Jordan Journal of Mathematics and Statistics (JJMS), 16(4), 2023, pp 649 - 668
DOI: https://doi.org/10.47013/16.4.3

TWO-DIMENSIONAL QUATERNIONIC FRACTIONAL MELLIN TRANSFORM OF A PARTICULAR ORDER

KHINAL PARMAR(1) AND V. R. LAKSHMI GORTY(2)

Abstract. Author introduces two-dimensional quaternionic fractional Mellin transform defined for a particular order α, β of integrable functions on $\mathbb{R}^2$ and prove its inversion formula using the relation between fractional Fourier transform and fractional Mellin transform. Properties like linearity, Parseval's formula and product theorem are obtained without any additional conditions. Applications of two dimensional quaternionic fractional Mellin transform are given to support the study.

1. Introduction

In 1998, O. Akay and G.F. Boudreaux-Bartels [1] developed a new fractional Mellin transform (FrMT) and fractional concepts based on the concept of fractional Fourier transform (FrFT). Authors in [2] studied on modified Mellin transform of generalized functions. Generalized two-dimensional was introduced in [19]. Digital computation of the fractional Mellin transform was analyzed in [4]. Authors in [6, 10] introduced quaternion Fourier transform and quaternionic Fourier-Mellin transform. The finite Mellin transform in quantum calculus and application were established in [13].
After introduction of non-commutative algebra of quaternions, many works of complex-valued valued functions were extended to to quaternion-valued functions. In recent study, few integral transforms have been extended to quaternion-valued functions [15, 16, 17, 18].
The content of the paper is organized as follows: In section 2, some basic facts of

2010 *Mathematics Subject Classification.* 44A35, 44A05, 46S10.
Key words and phrases. Fractional calculus, Mellin transform, convolution, partial differential equation.
Received: Aug. 29, 2022 Accepted: Jan. 3, 2023 .

quaternions and quaternion-valued functions are illustrated. In section 3, the two-dimensional quaternionic fractional Mellin transform (2D-QFrMT) is defined and its relation with two-dimensional quaternionic fraction Fourier transform (2D-QFrFT) is derived. In Section 4, various mathematical properties of 2D-QFrMT are obtained.In Section 5 and 6, Convolution type properties and Parseval's theorem are established. In section 7, the 2D-QFrMT developed during the study has been demonstrated with several applications.

The aim of the study is to establish the fractional Mellin transform in the class of quaternions. Majority of the properties are discussed in the study. This study will be useful for future researchers to bridge the gap and extend the applications of real-valued functions to quternion-valued functions.

2. Preliminary results

In quaternions, every element is a linear combination of a real scalar and three imaginary units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ with real coefficients .

Let q be a quaternion defined in

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_1 + \mathbf{k}q_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\} \tag{2.1}$$

be the division ring of quaternions, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy Hamilton's multiplication rules as in [9],

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.2}$$

The quaternion conjugate of q is defined by

$$\bar{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3;\ \ q_0, q_1, q_2, q_3 \in \mathbb{R}. \tag{2.3}$$

Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ holds its usual meaning as in [11], the inner product is given by

$$\langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathbb{R}^2} f(\mathrm{x})\overline{g(\mathrm{x})}d^2\mathrm{x}. \tag{2.4}$$

The norm of $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$\|f\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \langle f, f\rangle^{1/2}_{L^2(\mathbb{R}^2,\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\mathrm{x})|^2 d^2\mathrm{x}\right)^{1/2}. \tag{2.5}$$

Definition 2.1. For every function f as in [14], the fractional Mellin transform (FrMT) of order α is given by

$$\mathcal{M}_\alpha[f(t); s_\alpha] = \int_0^\infty f(t) t^{s_\alpha - 1} dt, \tag{2.6}$$

where $s_\alpha = a - iw^{\frac{1}{\alpha}}$ such that $a_1 < a < a_2$.

The largest open strip $(a_1,\ a_2)$ is the domain of the definition in which the integral converges. It is called the fundamental strip and is denoted by $St(a_1, a_2)$.
The inversion formula for (2.6) is

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \mathcal{M}_\alpha\left[f(t); s_\alpha\right] t^{-s_\alpha} ds_\alpha. \tag{2.7}$$

Relation between FrFT and FrMT was given in [14], for every $0 < \alpha \leq 1$

$$\mathcal{M}_\alpha[f(t); a - iw^{\frac{1}{\alpha}}] = \mathscr{F}_\alpha[f(e^{-x})e^{-ax}; w^{\frac{1}{\alpha}}]. \tag{2.8}$$

Definition 2.2. Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then two-dimensional quaternionic fractional Fourier transform (2D-QFrFT) of particular order α, β is defined in [16] as

$$\hat{f}_{\alpha,\beta}\left(w_1, w_2\right) = F_{\alpha,\beta}\left[f\left(x, y\right); w_1, w_2\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\mathbf{i} w_1^{\frac{1}{\alpha}} x} f\left(x, y\right) e^{\mathbf{j} w_2^{\frac{1}{\beta}} y} dx dy \tag{2.9}$$

where $w_1, w_2 > 0$ and $0 < \alpha, \beta \leq 1$.
The inversion formula is given by
(2.10)

$$F^{-1}_{\alpha,\beta}\left[\hat{f}_{\alpha,\beta}\left(w_1, w_2\right)\right] = \frac{1}{\left(2\pi\right)^2 \alpha\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\mathbf{i} w_1^{\frac{1}{\alpha}} x} w_1^{\frac{1-\alpha}{\alpha}} \hat{f}_{\alpha,\beta}\left(w_1, w_2\right) e^{-\mathbf{j} w_2^{\frac{1}{\beta}} x} w_2^{\frac{1-\beta}{\beta}} dw_1 dw_2.$$

3. Main Results

In this section, 2D-QFrMT for a quaternion-valued function is defined and the inversion formula is derived.

