Jordan Journal of Mathematics and Statistics (JJMS), 16(4), 2023, pp 617 - 647
DOI: https://doi.org/10.47013/16.4.2

WELL POSEDNESS AND STABILITY FOR THE NONLINEAR φ-CAPUTO HYBRID FRACTIONAL BOUNDARY VALUE PROBLEMS WITH TWO-POINT HYBRID BOUNDARY CONDITIONS

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Abstract. This article investigates into the study of nonlinear hybrid fractional boundary value problems, which involve φ-Caputo derivatives of fractional order and two-point hybrid boundary conditions. The author utilizes a fixed point theorem of Dhage to provide evidence for the existence and uniqueness of solutions, taking into consideration mixed Lipschitz and Caratheodory conditions. Additionally, the Ulam-Hyers types of stability are established in this context. The article concludes by introducing a class of fractional boundary value problems, which are dependent on the arbitrary values of φ and the boundary conditions chosen. The research presented in this article has the potential to be useful in various fields, such as engineering and science, where fractional differential equations are frequently used to model complex phenomena.

1. Introduction

The field of fractional calculus has gained significant attention in recent years due to its numerous applications in engineering and applied sciences. This field deals with integro-differential equations involving fractional derivatives in time, which are considered more realistic than those of integer order in time for describing many phenomena in nature. Fractional calculus has found applications in various fields such as signal processing, control theory, bioengineering and biomedical engineering,

2010 *Mathematics Subject Classification.* 34A40, 34A12, 34A99, 45D05.

Key words and phrases. Hybrid fractional differential equation, Boundary value problem, Green's function, Dhage fixed point theorem, φ-Caputo fractional derivatives, Existence Results, Hyers-Ulam stability of solutions.

Received: Aug. 28, 2022 Accepted: March 26, 2023 .

viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, and social sciences. Further information on this topic can be found in various references including [5], [7], [10], [23], [25], [28], [30], [33], and [40]. Recently, there has been significant interest in the quadratic perturbations of nonlinear differential equations, which are known as fractional hybrid differential equations. Many articles on the theory of hybrid differential equations can be found in the literature, see [11]-[16] and [39]. Investigations of hybrid differential equations are important as they include several dynamic systems as special cases, and further information on this topic can be found in various references including [4], [6], [12], [13], [14]-[19], [20], [25], [27], [34], [35], [39], and [40]. Additionally, the φ-fractional derivative has been considered in the literature as a generalization of the Riemann-Liouville derivative. This type of derivative has been reconsidered recently in [5], where the Caputo-type regularization of the existing definition and some interesting properties are provided. The φ-caputo fractional derivatives have been studied in various papers, including [1], [7], [8], [23], [24], and [29]. Moreover, the stability problem of differential equations has been extensively studied in the literature. The concept of Ulam stability in the case of Banach spaces was fostered by Hyers in [22], and Rassias gave an impressive speculation of the Ulam-Hyers stability of mappings by considering variables, which is referred to as Ulam-Hyers-Rassias stability. Recently, there has been a progression of papers dedicated to the examination of existence, uniqueness, and (UH) stability of solutions of fractional differential equations with different kinds of fractional derivatives, and further information on this topic can be found in the literature such as [37, 32].

Inspired by the above works, consider the following integral boundary fractional hybrid differential equations (IBFHDE for short) involving Caputo differential operators of order $1 < \alpha \leq 2$.

$$
\left\{
\begin{array}{l}
{}^{c}\mathfrak{D}^{\alpha,\varphi}\left(\dfrac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right)=\mathfrak{h}(\zeta,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(\zeta,\varkappa(\zeta)),\ \zeta\in[0,T],\\
\left.\dfrac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right|_{\zeta=0}=\frac{1}{\Gamma(\gamma)}\int_0^1\varphi'(s)\left(\varphi(1)-\varphi(s)\right)^{\gamma-1}\,\mathfrak{h}_1(s,\varkappa(s))\,ds,\\
\left.\dfrac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right|_{\zeta=T}=\frac{1}{\Gamma(\gamma)}\int_0^1\varphi'(s)\left(\varphi(1)-\varphi(s)\right)^{\gamma-1}\,\mathfrak{h}_2(s,\varkappa(s))\,ds,
\end{array}
\right.
\tag{1.1}
$$

where $\varphi(\zeta)$ is an increasing function with $\varphi'(\zeta) \neq 0 \ \forall \zeta \in I = [0,T]$, $1 < \beta < \alpha \leq 2$, $0 < \gamma < 1$, and ${}^{c}\mathfrak{D}^{\alpha,\varphi}$ is the φ-Caputo fractional derivative, $\Gamma(.)$ is the classical Gamma function, $\mathfrak{I}^{\beta,\varphi}$ is the left-sided φ-Riemann-Liouville fractional integral of order $\beta \in (0,1)$, with $\mathfrak{g} \in C(I \times R, R \setminus \{0\})$, $\mathfrak{h}, \mathfrak{u} \in C(I \times R, R)$, and $\mathfrak{f} \in C(I \times R, R)$. This equation has been developed and studied in recent years due to its wide range of applications in various fields of science and engineering. Here are some motivations for using this equation:

(1) *Anomalous diffusion modeling*: One of the main applications of fractional differential equations is in modeling anomalous diffusion phenomena. Anomalous diffusion is a type of random walk where the mean-squared displacement of a particle does not increase linearly with time, but rather exhibits a power-law behavior. This behavior is often observed in complex systems such as biological tissues, porous media, and disordered materials. The equation (1.1) has been used to model anomalous diffusion in these systems by incorporating fractional derivatives that capture the memory and long-range interactions of the particles. For further information, refer the readers to ([36], [38]).

(2) *Control and optimization*: Fractional differential equations have also been applied in control and optimization problems. The equation (1.1) can be used to model systems with memory effects and non-local interactions, which are often encountered in control and optimization problems. The fractional derivative and integral operators in the equation provide a means of controlling and optimizing these systems by adjusting the memory and interaction parameters. For further information, refer the readers to ([21], [26]) and the references therein.

(3) *Nonlinear dynamics*: The equation (1.1) has been used to study the dynamics of nonlinear systems. Nonlinear systems often exhibit complex behaviors such as chaos and bifurcations, which can be difficult to understand and analyze using traditional methods. The fractional derivative and integral operators in the equation provide a powerful tool for analyzing the dynamics of nonlinear systems by capturing the memory and long-range interactions of the system components. For more details, see ([2], [3]).

In summary, the equation (1.1) has many applications in modeling anomalous diffusion, control and optimization, and nonlinear dynamics. Its fractional derivative and integral operators provide a powerful tool for capturing the memory and long-range interactions of the system components, making it a useful tool for studying complex systems in various fields of science and engineering.

The present article is structured as follows: Section 1 introduces the aim of our research. In Section 2, we provide an overview of important background information that will be utilized throughout this work. Moving on to Section 3, we investigate the existence and uniqueness of solutions for the integral boundary value problem for the hybrid differential equation (IBFHDE) (1.1) with fractional order $\alpha \in (0,1)$ on the closed interval $[0,T]$. We consider mixed Lipschitz and Caratheodory conditions and apply Dhage's fixed point theorem for three operators in a Banach algebra X, as described in reference [16]. Furthermore, we explore the Ulam–Hyers stability for the (IBFHDE) (1.1). Lastly, in Section 4, we introduce certain classes of fractional derivatives by selecting appropriate values for $\varphi(\zeta)$ and taking other parameters into account. Our methodology yields several well-known investigations.

2. Preliminaries

Within this section, we will present a series of fundamental definitions and preliminary concepts that will be consistently applied throughout the entirety of our work.

Definition 2.1. For any real number $\alpha > 0$, the left-sided φ-Riemann-Liouville fractional integral of order α for an integrable function $\mathfrak{u} : I \to R$ with respect to another function $\varphi : I \to R$, which is an increasing differentiable function such that $\varphi'(\zeta) \neq 0$ for all $\zeta \in I = [0,T]$ is defined by:

$$\mathfrak{I}^{\alpha,\varphi}\mathfrak{u}(\zeta) = \frac{1}{\Gamma(\alpha)}\int_0^{\zeta} \varphi'(s)(\varphi(\zeta)-\varphi(s))^{\alpha-1}\mathfrak{u}(s)\,ds,$$

where Γ is the classical Euler Gamma function.

Definition 2.2. If $\mathfrak{n} \in N$ and $\varphi, \mathfrak{u} \in C^{\mathfrak{n}}(I,R)$ are two functions such that φ is increasing and $\varphi'(\zeta) \neq 0$ for all $\zeta \in I$, then the left-sided φ-Caputo fractional derivative of a function $\mathfrak{u}$ of order α is defined by:

$$\begin{aligned} {}^{c}\mathfrak{D}^{\alpha,\varphi}\mathfrak{u}(\zeta) &= \mathfrak{I}^{\mathfrak{n}-\alpha,\varphi}\left(\frac{1}{\varphi'(\zeta)}\frac{d}{d\zeta}\right)^{\mathfrak{n}}\mathfrak{u}(\zeta) \\ &= \frac{1}{\Gamma(\mathfrak{n}-\alpha)}\int_{0}^{\zeta}\varphi'(s)(\varphi(\zeta)-\varphi(s))^{\mathfrak{n}-\alpha-1}\mathfrak{u}_{\varphi}^{[\mathfrak{n}]}(s)\,ds \end{aligned}$$

where $\mathfrak{u}_{\varphi}^{[\mathfrak{n}]}(\zeta) = \left(\frac{1}{\varphi'(\zeta)}\frac{d}{d\zeta}\right)^{\mathfrak{n}}\mathfrak{u}(\zeta)$ and $\mathfrak{n} = [\alpha]+1$ for $\alpha \notin N$, and $\mathfrak{n} = \alpha$ for $\alpha \in N$.

For further properties of fractional calculus operators, see [28], [30], [31], and [33].

Now, denote by $X = C(I, R)$ to be the Banach algebra of all real-valued continuous functions from $I = [0, T]$ into R with the norm $\|\varkappa\| = \sup\{|\varkappa(\zeta)| : \zeta \in I\}$. Moreover, by $L^{1}(I, R)$, we denote by the space of Lebesgue integrable real-valued functions on I equipped with the L^{1}-norm $\|\varkappa\|_{L^{1}} = \int_{0}^{T}|\varkappa(s)|ds$.

Definition 2.3. [15] (Normed Algebra) If A is an algebra and $\|.\|$ is a norm on A satisfying $\|\varkappa.\mathfrak{z}\| \leq \|\varkappa\|.\|\mathfrak{z}\|$ for all $\varkappa, \mathfrak{z} \in A$, then $\|.\|$ is called an algebra norm and $(A, \|.\|)$ is called a normed algebra. A complete normed algebra is called a Banach algebra.

Definition 2.4. [15] Let X be a normed vector space. A mapping $T : X \to X$ is said to be Lipschitzian over a normed vector space X if there exists a constant $k \geq 0$ such that for all $\varkappa, \mathfrak{z} \in X$, the following inequality holds: $\|T\varkappa - T\mathfrak{z}\| \leq k(\|\varkappa - \mathfrak{z}\|)$. In other words, T is Lipschitzian if its Lipschitz constant k is finite.

Definition 2.5. [15] A mapping $\mathfrak{f} : I \times R \to R$ is said to satisfy a condition of L^{1}-Caratheodory or simply is called L^{1}-Caratheodory if

(1) $\zeta \to \mathfrak{f}(\zeta, \varkappa)$ is measurable for each $\varkappa \in R$,
(2) $\varkappa \to \mathfrak{f}(\zeta, \varkappa)$ is continuous almost everywhere for $\zeta \in I$, and
(3) for each real number $r > 0$ there exists a function $\mathfrak{g} \in L^{1}(I, R)$ such that $|\mathfrak{f}(\zeta, \varkappa)| \leq \mathfrak{g}(\zeta)$ a.e. $\zeta \in I$ for all $\varkappa \in R$ with $|\varkappa| \leq r$.

