem 3.3. *If the hypothesis (K) is given in Theorem 3.2, then the mapping T has a unique fixed point.*

Proof. Suppose that $s^{\star\star} \in X$ is another fixed point of T. By the hypothesis (K), we get $\alpha(s^\star, s^{\star\star}) \geq I$. By using (3.6), we can write

$$\begin{aligned} d_v(s^\star, s^{\star\star}) &= d_v(Ts^\star, Ts^{\star\star}) \\ &\preceq A_1 d_v(s^\star, Ts^\star) + A_2 d_v(s^{\star\star}, Ts^{\star\star}) + A_3 d_v(s^\star, s^{\star\star}). \end{aligned}$$

Hence, by Lemma 3.1, we conclude that $d_v(s^\star, s^{\star\star}) = \mathbf{0}$. This implies that $s^\star = s^{\star\star}$ □

In the following, we construct an example to illustrate Theorem 3.1, 3.2, and 3.3.

Example 3.3. Suppose that $X = \left\{ s_n = \frac{1}{5^n} : n \in \mathbb{N} \right\} \cup \{0\}$ and $d_v : X \times X \to \mathbb{R}^2$ is defined by $d_v(s, t) = (|s - t|, |s - t|)$ for each $s, t \in X$. We can easily show that (X, d_v) is a complete vector-valued metric space. Now, let $T : X \to X$ be defined by

$$Ts = \begin{cases} 0, & s = 0 \\ s_{n+1}, & s = s_n. \end{cases}$$

We show that T is a Reich-Perov contractive mapping in X. In particular, we construct three matrices

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, A_3 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}_0^+)$$

where $\rho(A_1)+\rho(A_2)+\rho(A_3) < 1$ such that (3.1) is satisfied. The proof is analogous to Example 2.5. Thus, we choose $a_1 = a_2 = a_3 = 1/4, b_1 = b_2 = b_3 = 0, c_1 = c_2 = c_3 = 0$, and $d_1 = d_2 = d_3 = 1/4$.

Next, the matrix $A = (I - A_2)^{-1}(A_1 + A_3) \in \mathcal{M}_{2\times 2}(\mathbb{R}_0^+)$ converges to zero. Consequently, T has a unique fixed point, $s = 0$ (according to Theorem 3.1).

Now, we define α-contractive mapping $\alpha : X \times X \to \mathcal{M}_{2\times 2}(\mathbb{R})$ where

$$\alpha(s,t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & s \geq t \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & s < t \end{cases}$$

$\forall s, t \in X$. By Definition 3.5, it follows that $s \geq t$, $\alpha(s,t) \geq I$ result in $\alpha(Ts, Tt) \geq I$. We conclude that T is an α-admissible function, and consequently, T is a Reich-Perov α-contractive mapping. By continuity of T on X and the existence of $s_0 \in X$ such that $\alpha(s_0, Ts_0) \geq I$, it follows from Theorem 3.2, we obtain T has a fixed point, i.e., $s = 0$. Finally, by applying hypothesis (K) and Theorem 3.3, we arrive at the uniqueness of T.

4. Future Research

Since the initial or boundary value problems for nonlinear differential systems can be represented as semilinear operator systems, such systems appear in various applications of mathematics. For example, various fixed point theorems such as Schauder, Leray–Schauder, Krasnoselskii, and Perov fixed point theorems were applied in the existence of solutions of such systems [12]. The applications lead us to a new idea for further investigation. In particular, about applying the

Reich-Perov and Reich Perov α-contractive fixed point theorems in vector-valued metric spaces.

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References

[1] S. Banach, Sur les operations dans les ensembles obstraits et. Leur. Application aux equation integrales, *Fund. Math.* **3**(1922),133–181.

[2] S. Reich, Some Remarks concerning contraction mappings. *Canad. Math. Bull.* **14**(1971), 121-124.

[3] R. Kannan, Some results on fixed points-II. *The American Mathematical Monthly* **76**(1969), 405-408.

[4] S.K. Chatterjea, Fixed point theorems. *Comptes Rendus de l'Academie bulgare des Sciences* **25**(1972), 727–730.

[5] L. Ȁiriȁ, On some with non-unique fixed points. *Publ. I'Institut MathÈMatique* **17**(1974), 52–58.

[6] B. E. Rhoades, A comparison of various definitions of contractive mappings, *Transactions of The American Mathematical Society* **226**(1977), 257-290.

[7] J. De Bakker and E. De Vink, Denotational models for programming languages: applications of Banach's fixed point theorem, *Topology and its Applications* **85**(1998), 35–52.

[8] C. Rousseau, *Banach fixed point theorem and applications*, Universite de Montreal, 2010.

[9] O. Ege and I. Karaca, Banach fixed point theorem for digital images, *Journal of Nonlinear Science and Applications* **8**(2015), 237-245.

[10] Md. Abdul Mannan, Md. R. Rahman, H. Akter, N. Nahar and S. Mondal, A Study of Banach Fixed Point Theorem and It's Applications, *American Journal of Computational Mathematics* **11**(2021), 157-174.

[11] A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Pvilizhen. Met. Reshen. Diff. Uvavn.* **2**(1964), 115-134.

[12] I. Altun, N. Hussain, M. Qasim and H. H. Al-Sulami, A New Fixed Point of Perov Type and its Application to a Semi Linear Operator System, *Journal Mathematics* **7**(2019), 1-10.

[13] I. Altun, M. Qasim, Application of Perov type fixed point results to complex partial differential equations, *Mathematical Methods in the Applied Sciences* **2**(2020), 2059-2070.

[14] F. Vetro, S. Radenovic, Some results of Perov type in rectangular cone metric space, *Journal of Fixed Point Theory and Applications* **2**(2018), 16p.

[15] R. S. Varga, *Matrix Iterative Analysis*, Springers Series in Computational Mathematics, Springer: Berlin, Germany, 2000.

(1) Department of Mathematics, Institut Teknologi Sepuluh Nopember, ITS Campus, Building F, 2nd Floor, Keputih, Sukolilo, Surabaya 60111, Indonesia.

Email address: sunarsini@matematika.its.ac.id

(2) Department of Mathematics, Institut Teknologi Sepuluh Nopember, ITS Campus, Building F, 2nd Floor, Keputih, Sukolilo, Surabaya 60111, Indonesia.

Email address: yunusm@matematika.its.ac.id

(3) Department of Mathematics, Institut Teknologi Sepuluh Nopember, ITS Campus, Building F, 2nd Floor, Keputih, Sukolilo, Surabaya 60111, Indonesia.

Email address: subiono2008@matematika.its.ac.id