Definition 3.1. For every $f \in L^2(\mathbb{R}^2, \mathbb{H})$, 2D-QFrMT of particular order α, β can be defined from [14, 20] as

$$\mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right); s_\alpha, s_\beta\right] = \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f\left(u, v\right) v^{s_\beta - 1} du dv \tag{3.1}$$

where, $s_\alpha = a - \mathbf{i}w_1^{\frac{1}{\alpha}}; s_\beta = b - \mathbf{j}w_2^{\frac{1}{\beta}}$ with $a \in St\left(a_1, a_2\right), b \in St\left(b_1, b_2\right)$ and $0 < \alpha, \beta \leq 1$.

Relationship between 2D-QFrMT and 2D-QFrFT

Using (2.8), the relation between 2D-QFrMT and 2D-QFrFT has been extended in $L^2(\mathbb{R}^2, \mathbb{H})$ for particular order α, β as follows:

Property 3.1. *For every* $0 < \alpha, \beta \leq 1$ *and* $s_\alpha = a - \mathbf{i}w_1^{\frac{1}{\alpha}},\ s_\beta = b - \mathbf{j}w_2^{\frac{1}{\beta}}$ *with* $a \in St\left(a_1,\ a_2\right),\ b \in St\left(b_1,\ b_2\right)$,

$$\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right); a - \boldsymbol{i}w_1^{\frac{1}{\alpha}}, b - \boldsymbol{j}w_2^{\frac{1}{\beta}}\right] = F_{\alpha,\beta}\left[f\left(e^{-x}, e^{-y}\right)e^{-ax}e^{-by}; w_1^{\frac{1}{\alpha}}, w_2^{\frac{1}{\beta}}\right]. \tag{3.2}$$

Proof. For every $a \in St\left(a_1, a_2\right), b \in St\left(b_1, b_2\right)$ and $u = e^{-x}, v = e^{-y}$;
and $dx = -u^{-1}du, dy = -v^{-1}dv$. Then

$$\begin{aligned}
F_{\alpha,\beta}\left[f\left(e^{-x}, e^{-y}\right)e^{-ax}e^{-by}; w_1^{\frac{1}{\alpha}}, w_2^{\frac{1}{\beta}}\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathbf{i}w_1^{\frac{1}{\alpha}}x} f\left(e^{-x}, e^{-y}\right)e^{-ax}e^{-by}e^{\mathbf{j}w_2^{\frac{1}{\beta}}y}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(a - \mathbf{i}w_1^{\frac{1}{\alpha}}\right)x} f\left(e^{-x}, e^{-y}\right) e^{-\left(b - \mathbf{j}w_2^{\frac{1}{\beta}}\right)y}dxdy \\
&= \int_0^{\infty}\int_0^{\infty} u^{s_\alpha - 1} f\left(u,v\right)v^{s_\beta - 1}dudv \\
&= \mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right); s_\alpha, s_\beta\right]
\end{aligned}$$

for every $0 < \alpha, \beta \leq 1$, $s_\alpha = a - \mathbf{i}w_1^{\frac{1}{\alpha}}, s_\beta = b - \mathbf{j}w_2^{\frac{1}{\beta}}$ with $a \in St\left(a_1, a_2\right), b \in St\left(b_1, b_2\right)$. □

Theorem 3.1. *The inversion formula of 2D-QFrMT can be obtained from 2D-QFrFT as*

$$f(u,v) = \frac{1}{(2\pi)^2 ij}\int_{b-j\infty}^{b+j\infty}\int_{a-i\infty}^{a+i\infty} u^{-s_\alpha}\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right); s_\alpha, s_\beta\right]v^{-s_\beta}ds_\alpha ds_\beta. \tag{3.3}$$

Proof. Consider

$$F_{\alpha,\beta}\left[f\left(e^{-x}, e^{-y}\right)e^{-ax}e^{-by}; w_1^{\frac{1}{\alpha}}, w_2^{\frac{1}{\beta}}\right] = \mathcal{M}_{\alpha,\beta}\left[f\left(e^{-x}, e^{-y}\right); a - \mathbf{i}w_1^{\frac{1}{\alpha}}, b - \mathbf{j}w_2^{\frac{1}{\beta}}\right].$$

Hence

$$f\left(e^{-x}, e^{-y}\right) e^{-ax} e^{-by} = \frac{1}{(2\pi)^2\alpha\beta} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x} w_1^{\frac{1-\alpha}{\alpha}} F_{\alpha,\beta}\left[f\left(e^{-x}, e^{-y}\right) e^{-ax} e^{-by}; w_1^{\frac{1}{\alpha}}, w_2^{\frac{1}{\beta}}\right]$$
$$\times e^{-\mathbf{j}w_2^{\frac{1}{\beta}}x} w_2^{\frac{1-\beta}{\beta}} dw_1 dw_2$$
$$= \frac{1}{(2\pi)^2\alpha\beta} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathbf{i}w_1^{\frac{1}{\alpha}}x} w_1^{\frac{1-\alpha}{\alpha}} \mathcal{M}_{\alpha,\beta}\left[f\left(e^{-x}, e^{-y}\right); a - \mathbf{i}w_1^{\frac{1}{\alpha}}, b - jw_2^{\frac{1}{\beta}}\right]$$
$$\times e^{-\mathbf{j}w_2^{\frac{1}{\beta}}x} w_2^{\frac{1-\beta}{\beta}} dw_1 dw_2.$$

Using $u = e^{-x}, v = e^{-y}; s_\alpha = a - \mathbf{i}w_1^{\frac{1}{\alpha}}, s_\beta = b - \mathbf{j}w_2^{\frac{1}{\beta}}$.
And $ds_\alpha = \frac{-\mathbf{i}}{\alpha} w_1^{\frac{1}{\alpha}-1} dw_1, ds_\beta = \frac{-\mathbf{j}}{\beta} w_2^{\frac{1}{\beta}-1} dw_2$, it follows that

$$f\left(u, v\right) u^a v^b = \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{\mathbf{i}w_1^{\frac{1}{\alpha}}} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right); s_\alpha, s_\beta\right] v^{\mathbf{j}w_2^{\frac{1}{\beta}}} ds_\alpha ds_\beta$$
$$f\left(u, v\right) = \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{\mathbf{i}w_1^{\frac{1}{\alpha}}} u^{-a} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right); s_\alpha, s_\beta\right] v^{\mathbf{j}w_2^{\frac{1}{\beta}}} v^{-b} ds_\alpha ds_\beta$$
$$= \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{-s_\alpha} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right); s_\alpha, s_\beta\right] v^{-s_\beta} ds_\alpha ds_\beta.$$

The inversion formula of 2D-QFrMT can be defined as

$$f(u, v) = \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty}\int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{-s_\alpha} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right); s_\alpha, s_\beta\right] v^{-s_\beta} ds_\alpha ds_\beta.$$

□

4. Properties of 2D-QFrMT

In this section, different characteristics and mathematical properties of 2D-QFrMT are proved.