Lemma 2.1. [5] Consider the real number $\alpha \in (0, 1]$, and let $\mathfrak{f} \in L^{1}(0, 1)$, Then,

(1) ${}^{c}\mathfrak{D}^{\alpha,\varphi}\mathfrak{I}^{\alpha,\varphi}\mathfrak{f}(\zeta) = \mathfrak{f}(\zeta)$ for all $\zeta \in I$.
(2) ${}^{c}\mathfrak{I}^{\alpha,\varphi}\,{}^{c}\mathfrak{D}^{\alpha,\varphi}\mathfrak{f}(\zeta) = \mathfrak{f}(\zeta) - \dfrac{\mathfrak{I}^{(1-\alpha),\varphi}\mathfrak{f}(\zeta)|_{\zeta=0}}{\Gamma(\alpha)}(\varphi(\zeta) - \varphi(0))^{\alpha-1}$ almost everywhere $\zeta \in I$.

Lemma 2.2. Let $\mathfrak{f} \in C\,(0, R)$ and $\alpha > 0$. Then, the differential equation ${}^{c}\mathfrak{D}_{a+}^{\alpha;\varphi}\mathfrak{f}(\zeta) = 0$ has a solution

$$\mathfrak{f}(\zeta) = c_0 + c_1\left(\varphi(\zeta) - \varphi(0)\right) + c_2\left(\varphi(\zeta) - \varphi(0)\right)^2 + ... + c_{\mathfrak{n}-1}\left(\varphi(\zeta) - \varphi(0)\right)^{\mathfrak{n}-1},$$

where $c_i \in R$, for all $i = 0, 1, 2, ..., \mathfrak{n} - 1$, such that $\mathfrak{n} = [\alpha] + 1$.

Lemma 2.3. [5] Let $\alpha, \beta \in R^+$, and $\mathfrak{f}(\zeta) \in L_1(I)$. Then, $\mathfrak{I}_{a+}^{\alpha,\varphi}\mathfrak{I}_{a+}^{\beta,\varphi}\mathfrak{f}(\zeta) = \mathfrak{I}_{a+}^{\beta,\varphi}\mathfrak{I}_{a+}^{\alpha,\varphi}\mathfrak{f}(\zeta) = \mathfrak{I}_{a+}^{\alpha+\beta,\varphi}\mathfrak{f}(\zeta)$, and $(\mathfrak{I}_{a+}^{\alpha,\varphi})^{\mathfrak{n}}\mathfrak{f}(\zeta) = \mathfrak{I}_{a+}^{\mathfrak{n}\alpha,\varphi}\mathfrak{f}(\zeta)$, where $\mathfrak{n} \in N$.

Definition 2.6. [7] Let X be any space and let $\mathfrak{f} : X \to X$. A point $\varkappa \in X$ is called *a fixed point* for mapping $\mathfrak{f}$ if $\varkappa = \mathfrak{f}(\varkappa)$.

Theorem 2.1. [16] Assume that S is a nonempty, closed, convex, and bounded subset of a Banach algebra X. Moreover, let $\mathcal{A} : X \to X$, $\mathcal{B} : S \to Y$, and $\mathcal{C} : X \to X$ be three operators satisfying the following conditions:

(1) $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants μ and σ, respectively,
(2) B is completely continuous.
(3) $\varkappa = \mathcal{A}\varkappa\mathcal{B}\mathfrak{z} + \mathcal{C}\varkappa$ implies that $\varkappa \in S$, for all $\mathfrak{z} \in S$.
(4) $\kappa\mathcal{N} + \mathcal{R} < \rho$ for $\rho > 0$, where $\mathcal{N} = \|\mathcal{B}(S)\|$.

Then the operator equation $\mathcal{A}\varkappa\mathcal{B}\mathfrak{z} + \mathcal{C}\varkappa = \varkappa$ has a solution in S.

Definition 2.7. The solution to the integral boundary value problem (IBFHDE) (1.1) is a continuous function $\varkappa \in C\,(I, R)$ that satisfies the boundary value problem (1.1) and such that $\zeta \to \dfrac{\varkappa(\zeta) - \mathfrak{f}(\zeta, \varkappa(\zeta))}{\mathfrak{g}(\zeta, \varkappa(\zeta))}$ is continuous for each $\varkappa \in R$.

3. Main Results

This section is focused on investigating the existence of solutions for the integral boundary value problem for hybrid differential equation with fractional order $\alpha \in (0, 1)$ (IBFHDE) (1.1) over the closed interval $[0, T]$ under mixed Lipschitz and Caratheodory conditions on the nonlinearities involved in it. To accomplish this objective, we utilize a fixed point theorem for three operators in a Banach algebra X, originally presented by Dhage[16].

To begin our analysis, we consider the following set of assumptions:

(A_0) For almost every $\zeta \in I$, the function $\varkappa \to \dfrac{\varkappa(\zeta) - \mathfrak{f}(\zeta, \varkappa(\zeta))}{\mathfrak{g}(\zeta, \varkappa(\zeta))}$ is continuous and increasing in R.

(A_1) The functions $\mathfrak{g} : I \times R \to R \setminus \{0\}$, and $\mathfrak{f} : I \times R \to R$ are continuous. There exist two positive functions $\mu(\zeta),\ \sigma(\zeta)$, with bounds $\|\mu\| = \sup\{\mu(\zeta) \,|\zeta \in I\}$ and $\|\sigma\| = \sup\{\sigma(\zeta)\,|\zeta \in I\}$ such that for all $\zeta \in I$, $\varkappa, \mathfrak{z} \in R$

$$|\mathfrak{g}(\zeta,\varkappa) - \mathfrak{g}(\zeta,\mathfrak{z})| \leq \mu(\zeta)\ |\varkappa - \mathfrak{z}|, \text{ and } |\mathfrak{f}(\zeta,\varkappa) - \mathfrak{f}(\zeta,\mathfrak{z})|\ \leq \sigma(\zeta)\ |\varkappa - \mathfrak{z}|.$$

(A_2) The functions $\mathfrak{h} : [0,T] \times R \to R$ and $\mathfrak{u} : [0,T] \times R \to R$ satisfy Caratheodory conditions, i.e, if $\mathfrak{h}$ and $\mathfrak{u}$ are measurable in ζ for any $\varkappa \in R$ and continuous in $\varkappa$ for almost all $\zeta \in [0,T]$, then there exist three functions $\zeta \to \boldsymbol{a}(\zeta), \zeta \to \boldsymbol{b}(\zeta)$ and $\zeta \to \boldsymbol{m}(\zeta)$ such that

$$|\mathfrak{h}(\zeta,\varkappa)| \leq\ \boldsymbol{a}(\zeta) + \boldsymbol{b}(\zeta)\ |\varkappa|,\ \forall\ (\zeta,\varkappa)\ \in I \times R,$$

$$|\mathfrak{u}(\zeta,\varkappa)| \leq \boldsymbol{m}(\zeta),\ \forall\ (\zeta,\varkappa)\ \in I \times R,$$

where $\boldsymbol{a}(.), \boldsymbol{m}(.) \in L^1$ and $\boldsymbol{b}(.)$ are measurable and bounded, and $\mathfrak{I}_c^{\gamma,\varphi}\boldsymbol{m}(.) \leq \mathcal{M}, \forall\gamma \leq \alpha, c \geq 0$.

(A_3) There exists a positive number ρ such that for $0 < \gamma < \beta < 1$,

$$\rho \geq \frac{\Gamma(\gamma+1)}{2\,(2k_1+k_2)\,\|\mu\|\,(\varphi\,(1)-\varphi\,(0))^{\gamma}}\left(1 - \|\sigma\| - \Re - \sqrt{\Delta}\right) \tag{3.1}$$

such that

$$\Re = \frac{\|\mu\|\,\aleph\Gamma(\gamma+1) + (2H_1+H_2)\,\|\mu\| + \widetilde{G}\,(2k_1+k_2)}{\Gamma(\gamma+1)}(\varphi\,(1)-\varphi\,(0))^{\gamma},$$

$$\aleph = G_0\,(\varphi\,(T)-\varphi\,(0))\left[\|\boldsymbol{a}\| + \mathcal{M}\,\|\boldsymbol{b}\|\,\frac{(\varphi\,(T)-\varphi\,(0))^{\beta-\gamma}}{\Gamma\,(\beta-\gamma+1)}\right],$$

$$\begin{aligned}\widetilde{G} &= \sup_{\zeta\in I}|\mathfrak{g}(\zeta,0)|,\\ \Delta &= \chi^2 - \eta,\end{aligned}$$

$$\chi = (\Re + \|\sigma\| - 1)\,,$$

$$\eta = \frac{4\,(2k_1+k_2)\,\|\mu\|\,(\varphi\,(1)-\varphi\,(0))^{\gamma}\left(F + \frac{(2H_1+H_2)\widetilde{G}(\varphi(1)-\varphi(0))^{\gamma}}{\Gamma(\gamma+1)} + \widetilde{G}(\varphi\,(1)-\varphi\,(0))^{\gamma}\right)}{\Gamma(\gamma+1)}.$$

(A_4) The functions $\mathfrak{h}_i : I \times R \to R$ for $i = 1,2$ are continuous and there exist constants $k_i \in [0,1)$ such that $|\mathfrak{h}_i(\zeta, \varkappa) - \mathfrak{h}_i(\zeta, \mathfrak{z})| \le k_i\ |\varkappa - \mathfrak{z}|$ for every $\zeta \in I$, and $\varkappa, \mathfrak{z} \in R$.

(A_5) There exist a positive number $G_0 = \max \{|G(\zeta, s)|,\ \text{for every}\ \ (\zeta, s) \in I \times I\}$.

(A_6) $\mathfrak{h}_i : I \times R \to R, i = 1,2$ are continuous and there exist constants $k_i \in [0,1)$ such that $|\mathfrak{h}_i(\zeta, \mathfrak{u}(\zeta)) - \mathfrak{h}_i(\zeta, \mathfrak{v}(\zeta))| \le k_i |\mathfrak{u} - \mathfrak{v}|$.

Remark 3.1. From assumptions (A_1) and (A_4), we deduce that for every $i = 1,2$:

$$|\mathfrak{h}_i(\zeta, \varkappa)| \le H_i + k_i |\varkappa(\zeta)|, \text{ where } H_i = \sup_{\zeta \in I} |\mathfrak{h}_i(\zeta, 0)|,$$

$$|\mathfrak{g}(\zeta, \varkappa)| \le \widetilde{G} + |\mu(\zeta)|\ |\varkappa(\zeta)|, \text{ where } \widetilde{G} = \sup_{\zeta \in I} |\mathfrak{g}(\zeta, 0)|,$$

$$|\mathfrak{f}(\zeta, \varkappa)| \le F + |\sigma(\zeta)|\ |\varkappa(\zeta)|, \text{ where } F = \sup_{\zeta \in I} |\mathfrak{f}(\zeta, 0)|.$$

Lemma 3.1. Let $\mathfrak{h}\left(\zeta, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(\zeta, \varkappa(\zeta))\right) \in C(I, R)$. Then, the (IBFHDE) (1.1) is equivalent to the following integral equation

$$\varkappa(\zeta) = \mathfrak{v}(\zeta, \varkappa(\zeta)) + \mathfrak{g}(\zeta, \varkappa(\zeta)) \int_0^T \varphi'(s)\, G(\zeta, s)\ \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right)\ ds, \tag{3.2}$$

where $\mathfrak{v}(\zeta, \varkappa(\zeta))$ is a continuous function in X such that

$$\mathfrak{v}(\zeta, \varkappa(\zeta)) = \mathfrak{f}(\zeta, \varkappa(\zeta)) + \frac{\mathfrak{g}(\zeta, \varkappa(\zeta))}{\Gamma(\gamma)} \left[\int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s))\, ds + \frac{\Theta(\zeta)}{\Theta(T)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \varkappa(s)) - \mathfrak{h}_1(s, \varkappa(s))]\, ds \right],$$

where $\Theta(\zeta) = \varphi(\zeta) - \varphi(0)$, and $G(\zeta, s)$ is the Green's function defined by

$$G(\zeta, s) = \begin{cases} \dfrac{(\varphi(\zeta) - \varphi(s))^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{\Theta(\zeta)(\varphi(T) - \varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}, & 0 \le s \le \zeta \le T \\ -\dfrac{\Theta(\zeta)(\varphi(T) - \varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}, & 0 \le \zeta \le s \le T, \end{cases}$$

Proof. Applying operation $\mathfrak{I}^{\alpha,\varphi}$ on the (IBFHDE) (1.1) and by Lemma (2.2), we obtain that

$$\frac{\varkappa(\zeta) - \mathfrak{f}(\zeta, \varkappa(\zeta))}{\mathfrak{g}(\zeta, \varkappa(\zeta))} = \mathfrak{I}_{0^+}^{\alpha,\varphi}\ \mathfrak{h}(\zeta, \mathfrak{I}^{\beta,\varphi}\ \mathfrak{u}(\zeta, \varkappa(\zeta)) + c_0 + c_1(\varphi(\zeta) - \varphi(0)),$$

where $c_0, c_1 \in R$.