Property 4.1 (Linearity property)**.** *Let $f_1, f_2 \in L^2(\mathbb{R}^2, \mathbb{H})$ and $l_1, l_2 \in \mathbb{R}$, then*

$$\mathcal{M}_{\alpha,\beta}\left[l_1 f_1 + l_2 f_2\right] = l_1 \mathcal{M}[f_1] + l_2 \mathcal{M}[f_2]. \tag{4.1}$$

Proof.

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[l_1 f_1 + l_2 f_2\right] &= \int_0^\infty \int_0^\infty u^{s_\alpha - 1}\left(l_1 f_1 + l_2 f_2\right) v^{s_\beta - 1} du dv \\ &= l_1 \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f_1 v^{s_\beta - 1} du dv + l_2 \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f_2 v^{s_\beta - 1} du dv \\ &= l_1 \mathcal{M}[f_1] + l_2 \mathcal{M}[f_2].\end{aligned}$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[l_1 f_1 + l_2 f_2\right] = l_1 \mathcal{M}[f_1] + l_2 \mathcal{M}[f_2].$$

□

Property 4.2 (Scaling property)**.** *Let* $f \in L^2(\mathbb{R}^2, \mathbb{H})$ *and* $p_1, p_2 \in \mathbb{R}$, *then*

$$\mathcal{M}_{\alpha,\beta}\left[f\left(p_1 t_1, p_2 t_2\right); s_\alpha, s_\beta\right] = \frac{1}{p_1^{s_\alpha}} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right)\right] \frac{1}{p_2^{s_\beta}}. \tag{4.2}$$

Proof.

$$\mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right); s_\alpha, s_\beta\right] = \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f\left(u, v\right) v^{s_\beta - 1} du dv.$$

Let $u = p_1 t_1, v = p_2 t_2; du = p_1 dt_1, dv = p_2 dt_2$. Then

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[f\left(p_1 t_1, p_2 t_2\right); s_\alpha, s_\beta\right] &= \int_0^\infty \int_0^\infty {t_1}^{s_\alpha - 1} f\left(p_1 t_1, p_2 t_2\right) {t_2}^{s_\beta - 1} dt_1 dt_2 \\ &= \int_0^\infty \int_0^\infty \left(\frac{u}{p_1}\right)^{s_\alpha - 1} f\left(u, v\right) \left(\frac{v}{p_2}\right)^{s_\beta - 1} \frac{du}{p_1} \frac{dv}{p_2} \\ &= \frac{1}{p_1^{s_\alpha}} \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f\left(u, v\right) v^{s_\beta - 1} du dv \frac{1}{p_2^{s_\beta}} \\ &= \frac{1}{p_1^{s_\alpha}} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right)\right] \frac{1}{p_2^{s_\beta}}.\end{aligned}$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[f\left(p_1 t_1, p_2 t_2\right); s_\alpha, s_\beta\right] = \frac{1}{p_1^{s_\alpha}} \mathcal{M}_{\alpha,\beta}\left[f\left(u, v\right)\right] \frac{1}{p_2^{s_\beta}}.$$

□

Property 4.3 (Multiplication by $(uv)^p$). *Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then*

$$\mathcal{M}_{\alpha,\beta}\left[(uv)^p f(u,v); s_\alpha, s_\beta\right] = \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha + p, s_\beta + p\right]. \tag{4.3}$$

Proof.

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[(uv)^p f(u,v); s_\alpha, s_\beta\right] &= \int_0^\infty \int_0^\infty u^{s_\alpha - 1}(uv)^p f(u,v)\, v^{s_\beta - 1} du dv \\ &= \int_0^\infty \int_0^\infty u^{s_\alpha + p - 1} f(u,v)\, v^{s_\beta + p - 1} du dv \\ &= \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha + p, s_\beta + p\right].\end{aligned}$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[(uv)^p f(u,v); s_\alpha, s_\beta\right] = \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha + p, s_\beta + p\right].$$

□

Property 4.4 (Multiplication by a power). *Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $0 \le q \le 1$, then*

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[(\log uv)^q f(u,v); s_\alpha, s_\beta\right] &= (1-q)!\left(\frac{d}{ds_\alpha}\right)^q \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right] \\ &\quad + (1-q)!\left(\frac{d}{ds_\beta}\right)^q \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right].\end{aligned} \tag{4.4}$$

Proof. As given in [14], we have $\left(\frac{d}{ds}\right)^q (u^{s-1}) = \frac{1}{(1-q)!} u^{s-1} (\log u)^q; 0 \le q \le 1$. Then $\mathcal{M}_{\alpha,\beta}\left[(\log uv)^q f(u,v); s_\alpha, s_\beta\right]$

$$\begin{aligned}&= \int_0^\infty \int_0^\infty u^{s_\alpha - 1}(\log u)^q f(u,v)\, v^{s_\beta - 1} du dv + \int_0^\infty \int_0^\infty u^{s_\alpha - 1}(\log v)^q f(u,v)\, v^{s_\beta - 1} du dv \\ &= (1-q)!\left(\frac{d}{ds_\alpha}\right)^q \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f(u,v)\, v^{s_\beta - 1} du dv \\ &\quad + (1-q)!\left(\frac{d}{ds_\beta}\right)^q \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f(u,v)\, v^{s_\beta - 1} du dv \\ &= (1-q)!\left(\frac{d}{ds_\alpha}\right)^q \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right] + (1-q)!\left(\frac{d}{ds_\beta}\right)^q \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right].\end{aligned}$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[(\log uv)^q f(u,v);s_\alpha,s_\beta\right]=(1-q)!\left(\frac{d}{ds_\alpha}\right)^q\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right]$$
$$+(1-q)!\left(\frac{d}{ds_\beta}\right)^q\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right].$$