Hence, the integral solution of (1.1) is

$$\varkappa(\zeta) = \mathfrak{f}(\zeta, \varkappa(\zeta)) + \mathfrak{g}(\zeta, \varkappa(\zeta)) \left[\begin{array}{c} \frac{1}{\Gamma(\alpha)} \int_0^\zeta \varphi'(\zeta)(\varphi(\zeta) - \varphi(s))^{\alpha-1} \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right) ds \\ + c_0 + c_1(\varphi(\zeta) - \varphi(0)) \end{array} \right].$$

Applying the boundary conditions of (1.1), we get

$$c_0 = \frac{1}{\Gamma(\gamma)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s)) ds,$$

and

$$c_1 = \frac{1}{\Theta(T)} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \varkappa(s)) - \mathfrak{h}_1(s, \varkappa(s))] ds - \frac{1}{\Gamma(\alpha)} \int_0^T \varphi'(s)(\varphi(T) - \varphi(s))^{\alpha-1} \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right) ds \right\},$$

where $\Theta(\zeta) = \varphi(\zeta) - \varphi(0)$.

This implies that

$$\begin{aligned}
\varkappa(\zeta) &= \mathfrak{f}(\zeta, \varkappa(\zeta)) \\
&+ \mathfrak{g}(\zeta, \varkappa(\zeta)) \left\{ \frac{1}{\Gamma(\alpha)} \int_0^\zeta \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha-1} \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right) ds \right. \\
&+ \frac{1}{\Gamma(\gamma)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s)) ds \\
&+ \frac{\Theta(\zeta)}{\Theta(T)} \left\{ \frac{1}{\Gamma(\gamma)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \varkappa(s)) - \mathfrak{h}_1(s, \varkappa(s))] ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^T \varphi'(s)(\varphi(T) - \varphi(s))^{\alpha-1} \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right) ds \right\} \\
&= \mathfrak{f}(\zeta, \varkappa(\zeta)) + \frac{\mathfrak{g}(\zeta, \varkappa(\zeta))}{\Gamma(\gamma)} \left[\int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s)) ds \right. \\
&\left. + \frac{\Theta(\zeta)}{\Theta(T)} \int_0^1 \varphi'(s)(\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \varkappa(s)) - \mathfrak{h}_1(s, \varkappa(s))] ds \right] \\
&+ \frac{\mathfrak{g}(\zeta, \varkappa(\zeta))}{\Gamma(\alpha)} \left[\int_0^\zeta \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha-1} \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right) ds \right. \\
&\left. - \frac{\Theta(\zeta)}{\Theta(T)} \left(\int_0^T \varphi'(s)(\varphi(T) - \varphi(s))^{\alpha-1} \mathfrak{h}\left(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right) ds \right] \\
&= \mathfrak{v}(\zeta, \varkappa(\zeta)) + \mathfrak{g}(\zeta, \varkappa(\zeta)) \int_0^T \varphi'(s) G(\zeta, s)\, \mathfrak{h}\left(\zeta, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(\zeta, \varkappa(\zeta))\right) ds.
\end{aligned}$$

□

3.1. **Existence of Solutions.** From Lemma (3.1), we obtain the following definition

Definition 3.1. By a mild solution of the (IBFHDE) (1.1), we mean a function $\varkappa \in C(I, R)$ satisfying integral equation (3.2) for all $\zeta \in I$.

Theorem 3.1. Assume that hypotheses $(A_0)-(A_3)$ and (A_6) hold. Then, the integral equation (3.2) has at least one mild solution defined in I.

Proof. Set $X = C(I, R)$ and define a subset S of X as $S = \{\varkappa \in X,\ \|\varkappa\| \leq \rho\}$, where ρ satisfies inequality (3.1). It is clear that S is a closed, convex, and bounded subset of the Banach space X. Define the following three operators: $\mathcal{A} : X \to X$, $\mathcal{B} : S \to X$ and $\mathcal{C} : X \to X$ as follows:

$$\mathcal{A}\varkappa(\zeta) = \mathfrak{g}(\zeta, \varkappa(\zeta)),\ \text{for } \zeta \in I \tag{3.3}$$

$$\mathcal{B}\varkappa(\zeta) = \int_0^T \varphi'(s)\, G(\zeta, s)\ \mathfrak{h}(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\, ds,\ \text{for } (\zeta, s) \in I \times I, \tag{3.4}$$

$$\mathcal{C}\varkappa(\zeta) = \mathfrak{v}(\zeta, \varkappa(\zeta)), \text{for } \zeta \in I. \tag{3.5}$$

Thus, the integral equation (3.2) is transformed into the following integral operator equation:

$$\varkappa(\zeta) = \mathcal{A}\varkappa(\zeta) \cdot \mathcal{B}\varkappa(\zeta) + \mathcal{C}\varkappa(\zeta),\ \text{for all } \zeta \in I. \tag{3.6}$$

In the following, we show that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy all the conditions of Lemma 2.1. The proof is accomplished in the following steps:

Step 1. Operators $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on X.

From assumption (A_1), it is obtained that for any $\varkappa, \mathfrak{z} \in X$,

$$\begin{aligned} |\mathcal{A}\varkappa(\zeta) - \mathcal{A}\mathfrak{z}(\zeta)| &= |\mathfrak{g}(\zeta, \varkappa(\zeta)) - \mathfrak{g}(\zeta, \mathfrak{z}(\zeta))| \\ &\leq \mu(\zeta)\ |\varkappa(\zeta) - \mathfrak{z}(\zeta)| \leq \|\mu\|\ \|\varkappa - \mathfrak{z}\| \end{aligned}$$

Thus, $\|\mathcal{A}\varkappa - \mathcal{A}\mathfrak{z}\| \leq \|\mu\|\ \|\varkappa - \mathfrak{z}\|$ for all $\varkappa, \mathfrak{z} \in X$ and this implies that $\mathcal{A}$ is a Lipschitzian on X with Lipschitz constant μ.

Moreover, $|\mathcal{C}\varkappa(\zeta) - \mathcal{C}\mathfrak{z}(\zeta)| = |\mathfrak{v}(\zeta, \varkappa(\zeta)) - \mathfrak{v}(\zeta, \mathfrak{z}(\zeta))|$ where;

$$\mathfrak{v}(\zeta, \varkappa(\zeta)) - \mathfrak{v}(\zeta, \mathfrak{z}(\zeta))$$

$$
\begin{aligned}
&= \mathfrak{f}(\zeta, \varkappa(\zeta)) - \mathfrak{f}(\zeta, \mathfrak{z}(\zeta)) \\
&+ \frac{\mathfrak{g}(\zeta, \varkappa(\zeta))}{\Gamma(\gamma)} \left[\int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s))\, ds \right. \\
&\left. + \frac{\Theta(\zeta)}{\Theta(T)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \varkappa(s)) - \mathfrak{h}_1(s, \varkappa(s))]\, ds \right] \\
&- \frac{\mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \left[\int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \mathfrak{z}(s))\, ds \right. \\
&\left. + \frac{\Theta(\zeta)}{\Theta(T)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \mathfrak{z}(s)) - \mathfrak{h}_1(s, \mathfrak{z}(s))]\, ds \right] \\
&= \mathfrak{f}(\zeta, \varkappa(\zeta)) - \mathfrak{f}(\zeta, \mathfrak{z}(\zeta)) + E_1(\zeta) + \frac{\Theta(\zeta)}{\Theta(T)} E_2(\zeta),
\end{aligned}
$$

where $E_1(\zeta)$ and $E_2(\zeta)$ are given as follows:

$$
\begin{aligned}
&E_1(\zeta) \\
&= \frac{\mathfrak{g}(\zeta, \varkappa(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s))\, ds \\
&- \frac{\mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \mathfrak{z}(s))\, ds \\
&= \frac{\mathfrak{g}(\zeta, \varkappa(\zeta)) - \mathfrak{g}(\zeta, \mathfrak{z}(\zeta)) + \mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s))\, ds \\
&- \frac{\mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \mathfrak{z}(s))\, ds \\
&= \frac{\mathfrak{g}(\zeta, \varkappa(\zeta)) - \mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \mathfrak{h}_1(s, \varkappa(s))\, ds \\
&+ \frac{\mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_1(s, \varkappa(s)) - \mathfrak{h}_1(s, \mathfrak{z}(s))]\, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&E_2(\zeta) \\
&= \frac{\mathfrak{g}(\zeta, \varkappa(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \varkappa(s)) - \mathfrak{h}_1(s, \varkappa(s))]\, ds \\
&- \frac{\mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} [\mathfrak{h}_2(s, \mathfrak{z}(s)) - \mathfrak{h}_1(s, \mathfrak{z}(s))]\, ds \\
&= \frac{\mathfrak{g}(\zeta, \varkappa(\zeta)) - \mathfrak{g}(\zeta, \mathfrak{z}(\zeta)) + \mathfrak{g}(\zeta, \mathfrak{z}(\zeta))}{\Gamma(\gamma)} \int_0^1 \varphi'(s) (\varphi(1) - \varphi(s))^{\gamma-1} \begin{bmatrix} \mathfrak{h}_2(s, \varkappa(s)) \\ -\mathfrak{h}_1(s, \varkappa(s)) \end{bmatrix} ds
\end{aligned}
$$

$$-\frac{\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}[\mathfrak{h}_2(s,\mathfrak{z}(s))-\mathfrak{h}_1(s,\mathfrak{z}(s))]\,ds$$
$$=\frac{\mathfrak{g}(\zeta,\varkappa(\zeta))-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}[\mathfrak{h}_2(s,\varkappa(s))-\mathfrak{h}_1(s,\varkappa(s))]\,ds$$
$$+\frac{\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}\begin{bmatrix}\mathfrak{h}_2(s,\varkappa(s))-\mathfrak{h}_1(s,\varkappa(s))\\-\mathfrak{h}_2(s,\mathfrak{z}(s))+\mathfrak{h}_1(s,\mathfrak{z}(s))\end{bmatrix}ds$$
$$=\frac{\mathfrak{g}(\zeta,\varkappa(\zeta))-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}[\mathfrak{h}_2(s,\varkappa(s))-\mathfrak{h}_1(s,\varkappa(s))]\,ds$$
$$+\frac{\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}[\mathfrak{h}_2(s,\varkappa(s))-\mathfrak{h}_2(s,\mathfrak{z}(s))]\,ds$$
$$+\frac{\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}[\mathfrak{h}_1(s,\mathfrak{z}(s))-\mathfrak{h}_1(s,\varkappa(s))].$$

By assumptions (A_1), (A_6), and Remark 3.1, it is obtained that

$$\begin{aligned}|E_1(\zeta)|&\le\frac{|\mathfrak{g}(\zeta,\varkappa(\zeta))-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}|\mathfrak{h}_1(s,\varkappa(s))|\,ds\\&+\frac{|\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}|\mathfrak{h}_1(s,\varkappa(s))-\mathfrak{h}_1(s,\mathfrak{z}(s))|\,ds\\&\le\frac{\mu(\zeta)\ |\varkappa-\mathfrak{z}|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}(H_1+k_1|\varkappa|)\,ds\\&+\frac{\widetilde{G}+\mu(\zeta)\ |\mathfrak{z}(\zeta)|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}k_1|\varkappa-\mathfrak{z}|\,ds\\&\le\left(\frac{\mu(\zeta)(H_1+k_1|\varkappa|)+k_1\left(\widetilde{G}+\|\mu\|\,|\mathfrak{z}|\right)}{\Gamma(\gamma+1)}(\varphi(1)-\varphi(0))^{\gamma}\right)\ |\varkappa-\mathfrak{z}|\\&\le c_1(\zeta)\ |\varkappa-\mathfrak{z}|,\end{aligned}$$

and

$$\begin{aligned}&|E_2(\zeta)|\\&\le\frac{|\mathfrak{g}(\zeta,\varkappa(\zeta))-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}\begin{pmatrix}|\mathfrak{h}_2(s,\varkappa(s))|\\+|\mathfrak{h}_1(s,\varkappa(s))|\end{pmatrix}ds\\&+\frac{|\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}\{|\mathfrak{h}_2(s,\varkappa(s))-\mathfrak{h}_2(s,\mathfrak{z}(s))|\}\,ds\\&+\frac{|\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|}{\Gamma(\gamma)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\gamma-1}\{|\mathfrak{h}_1(s,\mathfrak{z}(s))-\mathfrak{h}_1(s,\varkappa(s))|\}\,ds\end{aligned}$$

$$\leq \frac{\mu(\zeta)\ |\varkappa-\mathfrak{z}|}{\Gamma(\gamma)}\int_0^1 \varphi'(s)\left(\varphi(1)-\varphi(s)\right)^{\gamma-1}\left[(H_1+H_2)+(k_1+k_2)\left|\varkappa\right|\right]ds$$

$$+\frac{\widetilde{G}+\mu(\zeta)\ |\mathfrak{z}(\zeta)\,|}{\Gamma(\gamma)}\int_0^1 \varphi'(s)\left(\varphi(1)-\varphi(s)\right)^{\gamma-1}\left[k_2\left|\varkappa-\mathfrak{z}\right|+k_1\left|\mathfrak{z}-\varkappa\right|\right]ds$$

$$\leq \frac{\mu(\zeta)\ |\varkappa-\mathfrak{z}|}{\Gamma(\gamma+1)}\left(\varphi(1)-\varphi(s)\right)^{\gamma}\left((H_1+H_2)+(k_1+k_2)\left|\varkappa\right|\right)$$

$$+\frac{\widetilde{G}+\mu(\zeta)\ |\mathfrak{z}(\zeta)\,|}{\Gamma(\gamma+1)}\left(\varphi(1)-\varphi(s)\right)^{\gamma}(k_1+k_2)\left|\mathfrak{z}-\varkappa\right|$$

$$\leq \frac{\left(\begin{array}{c}\mu(\zeta)\left((H_1+H_2)+(k_1+k_2)\left|\varkappa\right|\right)\\ +\left(\widetilde{G}+\mu(\zeta)\ |\mathfrak{z}(\zeta)\,|\right)(k_1+k_2)\end{array}\right)}{\Gamma(\gamma+1)}\left(\varphi(1)-\varphi(s)\right)^{\gamma}\ \left|\varkappa-\mathfrak{z}\right|$$

$$\leq c_2(\zeta)\ |\varkappa-\mathfrak{z}|.$$

Hence, by assumption (A_1), we obtain that

$$\begin{aligned}\left|\mathfrak{v}(\zeta,\varkappa(\zeta))-\mathfrak{v}(\zeta,\mathfrak{z}(\zeta))\right| &\leq \left|\mathfrak{f}(\zeta,\varkappa(\zeta))-\mathfrak{f}(\zeta,\mathfrak{z}(\zeta))\right|+\left|E_1(\zeta)\right|+\frac{\Theta(\zeta)}{\Theta(T)}\left|E_2(\zeta)\right|\\ &\leq \left(\sigma(\zeta)+c_1(\zeta)+\frac{\Theta(\zeta)}{\Theta(T)}c_2(\zeta)\right)\ \left|\varkappa-\mathfrak{z}\right|.\end{aligned}$$

Taking supremum for all $\zeta\in[0,\zeta]$, we get

$$\left|\mathcal{C}\varkappa(\zeta)-\mathcal{C}\mathfrak{z}(\zeta)\right|\leq\left(\|\sigma\|+\|c_1\|+\|c_2\|\right)\ \left\|\varkappa-\mathfrak{z}\right\|,$$

where

$$\|c_1\|=\frac{\|\mu\|\left(H_1+k_1(\|\varkappa\|+\|\mathfrak{z}\|)+k_1\widetilde{G}\right.}{\Gamma(\gamma+1)}\left(\varphi(1)-\varphi(0)\right)^{\gamma},$$

and

$$\|c_2\|=\frac{\|\mu\|\left(H_1+H_2+(k_1+k_2)\left(\|\varkappa\|+\|\mathfrak{z}\|\right)\right)+(k_1+k_2)\,\widetilde{G}}{\Gamma(\gamma+1)}\left(\varphi(1)-\varphi(s)\right)^{\gamma}.$$

Therefore, operator $\mathcal{C}$ is a Lipschitzian mapping on X with Lipschitz constant $k=\|\sigma\|+\|c_1\|+\|c_2\|$.

Step 2. Operator $\mathcal{B}$ is continuous and compact on S.

First, operator $\mathcal{B}$ is continuous on X.

Let $\{\varkappa_{\mathfrak{n}}\}$ be a sequence in S that converges to point $\varkappa\in S$. Since $\mathfrak{u}(\zeta,\varkappa(\zeta))$ is continuous in X for all $\zeta\in T$, then (by assumption (A_2)) we have $\mathfrak{u}(\zeta,\varkappa_{\mathfrak{n}}(\zeta))$

converges to $\mathfrak{u}(\zeta, \varkappa(\zeta))$. Applying the Lebesgue dominated convergence theorem, it is obtained that

$$\lim_{\mathfrak{n}\to\infty} \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa_{\mathfrak{n}}(s)) = \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s)).$$

In addition, since $\mathfrak{h}(\zeta, \varkappa(\zeta))$ is continuous in $\varkappa$, then by the fractional-order integral properties, and by applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{\mathfrak{n}\to\infty} \mathcal{B}\varkappa_{\mathfrak{n}}(\zeta) &= \lim_{\mathfrak{n}\to\infty} \int_0^T \varphi'(s)\, G(\zeta, s)\, \mathfrak{h}(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa_{\mathfrak{n}}(s))ds \\ &= \int_0^T \varphi'(s)\, G(\zeta, s) \lim_{\mathfrak{n}\to\infty} \mathfrak{h}(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa_{\mathfrak{n}}(s))ds \\ &= \int_0^T \varphi'(s)\, G(\zeta, s)\, \mathfrak{h}(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))ds = \mathcal{B}\varkappa(\zeta).\end{aligned}$$

Thus, $\mathcal{B}\varkappa_{\mathfrak{n}} \to \mathcal{B}\varkappa$ as $\mathfrak{n} \to \infty$ uniformly on R^+. This implies that operator $\mathcal{B}$ is continuous on S for all $\zeta \in I$.

Next, operator $\mathcal{B}$ is compact on S since $\mathcal{B}(S)$ is uniformly bounded and equicontinuous in X:

Let $\varkappa \in S$ be arbitrary. By assumption (A_2), it is clear that

$$\begin{aligned}&|\mathcal{B}\varkappa(\zeta)| = \left|\int_0^T \varphi'(s)\, G(\zeta, s)\, \mathfrak{h}(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))ds\right| \\ &\leq \int_0^T \varphi'(s)\, |G(\zeta, s)|\, \left|\mathfrak{h}(s, \mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s, \varkappa(s))\right| ds \\ &\leq G_0 \int_0^T \varphi'(s)\, \big(\boldsymbol{a}(s) + \boldsymbol{b}(s)\, \mathfrak{I}^{\beta,\varphi}\, |\mathfrak{u}(s, \varkappa(s))|\big)\, ds \\ &\leq G_0 \int_0^T \varphi'(s)\, |\boldsymbol{a}(s)|\, ds + G_0 \int_0^T \varphi'(s)\, |\boldsymbol{b}(s)|\, \mathfrak{I}^{\beta,\varphi}\, |\mathfrak{u}(s, \varkappa(s))|\, ds \\ &\leq G_0\, \|\boldsymbol{a}\| \int_0^T \varphi'(s)\, ds + G_0\, \|\boldsymbol{b}\| \int_0^T \varphi'(s)\, \mathfrak{I}^{\beta,\varphi}\, |\mathfrak{u}(s, \varkappa(s))|\, ds \\ &\leq G_0\, \|\boldsymbol{a}\|\, (\varphi(T) - \varphi(0)) + G_0\, \|\boldsymbol{b}\| \int_0^T \varphi'(s)\, \mathfrak{I}^{\beta,\varphi}\boldsymbol{m}(s)\, ds \\ &\leq G_0\, \|\boldsymbol{a}\|\, (\varphi(T) - \varphi(0)) + G_0\, \|\boldsymbol{b}\|\, (\varphi(T) - \varphi(0))\;\; \mathfrak{I}^{\beta-\gamma,\varphi}\mathfrak{I}^{\gamma,\varphi}\boldsymbol{m}(\zeta) \\ &\leq G_0\, \|\boldsymbol{a}\|\, (\varphi(T) - \varphi(0)) \\ &+ G_0\, (\varphi(T) - \varphi(0))\;\; \|\boldsymbol{b}\|\;\; \mathcal{M} \int_0^\zeta \varphi'(s)\, \frac{(\varphi(\zeta) - \varphi(s))^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} ds.\end{aligned}$$

Taking supermom over all $\zeta \in I$, it is decuced that for all $\varkappa \in S$

$$\|\mathcal{B}\varkappa\| \leq G_0\left(\varphi\left(T\right)-\varphi\left(0\right)\right)\left(\ \|\boldsymbol{a}\|+\|\boldsymbol{b}\|\,\mathcal{M}\frac{\left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right).$$

Thus, operator $\mathcal{B}$ is uniformly bounded on S.

Now, $\mathcal{B}(S)$ is also an equicontinuous set in X.

Let $\zeta_1,\ \zeta_2 \in I$ such that $\zeta_1 < \zeta_2$. Then, for all $\varkappa \in S$ it is clear that

$$\begin{aligned}
&\mathcal{B}\varkappa(\zeta_2) - B\varkappa(\zeta_1)\\
&= \int_0^T \varphi'\left(s\right)G\left(\zeta_2,s\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))ds - \int_0^T \varphi'\left(s\right)G\left(\zeta_1,s\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))ds\\
&= \int_0^{\zeta_2} \varphi'\left(s\right)\left(\frac{\left(\varphi\left(\zeta_2\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\Theta\left(\zeta_2\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&+ \int_{\zeta_2}^T \varphi'\left(s\right)\left(-\frac{\Theta\left(\zeta_2\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&- \int_0^{\zeta_1} \varphi'\left(s\right)\left(\frac{\left(\varphi\left(\zeta_1\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\Theta\left(\zeta_1\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&- \int_{\zeta_1}^T \varphi'\left(s\right)\left(-\frac{\Theta\left(\zeta_1\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))\\
&= \int_0^{\zeta_1} \varphi'\left(s\right)\left(\frac{\left(\varphi\left(\zeta_2\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\Theta\left(\zeta_2\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&+ \int_{\zeta_1}^{\zeta_2} \varphi'\left(s\right)\left(\frac{\left(\varphi\left(\zeta_2\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\Theta\left(\zeta_2\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&+ \int_{\zeta_2}^T \varphi'\left(s\right)\left(-\frac{\Theta\left(\zeta_2\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&- \int_0^{\zeta_1} \varphi'\left(s\right)\left(\frac{\left(\varphi\left(\zeta_1\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\Theta\left(\zeta_1\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&- \int_{\zeta_1}^{\zeta_2} \varphi'\left(s\right)\left(-\frac{\Theta\left(\zeta_1\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds\\
&- \int_{\zeta_2}^T \varphi'\left(s\right)\left(-\frac{\Theta\left(\zeta_1\right)\left(\varphi\left(T\right)-\varphi\left(s\right)\right)^{\alpha-1}}{\Theta\left(T\right)\Gamma(\alpha)}\right)\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|\mathcal{B}\varkappa(\zeta_2) - B\varkappa(\zeta_1)|\\
&\leq \int_0^{\zeta_1} \varphi'\left(s\right)\left(\begin{array}{c} \frac{(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}}{\Gamma(\alpha)} \\ -\frac{(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)} \end{array}\right)\left|\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))\right|ds
\end{aligned}$$

$$+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\begin{pmatrix}\frac{(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}}{\Gamma(\alpha)}\\-\frac{(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\end{pmatrix}\left|\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))\right|ds$$

$$+\int_{\zeta_2}^{T}\varphi'(s)\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\left|\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s)))\right|ds$$

$$\leq\int_{0}^{\zeta_1}\varphi'(s)\begin{pmatrix}\frac{(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}}{\Gamma(\alpha)}\\-\frac{(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\end{pmatrix}\left(|\boldsymbol{a}(s)|+|\boldsymbol{b}(s)|\,\mathfrak{I}^{\beta,\varphi}|\mathfrak{u}(s,\varkappa(s))|\right)ds$$