□

Property 4.5. *(2D-QFrMT of derivatives)*

1. *Let* $f\in L^2(\mathbb{R}^2,\mathbb{H})$*, then*

$$\mathcal{M}_{\alpha,\beta}\left[\frac{\partial f(u,\ v)}{\partial u};s_\alpha,s_\beta\right]=(1-s_\alpha)\,\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha-1,s_\beta\right]\tag{4.5}$$

exists only if $\lim_{u\to\infty}u^{s_\alpha-1}f(u,v)$ *vanishes.*

Also

$$\mathcal{M}_{\alpha,\beta}\left[\frac{\partial f(u,\ v)}{\partial v};s_\alpha,s_\beta\right]=\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta-1\right](1-s_\beta)\tag{4.6}$$

exists only if $\lim_{v\to\infty}f(u,v)\,v^{s_\beta-1}$ *vanishes.*

2. *Let* $f\in L^2(\mathbb{R}^2,\mathbb{H})$*, then*

$$\mathcal{M}_{\alpha,\beta}\left[\frac{\partial^2 f(u,v)}{\partial u\partial v};s_\alpha,s_\beta\right]=(1-s_\alpha)\,\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha-1,s_\beta-1\right](1-s_\beta).\tag{4.7}$$

3. *For* $f\in L^2(\mathbb{R}^2,\mathbb{H})$*, then*

$$\mathcal{M}_{\alpha,\beta}\left[u\frac{\partial f(u,\ v)}{\partial u};s_\alpha,s_\beta\right]=-s_\alpha\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right].\tag{4.8}$$

Similarly we can write as

$$\mathcal{M}_{\alpha,\beta}\left[v\frac{\partial f(u,\ v)}{\partial v};s_\alpha,s_\beta\right]=-\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right]s_\beta.\tag{4.9}$$

4. *In general, we may write*

$$(i.)\ \mathcal{M}_{\alpha,\beta}\left[\frac{\partial^n f(u,\ v)}{\partial u^n};s_\alpha,s_\beta\right]=(-1)^n\frac{\Gamma(s_\alpha)}{\Gamma(s_\alpha-n)}\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha-n,s_\beta\right].\tag{4.10}$$

$$(ii.)\ \mathcal{M}_{\alpha,\beta}\left[u^n\frac{\partial^n f(u,\ v)}{\partial u^n};s_\alpha,s_\beta\right]=(-1)^n\frac{\Gamma(s_\alpha+n)}{\Gamma(s_\alpha)}\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right].\tag{4.11}$$

And also

$$(iii.)\ \mathcal{M}_{\alpha,\beta}\left[\frac{\partial^m f(u,\ v)}{\partial v^m};s_\alpha,s_\beta\right]=(-1)^m\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta-m\right]\frac{\Gamma(s_\beta)}{\Gamma(s_\beta-m)}.\tag{4.12}$$

$$\text{(4.13)}\quad (iv.)\ \mathcal{M}_{\alpha,\beta}\left[v^m \frac{\partial^m f(u,\ v)}{\partial v^m}; s_\alpha, s_\beta\right] = (-1)^m \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right] \frac{\Gamma(s_\beta+m)}{\Gamma(s_\beta)}.$$

And finally obtain

$$\text{(4.14)}\quad \begin{aligned} &(v.)\ \mathcal{M}_{\alpha,\beta}\left[\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m} f(u,v); s_\alpha, s_\beta\right] \\ &= (1-s_\alpha)(2-s_\alpha)\ldots..(n-s_\alpha)\mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha - n, s_\beta - m\right] \\ &\times (1-s_\beta)(2-s_\beta)\ldots..(m-s_\beta). \end{aligned}$$

Proof. 1. Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then the transform of f partially differentiated w.r.t. u can be obtained as

$$\mathcal{M}_{\alpha,\beta}\left[\frac{\partial f(u,v)}{\partial u}; s_\alpha, s_\beta\right] = \int_0^\infty\int_0^\infty u^{s_\alpha-1}\frac{\partial f(u,v)}{\partial u} v^{s_\beta-1} du dv.$$

This integral exists only if $u^{s_\alpha-1} f(u,v)$ vanishes at $u=0$ and as $u \to \infty$

$$\begin{aligned} &= -(s_\alpha - 1)\int_0^\infty\int_0^\infty u^{s_\alpha-2} f(u,v)\, v^{s_\beta-1} du dv \\ &= (1-s_\alpha)\mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha - 1, s_\beta\right]. \end{aligned}$$

Thus

$$\text{(4.15)}\quad \mathcal{M}_{\alpha,\beta}\left[\frac{\partial f(u,v)}{\partial u}; s_\alpha, s_\beta\right] = (1-s_\alpha)\mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha - 1, s_\beta\right].$$

Similarly if partially differentiated w.r.t. v , we get

$$\text{(4.16)}\quad \mathcal{M}_{\alpha,\beta}\left[\frac{\partial f(u,v)}{\partial v}; s_\alpha, s_\beta\right] = \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta - 1\right](1-s_\beta).$$

2. Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then the transform of f partially derivative w.r.t. v and u respectively can be obtained as

$$\begin{aligned} M_{\alpha,\beta}\left[\frac{\partial^2 f(u,v)}{\partial u \partial v}; s_\alpha, s_\beta\right] &= M_{\alpha,\beta}\left[\frac{\partial}{\partial u}\left(\frac{\partial f(u,v)}{\partial v}\right); s_\alpha, s_\beta\right] \\ &= (1-s_\alpha) M_{\alpha,\beta}\left[\frac{\partial f(u,v)}{\partial v}; s_\alpha - 1, s_\beta\right] \\ &= (1-s_\alpha) M_{\alpha,\beta}\left[f(u,v); s_\alpha - 1, s_\beta - 1\right](1-s_\beta). \end{aligned}$$