$$+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\begin{pmatrix}\frac{(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}}{\Gamma(\alpha)}\\-\frac{(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\end{pmatrix}\left(|\boldsymbol{a}(s)|+|\boldsymbol{b}(s)|\,\mathfrak{I}^{\beta,\varphi}|\mathfrak{u}(s,\varkappa(s))|\right)ds$$

$$+\int_{\zeta_2}^{T}\varphi'(s)\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\left(|\boldsymbol{a}(s)|+|\boldsymbol{b}(s)|\,\mathfrak{I}^{\beta,\varphi}|\mathfrak{u}(s,\varkappa(s))|\right)ds$$

$$\leq\|\boldsymbol{a}\|\left[\int_{0}^{\zeta_1}\varphi'(s)\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}\right]\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\Gamma(\alpha)}ds\right.$$

$$+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\frac{\begin{pmatrix}\Theta(T)(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\Gamma(\alpha)}ds$$

$$\left.+\int_{\zeta_2}^{T}\varphi'(s)\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}ds\right]$$

$$+\|\boldsymbol{b}\|\left[\int_{0}^{\zeta_1}\varphi'(s)\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}\right]\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\Gamma(\alpha)}\mathfrak{I}^{\beta,\varphi}|\mathfrak{u}(s,\varkappa(s))|\,ds\right.$$

$$+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\frac{\begin{pmatrix}\Theta(T)(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\Gamma(\alpha)}\mathfrak{I}^{\beta,\varphi}|\mathfrak{u}(s,\varkappa(s))|\,ds$$

$$\left.+\int_{\zeta_2}^{T}\varphi'(s)\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\mathfrak{I}^{\beta,\varphi}|\mathfrak{u}(s,\varkappa(s))|\,ds\right]$$

$$\leq\|\boldsymbol{a}\|\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(0))^{\alpha}-(\varphi(\zeta_1)-\varphi(0))^{\alpha}\right]\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(0))^{\alpha}\end{pmatrix}}{\Theta(T)\;\Gamma(\alpha+1)}$$

$$
+\|\boldsymbol{b}\|\left[\int_0^{\zeta_1}\varphi'(s)\,\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}\right]\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\,\Gamma(\alpha)}\,\mathfrak{I}^{\beta-\gamma,\varphi}\mathfrak{I}^{\gamma,\varphi}\boldsymbol{m}(s)\,ds\right.
$$

$$
+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\,\frac{\begin{pmatrix}\Theta(T)(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\,\Gamma(\alpha)}\,\mathfrak{I}^{\beta-\gamma,\varphi}\mathfrak{I}^{\gamma,\varphi}\boldsymbol{m}(s)\,ds
$$

$$
\left.+\int_{\zeta_2}^{T}\varphi'(s)\,\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\,\Gamma(\alpha)}\mathfrak{I}^{\beta-\gamma,\varphi}\mathfrak{I}^{\gamma,\varphi}\boldsymbol{m}(s)\,ds\right]
$$

$$
\leq\|\boldsymbol{a}\|\,\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(0))^{\alpha}-(\varphi(\zeta_1)-\varphi(0))^{\alpha}\right]\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(0))^{\alpha}\end{pmatrix}}{\Theta(T)\;\;\Gamma(\alpha+1)}
$$

$$
+\|\boldsymbol{b}\|\,\mathcal{M}\left[\int_0^{\zeta_1}\varphi'(s)\,\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}\right]\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\,\Gamma(\alpha)}\right.
$$

$$
\times\int_0^{s}\varphi'(\tau)\,\frac{(\varphi(s)-\varphi(\tau))^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}d\tau ds
$$

$$
+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\,\frac{\begin{pmatrix}\Theta(T)(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\,\Gamma(\alpha)}
$$

$$
\times\int_0^{s}\varphi'(\tau)\,\frac{(\varphi(s)-\varphi(\tau))^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}d\tau ds
$$

$$
\left.+\int_{\zeta_2}^{T}\varphi'(s)\,\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\,\Gamma(\alpha)}\int_0^{s}\varphi'(\tau)\,\frac{(\varphi(s)-\varphi(\tau))^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}d\tau ds\right]
$$

$$
\leq\|\boldsymbol{a}\|\,\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(0))^{\alpha}-(\varphi(\zeta_1)-\varphi(0))^{\alpha}\right]\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(0))^{\alpha}\end{pmatrix}}{\Theta(T)\;\;\Gamma(\alpha+1)}
$$

$$
+\|\boldsymbol{b}\|\,\mathcal{M}\left[\int_0^{\zeta_1}\varphi'(s)\,\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-(\varphi(\zeta_1)-\varphi(s))^{\alpha-1}\right]\\ -(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\,\Gamma(\alpha)}\right.
$$

$$
\times\frac{(\varphi(s)-\varphi(0))^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}ds
$$

$$+\int_{\zeta_1}^{\zeta_2}\varphi'(s)\frac{\begin{pmatrix}\Theta(T)(\varphi(\zeta_2)-\varphi(s))^{\alpha-1}-\\(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(s))^{\alpha-1}\end{pmatrix}}{\Theta(T)\Gamma(\alpha)}\frac{(\varphi(s)-\varphi(0))^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}ds$$

$$\left.+\int_{\zeta_2}^{T}\varphi'(s)\frac{(\Theta(\zeta_1)-\Theta(\zeta_2))(\varphi(T)-\varphi(s))^{\alpha-1}}{\Theta(T)\Gamma(\alpha)}\frac{(\varphi(s)-\varphi(0))^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}ds\right]$$

$$\leq\|\boldsymbol{a}\|\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(0))^{\alpha}-(\varphi(\zeta_1)-\varphi(0))^{\alpha}\right]\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(0))^{\alpha}\end{pmatrix}}{\Theta(T)\ \Gamma(\alpha+1)}$$

$$+\|\boldsymbol{b}\|\mathcal{M}\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(0))^{\alpha}-(\varphi(\zeta_1)-\varphi(0))^{\alpha}\right]\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(0))^{\alpha}\end{pmatrix}}{\Theta(T)\ \Gamma(\alpha+1)\Gamma(\beta-\gamma+1)}(\varphi(T)-\varphi(0))^{\beta-\gamma}$$

Therefore,

$$|\mathcal{B}\varkappa(\zeta_2)-\mathcal{B}\varkappa(\zeta_1)|\leq\varpi\left(\|\boldsymbol{a}\|+\frac{\|\boldsymbol{b}\|\mathcal{M}(\varphi(T)-\varphi(0))^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right),$$

where

$$\varpi=\frac{\begin{pmatrix}\Theta(T)\left[(\varphi(\zeta_2)-\varphi(0))^{\alpha}-(\varphi(\zeta_1)-\varphi(0))^{\alpha}\right]\\-(\Theta(\zeta_2)-\Theta(\zeta_1))(\varphi(T)-\varphi(0))^{\alpha}\end{pmatrix}}{\Theta(T)\ \Gamma(\alpha+1)}.$$

Thus, for any $\varepsilon>0$, there exists a positive number δ such that $|\zeta_2-\zeta_1|<\delta$ for all $\zeta_1,\zeta_2\in I$, which implies that $|B_\kappa(\zeta_2)-B_\kappa(\zeta_1)|<\varepsilon$ for all $\varkappa\in S$, then $\mathcal{B}(S)$ is an equicontinuous set in X. Furthermore, applying the Arzela-Ascoli theorem, we can conclude that $B(S)$ is uniformly bounded and equicontinuous in X, and hence it is a compact set. Consequently, we have established that the operator B is a complete and continuous operator on S.

Step 3. We will demonstrate that the operator $\varkappa=\mathcal{A}\varkappa\mathcal{B}\mathfrak{z}+\mathcal{C}\varkappa$ is bounded for all $\varkappa\in X$ and $\mathfrak{z}\in S$. Consider $\zeta\in I$, then

$$\begin{aligned}&|\varkappa(\zeta)|\\&\leq\ |\mathcal{A}\varkappa(\zeta)||\mathcal{B}\mathfrak{z}(\zeta)|+|\mathcal{C}\varkappa(\zeta)|\\&\leq|\mathfrak{g}(\zeta,\varkappa(\zeta))|\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \ |\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))|ds+|\mathfrak{v}(s,\varkappa(s))|\end{aligned}$$

$$\leq |\mathfrak{g}(\zeta,\varkappa(\zeta))| \int_0^T \varphi'(s) \, |G(\zeta,s)| \;\; |\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\; \mathfrak{u}(s,\varkappa(s))| ds + |\mathfrak{v}(\varkappa,\zeta)|$$

$$\leq G_0 |\mathfrak{g}(\zeta,\varkappa(\zeta))| \int_0^T \varphi'(s) \;\; |\mathfrak{h}(s,\mathfrak{I}^{\beta,\varphi}\; \mathfrak{u}(s,\varkappa(s))| ds + |\mathfrak{f}(\varkappa,\zeta)|$$

$$+ \frac{|\mathfrak{g}(\zeta,\varkappa(\zeta))|}{\Gamma(\gamma)} \int_0^1 \varphi'(s) \left(\varphi(1) - \varphi(s)\right)^{\gamma-1} |\mathfrak{h}_1(s,\varkappa(s))| \, ds$$

$$+ \frac{|\mathfrak{g}(\zeta,\varkappa(\zeta))|}{\Gamma(\gamma)} \frac{\Theta(\zeta)}{\Theta(T)} \int_0^1 \varphi'(s) \left(\varphi(1) - \varphi(s)\right)^{\gamma-1} |\mathfrak{h}_2(s,\varkappa(s)) - \mathfrak{h}_1(s,\varkappa(s))| \, ds$$

$$\leq G_0 (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \int_0^T \varphi'(s) \;\; |\boldsymbol{a}(s) + \boldsymbol{b}(s) \, \mathfrak{I}^{\beta,\varphi} \, |\mathfrak{u}(s,\varkappa(s))| \, ds$$

$$+ (F + |\sigma(\zeta)| \; |\varkappa(\zeta)|)$$

$$+ \frac{(\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|)}{\Gamma(\gamma)} \int_0^1 \varphi'(s) \left(\varphi(1) - \varphi(s)\right)^{\gamma-1} (k_1 |\varkappa| + H_1) \, ds$$

$$+ \frac{(\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|)}{\Gamma(\gamma)} \int_0^1 \varphi'(s) \left(\varphi(1) - \varphi(s)\right)^{\gamma-1} \begin{pmatrix} k_2 |\varkappa(\zeta)| + H_2 \\ + k_1 |\varkappa(\zeta)| + H_1 \end{pmatrix} ds$$

$$\leq G_0 (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \int_0^T \varphi'(s) \, G_0 \;\; \left(|\boldsymbol{a}(s) + \boldsymbol{b}(s) \, \mathfrak{I}^{\beta,\varphi} \boldsymbol{m}(s)\right) ds$$

$$+ (F + \|\sigma\| \, |\varkappa(\zeta)|)$$

$$+ (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \frac{\left(\varphi(1) - \varphi(0)\right)^{\gamma}}{\Gamma(\gamma+1)} \begin{pmatrix} (H_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) H_1) \\ + (k_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) k_1) |\varkappa(\zeta)| \end{pmatrix}$$

$$\leq (F + \|\sigma\| \, |\varkappa(\zeta)|)$$

$$+ G_0 (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \int_0^T \varphi'(s) \left(|\boldsymbol{a}(s) + \boldsymbol{b}(s) \, \mathfrak{I}^{\beta-\gamma,\varphi} \mathfrak{I}^{\gamma,\varphi} \boldsymbol{m}(s)\right) ds$$

$$+ (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \frac{\left(\varphi(1) - \varphi(0)\right)^{\gamma}}{\Gamma(\gamma+1)} \begin{pmatrix} (H_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) H_1) \\ + (k_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) k_1) |\varkappa(\zeta)| \end{pmatrix}$$

$$\leq (F + \|\sigma\| \, |\varkappa(\zeta)|)$$

$$+ G_0 (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \left(\|\boldsymbol{a}\| + \|\boldsymbol{b}\| \, \mathcal{M} \mathfrak{I}^{\beta-\gamma,\varphi}(s)\right) \left(\varphi(T) - \varphi(0)\right)$$

$$+ (\widetilde{G} + |\mu(\zeta)| \; |\varkappa(\zeta)|) \frac{\left(\varphi(1) - \varphi(0)\right)^{\gamma}}{\Gamma(\gamma+1)} \begin{pmatrix} (H_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) H_1) \\ + (k_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) k_1) |\varkappa(\zeta)| \end{pmatrix}$$

$$\leq (F + \|\sigma\| \, |\varkappa(\zeta)|)$$

$$+ G_0(\widetilde{G} + |\mu(\zeta)|\ |\varkappa(\zeta)|) \begin{pmatrix} \|\boldsymbol{a}\| \\ + \|\boldsymbol{b}\| \mathcal{M} \int_0^s \varphi'(\mathfrak{u}) \frac{(\varphi(\zeta)-\varphi(\mathfrak{u}))^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} d\mathfrak{u} \end{pmatrix} (\varphi(T) - \varphi(0))$$

$$+ (\widetilde{G} + |\mu(\zeta)|\ |\varkappa(\zeta)|) \frac{(\varphi(1) - \varphi(0))^\gamma}{\Gamma(\gamma+1)} \begin{pmatrix} (H_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) H_1) \\ +(k_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) k_1)|\varkappa(\zeta)| \end{pmatrix}$$

$$\leq (F + \|\sigma\|\, |\varkappa(\zeta)|)$$

$$+ G_0(\widetilde{G} + |\mu(\zeta)|\ |\varkappa(\zeta)|) \left(\|\boldsymbol{a}\| + \|\boldsymbol{b}\| \mathcal{M}\ \frac{(\varphi(T)-\varphi(0))^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right) (\varphi(T) - \varphi(0))$$

$$+ (\widetilde{G} + |\mu(\zeta)|\ |\varkappa(\zeta)|) \frac{(\varphi(1) - \varphi(0))^\gamma}{\Gamma(\gamma+1)} \begin{pmatrix} (H_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) H_1) \\ +(k_2 + \left(1 + \frac{\Theta(\zeta)}{\Theta(T)}\right) k_1)|\varkappa(\zeta)| \end{pmatrix}$$

By taking the supremum over all $\zeta \in I$, we obtain

$$\begin{aligned} \|\varkappa\| &\leq (F + \|\sigma\|\, \|\varkappa\|) \\ &\quad + G_0(\widetilde{G} + \|\mu\|\, \|\varkappa\|\, |) \left(\|\boldsymbol{a}\| + \|\boldsymbol{b}\| \mathcal{M}\ \frac{(\varphi(T)-\varphi(0))^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right) (\varphi(T) - \varphi(0)) \\ &\quad + (\widetilde{G} + \|\mu\|\ \ \|\varkappa\|) \frac{(\varphi(1) - \varphi(0))^\gamma}{\Gamma(\gamma+1)} ((H_2 + 2H_1) + (k_2 + 2k_1)\, \|\varkappa\|) \leq \rho. \end{aligned}$$

Accordingly,

$$\rho \geq \frac{\Gamma(\gamma+1)}{2\,(2k_1 + k_2)\, \|\mu\|\, (\varphi(1) - \varphi(0))^\gamma} \left(1 - \|\sigma\| - \Re - \sqrt{\Delta}\right),$$

where

$$\Re = \frac{\|\mu\|\, \aleph \Gamma(\gamma+1) + (2H_1 + H_2)\, \|\mu\| + \widetilde{G}\,(2k_1 + k_2)}{\Gamma(\gamma+1)} (\varphi(1) - \varphi(0))^\gamma,$$

$$\aleph = G_0\,(\varphi(T) - \varphi(0)) \left[\|\boldsymbol{a}\| + \mathcal{M}\, \|\boldsymbol{b}\|\, \frac{(\varphi(T) - \varphi(0))^{\beta-\gamma}}{\Gamma(\beta - \gamma + 1)}\right],$$

$$\Delta = (\Re + \|\sigma\| - 1)^2 - \eta,$$

so that

$$\eta = \frac{4\,(2k_1 + k_2)\, \|\mu\|\, (\varphi(1) - \varphi(0))^\gamma \left(F + \frac{(2H_1+H_2)\widetilde{G}(\varphi(1)-\varphi(0))^\gamma}{\Gamma(\gamma+1)} + \widetilde{G}(\varphi(1) - \varphi(0))^\gamma\right)}{\Gamma(\gamma+1)}.$$

Therefore, $\varkappa \in S$.

Step 4. As a last step, we show that $l\mathcal{N} + r < \rho$, such that

$$\mathcal{N} \quad = \quad \|\mathcal{B}(S)\| = \sup_{\varkappa \in S} \left\{ \sup_{\zeta \in I} |\mathcal{B}\varkappa(\zeta)| \right\}$$

$$\leq \ G_0 (\varphi(T) - \varphi(0)) \left[\|\boldsymbol{a}\| + \mathcal{M} \|\boldsymbol{b}\| \frac{(\varphi(T) - \varphi(0))^{\beta-\gamma}}{\Gamma(^{\beta-\gamma} + 1)} \right],$$

By assumption (A_3), we have $\mathcal{LM} + \mathcal{K} \ < 1$,where

$$\mathcal{L} = 2\|b\| G_0 \|\mu\| \Gamma(\gamma+1) \left[\begin{array}{c} (1 - \|\sigma\|)\Gamma(\gamma+1) \\ -2\widetilde{G}(2k_1 + k_2)(\varphi(1) - \varphi(0))^{\gamma} \end{array} \right] (\varphi(T) - \varphi(0))^{\beta+\gamma+1},$$

and $\mathcal{K} = 1 - a_0 (a_1(\varphi(1) - \varphi(0))^{2\gamma} + a_2(\varphi(1) - \varphi(0))^{\gamma}\Gamma(\gamma+1) + a_3)$, where

$$a_0 = \frac{\Gamma(\beta - \gamma + 1)(\varphi(T) - \varphi(0))^{2\gamma}}{(\varphi(1) - \varphi(0))^{\gamma}},$$

$$a_1 = \left[\begin{array}{c} 4\|a\| G_0 \widetilde{G}(2k_1 + k_2)\mu(\varphi(T) - \varphi(0))\Gamma(\gamma+1) \\ +4(2H_1 + H_2)\widetilde{G}(2k_1 + k_2) \end{array} \right] \|\mu\|,$$

$$a_2 = \left(\begin{array}{c} [2\|a\| G_0 (\|\sigma\| - 1)(\varphi(T) - \varphi(0))] \|\mu\| \Gamma(\gamma+1) + 4F(2k_1 + k_2)\|\mu\| \\ +2(2H_1 + H_2)\|\mu\|(\|\sigma\| - 1) + 2\widetilde{G}(2k_1 + k_2)(\|\sigma\| - 1) \end{array} \right),$$

$$a_3 = 3(\|\sigma\| - 1)^2 \Gamma(\gamma+1)^2.$$

Hence, the last condition of Theorem 2.1 is satisfied with

$$l = \left(\|\mu\| \|\varkappa\| + \widetilde{G} \right) (\varphi(1) - \varphi(0))^{\gamma},$$

and

$$r = (\|\sigma\| \|\varkappa\| + F) + \left(\|\mu\| \|\varkappa\| + \widetilde{G} \right) (\varphi(1) - \varphi(0))^{\gamma} \frac{(2k_1 + k_2)\|\varkappa\| + (2H_1 + H_2)}{\Gamma(\gamma+1)}.$$

As a result of fulfilling all the requirements stated in Theorem 2.1, the operator equation $\varkappa = \mathcal{A}\varkappa\mathcal{B}\mathfrak{z} + \mathcal{C}\varkappa$ possesses a solution in the set S. Consequently, it can be concluded that problem (1.1) has at least one mild solution on I. This statement serves as the conclusion of the proof. □

3.2. **Uniqueness of Solutions.** The succeeding text outlines the sufficient conditions for the uniqueness of the solution of the quadratic functional integral equation (3.2).

Consider the following assumption:

(A_7) Let $\mathfrak{h}:[0,T]\times R\to R$ and $\mathfrak{u}:[0,T]\times R\to R$ be a continuous functions satisfying the Lipschitz condition and there exists two positive functions $w(\zeta),\ \theta(\zeta)$ with bounded $\|w\|$ and $\|\theta\|$, such that

$$|\mathfrak{h}(\zeta,\varkappa)-\mathfrak{h}(\zeta,\mathfrak{z})|\leq w(\zeta)|\varkappa-\mathfrak{z}|, \text{ and } |\mathfrak{u}(\zeta,\varkappa)-\mathfrak{u}(\zeta,\mathfrak{z})|\leq \theta(\zeta)|\varkappa-\mathfrak{z}|,$$

with $H=\sup_{\zeta\in I}|\mathfrak{h}(\zeta,0)|$, and $U=\sup_{\zeta\in I}|\mathfrak{u}(\zeta,0)|$.

Theorem 3.2. Assume that (A_0), (A_1), (A_3), (A_6) and (A_7) hold, then the solution $\varkappa\in C[0,T]$ of the quadratic functional integral equation (3.2) is unique, if

$$\left(\begin{array}{c}\|\mu\|+\frac{(\widetilde{G}+\|\mu\|\|\varkappa\|)}{\Gamma(\gamma+1)}(2k_1+k_2)+\|\mu\|\,G_0\left[\|w\|\,\mathcal{M}\frac{T^{\beta-\gamma+1}}{\Gamma(\beta-\gamma+1)}+H\ T\right]\\ +\left[\|\mu\|\,\|\mathfrak{z}\|+\widetilde{G}\right]\|w\|\,\|\theta\|\,G_0\frac{T^{\beta+1}}{\Gamma(\beta+1)}\end{array}\right)<1. \tag{3.7}$$

Proof. Suppose that $\varkappa(\zeta)$ and $\mathfrak{z}(\zeta)$ are two solutions of (3.2), then from Theorem 3.1, we have

$$\begin{aligned}
&|\varkappa(\zeta)-\mathfrak{z}(\zeta)|\\
&\leq|\mathfrak{v}(\zeta,\varkappa(\zeta))-\mathfrak{v}(\zeta,\mathfrak{z}(\zeta))|\\
&+\left|\mathfrak{g}(\zeta,\varkappa(\zeta))\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))\right)\,ds\right.\\
&\qquad\left.-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\mathfrak{z}(s))\right)\,ds\right|\\
&\leq|\mathfrak{v}(\zeta,\varkappa(\zeta))-\mathfrak{v}(\zeta,\mathfrak{z}(\zeta))|\\
&+|\mathfrak{g}(\zeta,\varkappa(\zeta))-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \left|\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi,}\mathfrak{u}(s,\varkappa(s))\right)\right|\,ds\\
&+|\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \left|\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))\right)-\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\mathfrak{z}(s))\right)\right|\,ds\\
&\leq\left(\sigma(\zeta)+c_1(\zeta)+\frac{\Theta(\zeta)}{\Theta(T)}c_2(\zeta)\right)\ |\varkappa(\zeta)-\mathfrak{z}(\zeta)|\\
&+|\mathfrak{g}(\zeta,\varkappa(\zeta))-\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))|\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \left[\left|\begin{array}{c}\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))\right)\\-\mathfrak{h}(s,0)+|\mathfrak{h}(s,0)|\end{array}\right|\right]ds\\
&+\left[|\mathfrak{g}(\zeta,\mathfrak{z}(\zeta))-\mathfrak{g}(\zeta,0)|+|\mathfrak{g}(\zeta,0)|\right]\int_0^T\varphi'(s)\,|G(\zeta,s)|\ \left[\begin{array}{c}\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\varkappa(s))\right)\\-\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}(s,\mathfrak{z}(s))\right)\end{array}\right]\ ds\\
&\leq\left(\sigma(\zeta)+c_1(\zeta)+\frac{\Theta(\zeta)}{\Theta(T)}c_2(\zeta)\right)\ |\varkappa(\zeta)-\mathfrak{z}(\zeta)|
\end{aligned}$$

$$+\left|\mu\left(\zeta\right)\right|\left|\varkappa(\zeta)-\mathfrak{z}(\zeta)\right|\int_{0}^{T}\varphi'\left(s\right)\left|G(\zeta,s)\right|\left[\left|w(\zeta)\right|\ \mathfrak{I}^{\beta,\varphi}\ \left|\mathfrak{u}\left(s,\varkappa\left(s\right)\right)\right|+H\right]ds$$

$$+\left[\left|\mu\left(\zeta\right)\right|\left|\mathfrak{z}\left(\zeta\right)\right|+\widetilde{G}\right]\int_{0}^{T}\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|w(\zeta)\right|\ \mathfrak{I}^{\beta,\varphi}\ \left|\mathfrak{u}\left(\zeta,\varkappa\left(\zeta\right)\right)-\mathfrak{u}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\right|\ ds$$