Thus

$$M_{\alpha,\beta}\left[\frac{\partial^2 f(u,v)}{\partial u\partial v};s_\alpha,s_\beta\right]=(1-s_\alpha)M_{\alpha,\beta}\left[f(u,v);s_\alpha-1,s_\beta-1\right](1-s_\beta). \tag{4.17}$$

3. Let $f\in L^2(\mathbb{R}^2,\mathbb{H})$, then the transform of f partially differentiated w.r.t u and multiplied with u can be obtained as

$$\mathcal{M}_{\alpha,\beta}\left[u\frac{\partial f(u,v)}{\partial u};s_\alpha,s_\beta\right]=\int_0^\infty\int_0^\infty u^{s_\alpha}\frac{\partial f(u,v)}{\partial u}v^{s_\beta-1}dudv.$$

This integral exists only if $u^{s_\alpha}f(u,v)$ vanishes at $u=0$ and as $u\to\infty$

$$\begin{aligned}&=-s_\alpha\int_0^\infty\int_0^\infty u^{s_\alpha-1}f(u,v)v^{s_\beta-1}dudv\\&=-s_\alpha\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right].\end{aligned} \tag{4.18}$$

Similarly if partially differentiated w.r.t. v and multiplied with v, we get

$$\mathcal{M}_{\alpha,\beta}\left[v\frac{\partial f(u,v)}{\partial v};s_\alpha,s_\beta\right]=-\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha,s_\beta\right]s_\beta. \tag{4.19}$$

4.(i.) For $n=1$, the result is true from (4.5)

For $n=2$, we get

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[\frac{\partial^2 f(u,v)}{\partial u^2};s_\alpha,s_\beta\right]&=\int_0^\infty\int_0^\infty u^{s_\alpha-1}\frac{\partial^2 f(u,v)}{\partial u^2}v^{s_\beta-1}dudv\\&=-(s_\alpha-1)\int_0^\infty\int_0^\infty u^{s_\alpha-2}\frac{\partial f(u,v)}{\partial u}v^{s_\beta-1}dudv\\&=(s_\alpha-1)(s_\alpha-2)\int_0^\infty\int_0^\infty u^{s_\alpha-3}f(u,v)v^{s_\beta-1}dudv\\&=(s_\alpha-1)(s_\alpha-2)\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha-2,s_\beta\right].\end{aligned}$$

Assume the result is true for $n=k-1$

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[\frac{\partial^{((k-1))} f(u,v)}{\partial u^{(k-1)}};s_\alpha,s_\beta\right]=&(-1)^{(k-1)}(s_\alpha-1)(s_\alpha-2)\cdots(s_\alpha-(k-1))\\&\times\mathcal{M}_{\alpha,\beta}\left[f(u,v);s_\alpha-(k-1),s_\beta\right].\end{aligned}$$

By method of induction for $n = k$, we get

$$\mathcal{M}_{\alpha,\beta}\left[\frac{\partial^k f(u,v)}{\partial u^k}; s_\alpha, s_\beta\right] = (-1)^k (s_\alpha - 1)(s_\alpha - 2)\cdots(s_\alpha - k)$$
$$\times \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha - k, s_\beta\right]$$
$$= (-1)^k \frac{\Gamma(s_\alpha)}{\Gamma(s_\alpha - n)} \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha - k, s_\beta\right].$$

Thus the result is true $\forall\ n$.

Hence proved.

$(ii.)$ For $n = 1$, the result is true from (4.8).

For $n = 2$, we get

$$\mathcal{M}_{\alpha,\beta}\left[u^2 \frac{\partial^2 f(u,v)}{\partial u^2}; s_\alpha, s_\beta\right] = \int_0^\infty \int_0^\infty u^{s_\alpha+1} \frac{\partial^2 f(u,v)}{\partial u^2} v^{s_\beta - 1} du dv$$
$$= -(s_\alpha + 1) \int_0^\infty \int_0^\infty u^{s_\alpha} \frac{\partial f(u,v)}{\partial u} v^{s_\beta - 1} du dv$$
$$= (s_\alpha + 1)(s_\alpha) \int_0^\infty \int_0^\infty u^{s_\alpha - 1} f(u,v) v^{s_\beta - 1} du dv$$
$$= (s_\alpha + 1)(s_\alpha) \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right].$$

Assume the result is true for $n = k - 1$

$$\mathcal{M}_{\alpha,\beta}\left[u^{(k-1)} \frac{\partial^{((k-1))} f(u,v)}{\partial u^{(k-1)}}; s_\alpha, s_\beta\right] = (-1)^{(k-1)} (s_\alpha + (k-2))(s_\alpha + (k-1))\cdots(s_\alpha)$$
$$\times \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right].$$

By method of induction for $n = k$, we get

$$\mathcal{M}_{\alpha,\beta}\left[u^k \frac{\partial^k f(u,v)}{\partial u^k}; s_\alpha, s_\beta\right] = (-1)^k (s_\alpha + (k-1))(s_\alpha + k)\cdots(s_\alpha)$$
$$\times \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right]$$
$$= (-1)^k \frac{\Gamma(s_\alpha + n)}{\Gamma(s_\alpha)} \mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right].$$

Thus the result is true $\forall\ n$.

Hence proved.

Similarly, the results (*iii.*) and (*iv.*) can be proved.

(*v.*) For $n = 1$ and $m = 1$, we have

$$\begin{aligned}
M_{\alpha,\beta}\left[\frac{\partial}{\partial u}\frac{\partial}{\partial v}f(u,v);s_\alpha,s_\beta\right] &= \int_0^\infty\int_0^\infty u^{s_\alpha-1}\frac{\partial}{\partial u}\frac{\partial}{\partial v}f(u,v)\,v^{s_\beta-1}dudv\\
&= -(s_\alpha-1)\int_0^\infty\int_0^\infty u^{s_\alpha-2}\frac{\partial}{\partial v}f(u,v)\,v^{s_\beta-1}dudv\\
&= (1-s_\alpha)\int_0^\infty\int_0^\infty u^{s_\alpha-2}\frac{\partial}{\partial v}f(u,v)\,v^{s_\beta-2}dudv\,(1-s_\beta)\\
&= (1-s_\alpha)\,M_{\alpha,\beta}\left[f(u,v);s_\alpha-1,s_\beta-1\right](1-s_\beta)
\end{aligned}$$

For $n = 2$ and $m = 2$, we get

$$\begin{aligned}
&M_{\alpha,\beta}\left[\frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial v^2}f(u,v);s_\alpha,s_\beta\right]\\
&= \int_0^\infty\int_0^\infty u^{s_\alpha-1}\frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial v^2}f(u,v)\,v^{s_\beta-1}dudv\\
&= -(s_\alpha-1)\int_0^\infty\int_0^\infty u^{s_\alpha-2}\frac{\partial}{\partial u}\frac{\partial^2}{\partial v^2}f(u,v)\,v^{s_\beta-1}dudv\\
&= (1-s_\alpha)(2-s_\alpha)\int_0^\infty\int_0^\infty u^{s_\alpha-3}\frac{\partial^2}{\partial v^2}f(u,v)\,v^{s_\beta-1}dudv\\
&= (1-s_\alpha)(2-s_\alpha)\int_0^\infty\int_0^\infty u^{s_\alpha-3}\frac{\partial}{\partial v}f(u,v)\,v^{s_\beta-2}dudv\,(1-s_\beta)\\
&= (1-s_\alpha)(2-s_\alpha)\int_0^\infty\int_0^\infty u^{s_\alpha-3}f(u,v)\,v^{s_\beta-3}dudv\,(1-s_\beta)(2-s_\beta)\\
&= (1-s_\alpha)(2-s_\alpha)\,M_{\alpha,\beta}\left[f(u,v);s_\alpha-2,s_\beta-2\right](1-s_\beta)(2-s_\beta)
\end{aligned}$$

Assume the result is true for $n - 1$ and $m - 1$.

By method of induction, we get

$$\mathcal{M}_{\alpha,\beta}\left[\frac{\partial^n}{\partial u^n}\frac{\partial^m}{\partial v^m}f\left(u,v\right);s_\alpha,s_\beta\right]$$
$$=\left(1-s_\alpha\right)\left(2-s_\alpha\right).....\left(n-s_\alpha\right)\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right);s_\alpha-n,s_\beta-m\right]$$
$$\times\left(1-s_\beta\right)\left(2-s_\beta\right).....\left(m-s_\beta\right).$$

Thus the result is true $\forall\ n, m$.

Hence proved. ☐

Property 4.6. *(2D-QFrMT of integrals)*
Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then

$$\mathcal{M}_{\alpha,\beta}\left[\int_0^u\int_0^v f\left(p,q\right)dpdq;s_\alpha,s_\beta\right]=\frac{1}{s_\alpha s_\beta}\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right);s_\alpha+1,s_\beta+1\right]. \tag{4.20}$$

Proof. Analogous to the fundamental theorem of calculus as stated in [8]; considering

$$F_1\left(u,v\right)=\int_0^u f\left(p,v\right)dp \tag{4.21}$$

such that ${F_1}'\left(u,v\right)=f\left(u,v\right)$ with $F_1\left(0,v\right)=0$, can be written as

$$\mathcal{M}_{\alpha,\beta}\left[F'\left(u,v\right)=f\left(u,v\right);s_\alpha,s_\beta\right]=-\left(s_\alpha-1\right)\mathcal{M}_{\alpha,\beta}\left[\int_0^u f\left(p,v\right)dp;s_\alpha-1,s_\beta\right].$$

Replace s_α by $s_\alpha+1$;

$$\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right);s_\alpha+1,s_\beta\right]=-\left(s_\alpha\right)\mathcal{M}_{\alpha,\beta}\left[\int_0^u f\left(p,v\right)dp;s_\alpha,s_\beta\right]$$

$$\mathcal{M}_{\alpha,\beta}\left[\int_0^u f\left(p,v\right)dp;s_\alpha,s_\beta\right]=-\frac{1}{s_\alpha}\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right);s_\alpha+1,s_\beta\right].$$

Analogous to (4.21), consider

$$F_1\left(p,v\right)=\int_0^v f\left(p,q\right)dq \tag{4.22}$$

such that ${F_1}'\left(p,v\right)=f\left(p,v\right)$ with $F_1\left(p,0\right)=0$, follows

$$\mathcal{M}_{\alpha,\beta}\left[{F_1}'\left(p,v\right)=f\left(p,v\right);s_\alpha,s_\beta\right]=-\left(s_\beta-1\right)\mathcal{M}_{\alpha,\beta}\left[\int_0^v f\left(p,q\right)dq;s_\alpha,s_\beta-1\right].$$

Replace s_β by $s_\beta+1$

$$\mathcal{M}_{\alpha,\beta}\left[f\left(p,v\right);s_\alpha,s_\beta+1\right]=-\left(s_\beta\right)\mathcal{M}_{\alpha,\beta}\left[\int\limits_0^v f\left(p,q\right)dq;s_\alpha,s_\beta\right]$$

$$\mathcal{M}_{\alpha,\beta}\left[\int\limits_0^v f\left(p,q\right)dq;s_\alpha,s_\beta\right]=-\frac{1}{s_\beta}\mathcal{M}_{\alpha,\beta}\left[f\left(p,v\right);s_\alpha,s_\beta+1\right].$$