$$\leq\left(\sigma(\zeta)+c_{1}\left(\zeta\right)+\frac{\Theta\left(\zeta\right)}{\Theta\left(T\right)}c_{2}\left(\zeta\right)\right)\ \left|\varkappa\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|$$

$$+\left|\mu\left(\zeta\right)\right|\left|\varkappa(\zeta)-\mathfrak{z}(\zeta)\right|\int_{0}^{T}\left|G(\zeta,s)\right|\ \left[\left|w(\zeta)\right|\mathfrak{I}^{\beta-\gamma,\varphi}\mathfrak{I}^{\gamma,\varphi}\boldsymbol{m}\left(s\right)+H\right]ds$$

$$+\left[\left|\mu\left(\zeta\right)\right|\left|\mathfrak{z}\left(\zeta\right)\right|+\widetilde{G}\right]\int_{0}^{T}\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|w(\zeta)\right|\ \left|\theta\left(\zeta\right)\right|\mathfrak{I}^{\beta,\varphi}\left|\varkappa\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|\ ds$$

$$\leq\left(\sigma(\zeta)+c_{1}\left(\zeta\right)+\frac{\Theta\left(\zeta\right)}{\Theta\left(T\right)}c_{2}\left(\zeta\right)\right)\ \left|\varkappa\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|$$

$$+\left|\mu\left(\zeta\right)\right|\left|\varkappa(\zeta)-\mathfrak{z}(\zeta)\right|\int_{0}^{T}\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left[\left|w(\zeta)\right|\mathfrak{I}^{\beta-\gamma,\varphi}\mathcal{M}+H\right]ds$$

$$+\left[\left|\mu\left(\zeta\right)\right|\left|\mathfrak{z}\left(\zeta\right)\right|+\widetilde{G}\right]\int_{0}^{T}\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|w(\zeta)\right|\ \left|\theta\left(\zeta\right)\right|\mathfrak{I}^{\beta,\varphi}\left|\varkappa\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|\ ds.$$

Taking supremum over $\zeta\in I$, we get

$$\begin{aligned}
&\left\|\varkappa-\mathfrak{z}\right\|\\
&\leq\left(\left\|\sigma\right\|+\left\|c_{1}\right\|+\left\|c_{2}\right\|\right)\ \left\|\varkappa-\mathfrak{z}\right\|\\
&+\left\|\mu\right\|\left\|\varkappa-\mathfrak{z}\right\|G_{0}\int_{0}^{T}\varphi'\left(s\right)\left[\left\|w\right\|\mathcal{M}\mathfrak{I}^{\beta-\gamma,\varphi}+H\right]ds\\
&+\left[\left\|\mu\right\|\left\|\mathfrak{z}\right\|+\widetilde{G}\right]\left\|w\right\|\left\|\theta\right\|\left\|\varkappa-\mathfrak{z}\right\|G_{0}\int_{0}^{T}\varphi'\left(s\right)\mathfrak{I}^{\beta,\varphi}\ ds\\
&\leq\left(\left\|\sigma\right\|+\left\|c_{1}\right\|+\left\|c_{2}\right\|\right)\ \left\|\varkappa-\mathfrak{z}\right\|\\
&+\left\|\mu\right\|\left\|\varkappa-\mathfrak{z}\right\|G_{0}\left[\begin{array}{c}\left\|w\right\|\mathcal{M}\int_{0}^{T}\varphi'\left(s\right)\int_{0}^{s}\varphi'\left(\tau\right)\frac{\left(\varphi(s)-\varphi(\tau)\right)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}d\tau ds\\+H\int_{0}^{T}\varphi'\left(s\right)ds\end{array}\right]\\
&+\left[\left\|\mu\right\|\left\|\mathfrak{z}\right\|+\widetilde{G}\right]\left\|w\right\|\left\|\theta\right\|\left\|\varkappa-\mathfrak{z}\right\|G_{0}\int_{0}^{T}\varphi'\left(s\right)\int_{0}^{s}\varphi'\left(\tau\right)\frac{\left(\varphi\left(s\right)-\varphi\left(\tau\right)\right)^{\beta-1}}{\Gamma\left(\beta\right)}d\tau\ ds\\
&\leq\left(\left\|\sigma\right\|+\left\|c_{1}\right\|+\left\|c_{2}\right\|\right)\ \left\|\varkappa-\mathfrak{z}\right\|\\
&+\left\|\mu\right\|\left\|\varkappa-\mathfrak{z}\right\|G_{0}\ \left[\left\|w\right\|\mathcal{M}\frac{\left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\beta-\gamma+1}}{\Gamma\left(\beta-\gamma+1\right)}+H\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)\right]\\
&+\left[\left\|\mu\right\|\left\|\mathfrak{z}\right\|+\widetilde{G}\right]\left\|w\right\|\left\|\theta\right\|\left\|\varkappa-\mathfrak{z}\right\|G_{0}\frac{\left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\beta+1}}{\Gamma\left(\beta+1\right)}.
\end{aligned}$$

Then,

$$\|\varkappa-\mathfrak{z}\|\left(\begin{array}{c} 1-(\|\sigma\|+\|c_1\|+\|c_2\|) \\ -\|\mu\|\,G_0\,\left[\|w\|\,\mathcal{M}\frac{(\varphi(T)-\varphi(0))^{\beta-\gamma+1}}{\Gamma(\beta-\gamma+1)}+H\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)\right] \\ -\left[\|\mu\|\,\|\mathfrak{z}\|+\widetilde{G}\right]\|w\|\,\|\theta\|\,\|\varkappa-\mathfrak{z}\|\,G_0\frac{(\varphi(T)-\varphi(0))^{\beta+1}}{\Gamma(\beta+1)} \end{array}\right)\leq 0.$$

This proves the uniqueness of the solution of quadratic integral equation (3.2). □

3.3. **Ulam Stability of Solutions.** In the following, we study the Ulam stability for the (IBFHDE) (1.1). Let $\epsilon>0$ and $\Phi: I\to R^{+}$ be a continuous function and consider the following inequalities:

$$\left|{}^{c}\mathfrak{D}^{\alpha,\varphi}\left(\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right)-\mathfrak{h}(\zeta,\mathfrak{I}^{\beta,\varphi}\ \mathfrak{u}(\zeta,\varkappa(\zeta))\right|\leq\epsilon,\ \zeta\in I \tag{3.8}$$

$$\left|{}^{c}\mathfrak{D}^{\alpha,\varphi}\left(\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right)-\mathfrak{h}(\zeta,\mathfrak{I}^{\beta,\varphi}\ \mathfrak{u}(\zeta,\varkappa(\zeta))\right|\leq\Phi(\zeta),\ \zeta\in I \tag{3.9}$$

$$\left|{}^{c}\mathfrak{D}^{\alpha,\varphi}\left(\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right)-\mathfrak{h}(\zeta,\mathfrak{I}^{\beta,\varphi}\ \mathfrak{u}(\zeta,\varkappa(\zeta))\right|\leq\epsilon\ \Phi(\zeta),\ \zeta\in I. \tag{3.10}$$

Definition 3.2. The problem (IBFHDE) (1.1) is said to be Ulam-Hyers stable if there exists a real number $c_{\mathfrak{f}}>0$ such that for each $\epsilon>0$ and for each solution $\mathfrak{z}\in C(I,R)$ of the inequality (3.8) there exists a solution $\varkappa\in C(I,R)$ of (1.1) with

$$|\mathfrak{z}(\zeta)-\varkappa(\zeta)|\leq\epsilon\ \ c_{\mathfrak{f}},\ \zeta\in I.$$

Definition 3.3. The problem (IBFHDE) (1.1) is said to be generalized Ulam-Hyers stable if there exists $c_{\mathfrak{f}}\in C(R_+,R_+)$ with $c_{\mathfrak{f}}(0)=0$ such that for each $\epsilon>0$ and for each solution $\mathfrak{z}\in C(I,R)$ of the inequality (3.8) there exists a solution $\varkappa\in C(I,R)$ of (1.1) with

$$|\mathfrak{z}(\zeta)-\varkappa(\zeta)|\leq c_{\mathfrak{f}}(\epsilon),\ \zeta\in I.$$

Definition 3.4. The problem (IBFHDE) (1.1) is said to be Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{\mathfrak{f},\Phi}>0$ such that for each $\epsilon>0$ and for each solution $\mathfrak{z}\in C(I,R)$ of the inequality (3.9) there exists a solution $\varkappa\in C(I,R)$ of (1.1) with

$$|\mathfrak{z}(\zeta)-\varkappa(\zeta)|\leq\epsilon c_{\mathfrak{f},\Phi}\Phi(\zeta),\ \zeta\in I.$$

The problem (IBFHDE) (1.1) is said to be generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{\mathfrak{f},\Phi} > 0$ such that for each solution $\mathfrak{z} \in C(I,R)$ of the inequality (3.10) there exists a solution $\varkappa \in C(I,R)$ of (1.1) with

$$|\mathfrak{z}(\zeta) - \varkappa(\zeta) > | \leq c_{\mathfrak{f},\Phi}\Phi(\zeta),\ \zeta \in I.$$

3.4. **Ulam-Hyers Stability of Solutions.** In the following, we study the Ulam-Hyers stability for (IBFHDE) (1.1).

Theorem 3.3. Let the assumptions of Theorem 3.2 be satisfied. Then problem (IBFHDE) (1.1) is Ulam-Hyers stable.

Proof. Let $\epsilon > 0$ and let $\omega \in C(I,R)$ be a function which satisfies inequality (3.8),

$$\left| {}^{c}\mathfrak{D}^{\alpha,\varphi}\left(\frac{\omega(\zeta) - \mathfrak{f}(\zeta,\omega(\zeta))}{\mathfrak{g}(\zeta,\omega(\zeta))}\right) - \mathfrak{h}(\zeta,\mathfrak{I}^{\beta,\varphi}\ \mathfrak{u}(\zeta,\omega(\zeta))\right| \leq \epsilon,\ \zeta \in I, \tag{3.11}$$

and let $\mathfrak{z} \in C(I,R)$ be the unique solution of (IBFHDE) (1.1) which is by Lemma 3.1 is equivalent to the fractional order integral equation

$$\mathfrak{z}(\zeta) = \mathfrak{v}(\zeta,\mathfrak{z}(\zeta)) + \mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(\zeta,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\right)\ ds.$$

Applying $\mathfrak{I}^{\alpha,\varphi}$ on both sides of (3.11), we get

$$|\omega(\zeta) - \mathfrak{v}(\zeta,\omega(\zeta)) - \mathfrak{g}\left(\zeta,\omega\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)\ ds| \leq \frac{\epsilon\ \left(\varphi\left(T\right) - \varphi\left(0\right)\right)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.12}$$