Now,

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}\left[\int\limits_0^u\int\limits_0^v f\left(p,q\right)dpdq;s_\alpha,s_\beta\right]&=\mathcal{M}_{\alpha,\beta}\left[\int\limits_0^u F_1\left(p,v\right)dp;s_\alpha,s_\beta\right]\\&=\frac{-1}{s_\alpha}\mathcal{M}_{\alpha,\beta}\left[F_1\left(u,v\right);s_\alpha+1,s_\beta\right]\\&=\frac{-1}{s_\alpha}\mathcal{M}_{\alpha,\beta}\left[\int\limits_0^v f\left(u,q\right)dq;s_\alpha+1,s_\beta\right]\\&=\frac{1}{s_\alpha s_\beta}\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right);s_\alpha+1,s_\beta+1\right].\end{aligned}$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[\int\limits_0^u\int\limits_0^v f\left(p,q\right)dpdq;s_\alpha,s_\beta\right]=\frac{1}{s_\alpha s_\beta}\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right);s_\alpha+1,s_\beta+1\right].\tag{4.23}$$

□

5. Convolution type properties

Let $f,\ g$ be the quaternion-valued functions, then convolution operations associated with 2D-QFrMT can be defined using [5, 12] as

$$\left(f*g\right)\left(u,\ v\right)=\int\limits_0^\infty\int\limits_0^\infty f\left(\frac{u}{r},\ \frac{v}{t}\right)g\left(r,\ t\right)\frac{dr}{r}\frac{dt}{t}.\tag{5.1}$$

$$\left(f\circ g\right)\left(u,\ v\right)=\int\limits_0^\infty\int\limits_0^\infty f\left(ru,\ tu\right)g\left(r,\ t\right)drdt.\tag{5.2}$$

Property 5.1.

a) $\mathcal{M}_{\alpha,\beta}\left[\left(f*g\right)\right]=\mathcal{M}_{\alpha,\beta}\left[f;s_\alpha,s_\beta\right]\mathcal{M}_{\alpha,\beta}\left[g;s_\alpha,s_\beta\right].$

b) $\mathcal{M}_{\alpha,\beta}\left[\left(f\circ g\right)\right]=\mathcal{M}_{\alpha,\beta}\left[g;1-s_\alpha,1-s_\beta\right]\mathcal{M}_{\alpha,\beta}\left[f;s_\alpha,s_\beta\right].$

Proof.

$$a)\ \mathcal{M}_{\alpha,\beta}\left[(f*g)(u,v)\right]=\mathcal{M}_{\alpha,\beta}\left[\int_0^\infty\int_0^\infty f\left(\frac{u}{r},\frac{v}{t}\right)g(r,t)\frac{dr}{r}\frac{dt}{t}\right]$$

$$=\int_0^\infty\int_0^\infty u^{s_\alpha-1}v^{s_\beta-1}dudv\int_0^\infty\int_0^\infty f\left(\frac{u}{r},\frac{v}{t}\right)g(r,t)\frac{dr}{r}\frac{dt}{t}$$

$$=\int_0^\infty\int_0^\infty u^{s_\alpha-1}f\left(\frac{u}{r},\frac{v}{t}\right)v^{s_\beta-1}dudv\int_0^\infty\int_0^\infty g(r,t)\frac{dr}{r}\frac{dt}{t}.$$

Put $\frac{u}{r}=\xi$ and $\frac{v}{t}=\eta$

$$=\int_0^\infty\int_0^\infty (r\xi)^{s_\alpha-1}f(\xi,\eta)(t\eta)^{s_\beta-1}rd\xi\ td\eta\int_0^\infty\int_0^\infty g(r,t)\frac{dr}{r}\frac{dt}{t}$$

$$=\int_0^\infty\int_0^\infty \xi^{s_\alpha-1}f(\xi,\eta)\,\eta^{s_\beta-1}d\xi d\eta\int_0^\infty\int_0^\infty r^{s_\alpha-1}f(r,t)\,t^{s_\beta-1}drdt$$

$$=\mathcal{M}_{\alpha,\beta}\left[f(\xi,\eta);s_\alpha,s_\beta\right]\mathcal{M}_{\alpha,\beta}\left[g(r,t);s_\alpha,s_\beta\right].$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[(f*g)\right]=\mathcal{M}_{\alpha,\beta}\left[f;s_\alpha,s_\beta\right]\mathcal{M}_{\alpha,\beta}\left[g;s_\alpha,s_\beta\right]. \tag{5.3}$$

$$b)\ \mathcal{M}_{\alpha,\beta}\left[(f\circ g)(u,v)\right]=\mathcal{M}_{\alpha,\beta}\left[\int_0^\infty\int_0^\infty f(ru,tv)g(r,t)\,drdt\right]$$

$$=\int_0^\infty\int_0^\infty u^{s_\alpha-1}v^{s_\beta-1}dudv\int_0^\infty\int_0^\infty f(ru,tv)\,g(r,t)\,drdt.$$

Put $ru=\xi$ and $tv=\eta$

$$=\int_0^\infty\int_0^\infty g(r,t)\,drdt\int_0^\infty\int_0^\infty f(\xi,\eta)\,\xi^{s_\alpha-1}\eta^{s_\beta-1}r^{1-s_\alpha}t^{1-s_\beta}\frac{d\xi}{r}\frac{d\eta}{t}$$

$$=\int_0^\infty\int_0^\infty r^{1-s_\alpha-1}g(r,t)\,t^{1-s_\beta-1}drdt\int_0^\infty\int_0^\infty \xi^{s_\alpha-1}f(\xi,\eta)\,\eta^{s_\beta-1}d\xi d\eta$$

$$=\mathcal{M}_{\alpha,\beta}\left[g(r,t);1-s_\alpha,1-s_\beta\right]\mathcal{M}_{\alpha,\beta}\left[f(\xi,\eta);s_\alpha,s_\beta\right].$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[(f\circ g)\right]=\mathcal{M}_{\alpha,\beta}\left[g;1-s_\alpha,1-s_\beta\right]\mathcal{M}_{\alpha,\beta}\left[f;s_\alpha,s_\beta\right]. \tag{5.4}$$