This implies that for each $\zeta \in I$, we have:

$$\begin{aligned}
&|\omega(\zeta) - \mathfrak{z}(\zeta)| \\
&= |\omega(\zeta) - \mathfrak{v}(\zeta,\mathfrak{z}(\zeta)) - \mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\mathfrak{z}\left(s\right)\right)\right)\ ds| \\
&= |\omega(\zeta) - \mathfrak{v}(\zeta,\omega(\zeta)) - \mathfrak{g}\left(\zeta,\omega\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)\ ds \\
&+ \mathfrak{v}(\zeta,\omega(\zeta)) + \mathfrak{g}\left(\zeta,\omega\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)\ ds \\
&- \mathfrak{v}(\zeta,\mathfrak{z}(\zeta)) - \mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\mathfrak{z}\left(s\right)\right)\right)\ ds| \\
&\leq |\omega(\zeta) - \mathfrak{v}(\zeta,\omega(\zeta)) - \mathfrak{g}\left(\zeta,\omega\left(\zeta\right)\right)\int_0^T \varphi'\left(s\right)G(\zeta,s)\ \mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)\ ds|
\end{aligned}$$

$$+\left|\mathfrak{v}(\zeta,\omega(\zeta))-\mathfrak{v}(\zeta,\mathfrak{z}(\zeta))\right|$$

$$+\left|\mathfrak{g}\left(\zeta,\omega\left(\zeta\right)\right)-\mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\right|\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)\right|\ ds$$

$$+\left|\mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\right|\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)-\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\mathfrak{z}\left(s\right)\right)\right)\right|\ ds$$

$$\leq\frac{\epsilon\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\alpha}}{\Gamma(\alpha+1)}+\left(\sigma(\zeta)+c_1\left(\zeta\right)+\frac{\Theta\left(\zeta\right)}{\Theta\left(T\right)}c_2\left(\zeta\right)\right)\ \left|\omega\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|$$

$$+\left|\mathfrak{g}\left(\zeta,\omega\left(\zeta\right)\right)-\mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)\right|\int_0^T\left|G(\varphi'\left(s\right)\zeta,s)\right|\ \left[\begin{array}{c}\left|\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)-\mathfrak{h}\left(s,0\right)\right|\\+\left|\mathfrak{h}\left(s,0\right)\right|\end{array}\right]ds$$

$$+\left|\mathfrak{g}\left(\zeta,\mathfrak{z}\left(\zeta\right)\right)-\mathfrak{g}\left(\zeta,0\right)+\mathfrak{g}\left(\zeta,0\right)\right|\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|\begin{array}{c}\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\omega\left(s\right)\right)\right)\\-\mathfrak{h}\left(s,\mathfrak{I}^{\beta,\varphi}\mathfrak{u}\left(s,\mathfrak{z}\left(s\right)\right)\right)\end{array}\right|\ ds$$

$$\leq\frac{\epsilon\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\alpha}}{\Gamma(\alpha+1)}+\left(\sigma(\zeta)+c_1\left(\zeta\right)+\frac{\Theta\left(\zeta\right)}{\Theta\left(T\right)}c_2\left(\zeta\right)\right)\ \left|\omega\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|$$

$$+\left|\mu\left(\zeta\right)\right|\left|\omega(\zeta)-\mathfrak{z}(\zeta)\right|\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left[\left|w(s)\right|\ \mathfrak{I}^{\beta,\varphi}\ \left|\mathfrak{u}\left(s,\omega\left(s\right)\right)\right|+H\right]ds$$

$$+\left[\left|\mu\left(\zeta\right)\right|\left|\mathfrak{z}\left(\zeta\right)\right|+\widetilde{G}\right]\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|w(s)\right|\ \mathfrak{I}^{\beta,\varphi}\ \left|\mathfrak{u}\left(s,\omega\left(s\right)\right)-\mathfrak{u}\left(s,\mathfrak{z}\left(s\right)\right)\right|\ ds$$

$$\leq\frac{\epsilon\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\alpha}}{\Gamma(\alpha+1)}+\left(\sigma(\zeta)+c_1\left(\zeta\right)+\frac{\Theta\left(\zeta\right)}{\Theta\left(T\right)}c_2\left(\zeta\right)\right)\ \left|\omega\left(\zeta\right)-\mathfrak{z}\left(\zeta\right)\right|$$

$$+\left|\mu\left(\zeta\right)\right|\left|\varkappa(\zeta)-\mathfrak{z}(\zeta)\right|\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left[\left|w(s)\right|\mathfrak{I}^{\beta-\gamma,\varphi}\mathfrak{I}^{\gamma,\varphi}\boldsymbol{m}\left(s\right)+H\right]ds$$

$$+\left[\left|\mu\left(\zeta\right)\right|\left|\mathfrak{z}\left(\zeta\right)\right|+\widetilde{G}\right]\int_0^T\varphi'\left(s\right)\left|G(\zeta,s)\right|\ \left|w(s)\right|\ \left|\theta\left(s\right)\right|\mathfrak{I}^{\beta,\varphi}\left|\omega\left(s\right)-\mathfrak{z}\left(s\right)\right|\ ds.$$

Taking supremum for all $\zeta\in I$, we get

$$\begin{aligned}\left\|\omega-\mathfrak{z}\right\|\leq&\frac{\epsilon\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\alpha}}{\Gamma(\alpha+1)}+\left(\left\|\sigma\right\|+\left\|c_1\right\|+\left\|c_2\right\|\right)\ \left\|\omega-\mathfrak{z}\right\|\\&+\left\|\mu\right\|G_0\ \left[\left\|w\right\|\mathcal{M}\frac{\left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\beta-\gamma+1}}{\Gamma\left(\beta-\gamma+1\right)}+H\ \left(\varphi\left(T\right)-\varphi\left(0\right)\right)\right]\left\|\omega-\mathfrak{z}\right\|\\&+\left[\left\|\mu\right\|\left\|\mathfrak{z}\right\|+\widetilde{G}\right]\left\|w\right\|\left\|\theta\right\|G_0\frac{\left(\varphi\left(T\right)-\varphi\left(0\right)\right)^{\beta+1}}{\Gamma\left(\beta+1\right)}\left\|\omega-\mathfrak{z}\right\|.\end{aligned}$$

Let

$$\eta=1-\left(\left\|\sigma\right\|+\left\|c_1\right\|+\left\|c_2\right\|\right)$$

$$- \|\mu\| \, G_0 \, \left[\|w\| \, \mathcal{M} \frac{(\varphi(T) - \varphi(0))^{\beta-\gamma+1}}{\Gamma(\beta-\gamma+1)} + H \, (\varphi(T) - \varphi(0)) \right]$$

$$- \left[\|\mu\| \, \|\mathfrak{z}\| + \widetilde{G} \right] \|w\| \, \|\theta\| \, G_0 \frac{(\varphi(T) - \varphi(0))^{\beta+1}}{\Gamma(\beta+1)},$$

we get

$$\|\omega - \mathfrak{z}\| \leq \left(\frac{\epsilon \, (\varphi(T) - \varphi(0))^{\alpha}}{\Gamma(\alpha+1)} \eta^{-1} \right) \epsilon = c_{\mathfrak{f}} \, \epsilon \, .$$

Therefore, problem (IBFHDE) (1.1) is Ulam-Hyers stable. This completes the proof. □

Remark 3.2. If we put $\Phi(\epsilon) = c_{\mathfrak{f}} \, \epsilon$, then $\Phi(0) = 0$ which yields that the (IBFHDE) (1.1) is generalized Ulam-Hyers stable. Moreover, it is easy to show that if Φ is an increasing function, $\lambda_\Phi > 0$, such that, for each $\zeta \in I$ we have $\mathfrak{I}^{\alpha}\Phi \leq \lambda_\Phi \Phi$, then the problem (IBFHDE) (1.1) is Ulam-Hyers-Rassias stable with respect to Φ and with a real constant $c_{\mathfrak{f},\Phi} = \frac{\lambda_\Phi}{\eta}$.

4. Special Cases

Within this segment, we introduce a set of fractional derivatives that rely on the selection of the arbitrary value for $\varphi(\zeta)$, the consideration of β, and the boundary conditions.

- If $\varphi(\zeta) = \zeta$, then the results presented in this paper are consistent with those found in Awad's [9] research on implicit fractional-order differential problems:

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha} \left(\frac{\varkappa(\zeta) - \mathfrak{f}(\zeta, \varkappa(\zeta))}{\mathfrak{g}(\zeta, \varkappa(\zeta))} \right) = \mathfrak{h}(\zeta, \mathfrak{I}^{\beta}\mathfrak{u}(\zeta, \varkappa(\zeta)), \ \zeta \in [0, T], \\ \left. \frac{\varkappa(\zeta) - \mathfrak{f}(\zeta, \varkappa(\zeta))}{\mathfrak{g}(\zeta, \varkappa(\zeta))} \right|_{\zeta=0} = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} \, \mathfrak{h}_1(s, \varkappa(s)) \, ds, \\ \left. \frac{\varkappa(\zeta) - \mathfrak{f}(\zeta, \varkappa(\zeta))}{\mathfrak{g}(\zeta, \varkappa(\zeta))} \right|_{\zeta=T} = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} \, \mathfrak{h}_2(s, \varkappa(s)) \, ds, \end{cases}$$

- When $\varphi(\zeta) = \zeta$, $\mathfrak{f}(\zeta, \varkappa(\zeta)) = 0$, $\beta \to 0$, and $\mathfrak{h}_i(s, \varkappa(s)) = 0$ for $i = 1, 2$, then we get the fractional hybrid differential equations involving Riemann-Liouville differential operators $\mathfrak{D}^{\alpha} \left(\frac{\varkappa(\zeta)}{\mathfrak{f}(\zeta, \varkappa(\zeta))} \right) = \mathfrak{g}(\zeta, \varkappa(\zeta))$ a.e. $\zeta \in I$, with $\varkappa(0) = 0$ studied by Sun *et al.* [35]:

$$\mathfrak{D}^{\alpha} \left(\frac{\varkappa(\zeta)}{-\mathfrak{f}(\zeta, \varkappa(\zeta))} \right) = \mathfrak{g}(\zeta, \varkappa(\zeta)) \text{ a.e. } \zeta \in I,$$

$$\varkappa(0) = 0$$

- When $\varphi(\zeta)=\zeta$, $\mathfrak{f}(\zeta,\varkappa(\zeta))=0$, $\beta\to 0$, $\mathfrak{h}_1(s,\varkappa(s))=\mathfrak{h}_2(s,\varkappa(s))=\Gamma(\gamma)(1-s)^{1-\gamma}$, $\left.\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right|_{\zeta=0}=-\frac{c}{2a}$, and $\left.\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right|_{\zeta=T}=-\frac{c}{2b}$, where a,b, and c are real constants with $a+b\neq 0$, we obtain the boundary value problems for hybrid differential equations with fractional orders studied by Hilal *at el.* [20]:

$$\mathfrak{D}^{\alpha}\left(\frac{\varkappa(\zeta)}{\mathfrak{f}(\zeta,\varkappa(\zeta))}\right)=\mathfrak{g}(\zeta,\varkappa(\zeta)) \text{ a.e. } \zeta\in I,$$
$$a\frac{\varkappa(0)}{-\mathfrak{f}(0,\varkappa(0))}+b\frac{\varkappa(T)}{-\mathfrak{f}(T,\varkappa(T))}=c$$

- When $\varphi(\zeta)=\zeta$, $\beta\to 0$, $T=1$, and $\mathfrak{h}_i(s,\varkappa(s))=0$ for $i=1,2$, we obtain the the hybrid fractional differential equation studied by Ullah at el. [34]:

$$\mathfrak{D}^{\alpha}\left(\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right)=\mathfrak{h}(\zeta,\varkappa(\zeta)),\ \zeta\in I$$
$$\left.\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right|_{\zeta=0}=\left.\frac{\varkappa(\zeta)-\mathfrak{f}(\zeta,\varkappa(\zeta))}{\mathfrak{g}(\zeta,\varkappa(\zeta))}\right|_{\zeta=1}=0$$

- Finally, when $\varphi(\zeta)=\zeta$, $\varkappa(\zeta)\to\varkappa(\Theta_i(\zeta))$, $\mathfrak{g}(0,\varkappa(0))=1$, and $\mathfrak{h}_i(s,\varkappa(s))=0$ for $i=1,2$, we obtain the following fractional hybrid differential equations involving the Riemann-Liouville differential operators on the delay functions studied by Al Issa at el. [6]:

$$\begin{cases}\mathfrak{D}^{\alpha}\left(\frac{\varkappa(\zeta)-k(\zeta,\varkappa(\Theta_1(\zeta)))}{\mathfrak{g}(\zeta,\varkappa(\Theta_2(\zeta)))}\right)=\mathfrak{f}(\zeta,\mathfrak{I}^{\beta}\mathfrak{u}(\zeta,\varkappa(\Theta_3(\zeta)))), \mathfrak{f}or\zeta\in I=[0,T],\\ \varkappa(0)=k(0,\varkappa(0)).\end{cases}$$

5. Conclusion

In summary, the article presents a study on nonlinear hybrid fractional boundary value problems involving φ-Caputo derivatives of fractional order and with two-point hybrid boundary conditions. The author uses a fixed point theorem of Dhage to prove the existence and uniqueness of solutions under mixed Lipschitz and Caratheodory conditions. Furthermore, the Ulam-Hyers types of stability is established. Lastly, the article presents a class of fractional boundary value problems based on the choices for the arbitrary values of φ and the boundary conditions.

For future work, the research may be extended to investigate more general classes of nonlinear fractional boundary value problems with various types of boundary conditions. In addition, numerical methods can be also explored for solving such problems.

Additionally, applications of the developed mathematical results can be considered to different areas of science and engineering, such as physics, biology, or finance.

Acknowledgement

The author would like to thank the editor and the referees for their valuable suggestions and careful readings towards the improvement of the paper

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