□

6. Parseval theorem

Theorem 6.1. *Let* $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ *and*

$$\mathcal{M}_{\alpha,\beta}\left[f(u,v); s_\alpha, s_\beta\right] = \tilde{f}\left(s_\alpha, s_\beta\right); \mathcal{M}_{\alpha,\beta}\left[g(u,v); s_\alpha, s_\beta\right] = \tilde{g}\left(s_\alpha, s_\beta\right)$$

then

$$\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right)g\left(u,v\right)\right] = \frac{1}{(2\pi)^2 \boldsymbol{ij}} \int\limits_{b-\boldsymbol{j}\infty}^{b+\boldsymbol{j}\infty} \int\limits_{a-\boldsymbol{i}\infty}^{a+\boldsymbol{i}\infty} \tilde{f}\left(s_\alpha, s_\beta\right)\tilde{g}\left(t_\alpha - s_\alpha, t_\beta - s_\beta\right) ds_\alpha ds_\beta. \tag{6.1}$$

Proof.

$$\begin{aligned}
\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right)g\left(u,v\right)\right] &= \int\limits_0^\infty \int\limits_0^\infty u^{t_\alpha - 1} f\left(u,v\right) g\left(u,v\right) v^{t_\beta - 1} du dv \\
&= \int\limits_0^\infty \int\limits_0^\infty u^{t_\alpha - 1} g\left(u,v\right) v^{t_\beta - 1} du dv \\
&\quad \times \frac{1}{(2\pi)^2 \mathbf{ij}} \int\limits_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty} \int\limits_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{-s_\alpha} \tilde{f}\left(s_\alpha, s_\beta\right) v^{-s_\beta} ds_\alpha ds_\beta \\
&= \frac{1}{(2\pi)^2 \mathbf{ij}} \int\limits_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty} \int\limits_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} \tilde{f}\left(s_\alpha, s_\beta\right)\tilde{g}\left(t_\alpha - s_\alpha, t_\beta - s_\beta\right) ds_\alpha ds_\beta.
\end{aligned}$$

Thus

$$\mathcal{M}_{\alpha,\beta}\left[f\left(u,v\right)g\left(u,v\right)\right] = \frac{1}{(2\pi)^2 \mathbf{ij}} \int\limits_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty} \int\limits_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} \tilde{f}\left(s_\alpha, s_\beta\right)\tilde{g}\left(t_\alpha - s_\alpha, t_\beta - s_\beta\right) ds_\alpha ds_\beta.$$

□

7. Applications

a.) Here 2D-QFrMT is applied to the partial differential equation as in [14],

$$u^2 v^2 \frac{\partial^2 f\left(u,v\right)}{\partial u \partial v} + uvf\left(u,v\right) = \delta\left(u+h, v+k\right) \tag{7.1}$$

where $\delta(u+h, v+k)$ is Dirac-delta function.

Solution. By applying 2D-QFrMT on both sides of (7.1) as given in [3], we obtain

$$(s_\alpha+1)(s_\beta+1)\mathcal{M}_{\alpha,\beta}[f(u,v); s_\alpha+1, s_\beta+1] + \mathcal{M}_{\alpha,\beta}[f(u,v); s_\alpha+1, s_\beta+1] = h^{s_\alpha-1}k^{s_\beta-1}. \tag{7.2}$$

By solving (7.2), we get

$$\mathcal{M}_{\alpha,\ \beta}[f(u,v); s_\alpha+1, s_\beta+1] = \frac{h^{s_\alpha-1}k^{s_\beta-1}}{1+(s_\alpha+1)(s_\beta+1)}. \tag{7.3}$$

By using the inversion formula of 2D-QfrMT as given in (3.3), the solution is obtained as

$$f(u,v) = \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty} \int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{-(s_\alpha+1)} \frac{h^{s_\alpha-1}k^{s_\beta-1}}{1+(s_\alpha+1)(s_\beta+1)} v^{-(s_\beta+1)} ds_\alpha ds_\beta. \tag{7.4}$$

b.) **Quaternion wave equation:**

For quaternion-valued function, the wave equation can be given as follows:

$$\zeta_{tt} - A^2\triangle^2\zeta = 0 \tag{7.5}$$

and

$$\zeta = f(u,\ v),\ \zeta_t = 0;\ \text{when } t = 0, \tag{7.6}$$

where A is a constant, $f \in L^2(\mathbb{R}^2,\ \mathbb{H})$ and $\triangle^2 = u^2\dfrac{\partial^2}{\partial^2 u} + v^2\dfrac{\partial^2}{\partial^2 v}$.

Applying 2D-QFrMT on both sides of (7.5), we get

$$\mathcal{M}_{\alpha,\ \beta}[\zeta_{tt}] - A^2\left\{(s_\alpha+1)(s_\alpha)\mathcal{M}_{\alpha,\ \beta}[\zeta] + \mathcal{M}_{\alpha,\ \beta}[\zeta](s_\beta+1)(s_\beta)\right\} = 0$$

$$\mathcal{M}_{\alpha,\ \beta}[\zeta_{tt}] - A^2\left\{s_\alpha^2 + s_\alpha + s_\beta^2 + s_\beta\right\}\mathcal{M}_{\alpha,\ \beta}[\zeta] = 0$$

$$\mathcal{M}_{\alpha,\ \beta}[\zeta_{tt}] - A^2S^2\mathcal{M}_{\alpha,\ \beta}[\zeta] = 0. \tag{7.7}$$

By interchanging differentiation with 2D-QFrMT, we get general solution of (7.7) as

$$\mathcal{M}_{\alpha,\ \beta}[\zeta] = c_1e^{-ASt} + c_2e^{ASt}. \tag{7.8}$$

By imposing the initial conditions, we get

$$\mathcal{M}_{\alpha,\ \beta}[\zeta] = \left(1 - \frac{1}{2AS}\right)e^{-ASt}\mathcal{M}_{\alpha,\ \beta}[f(u,\ v)] + \frac{1}{2AS}e^{ASt}\mathcal{M}_{\alpha,\ \beta}[f(u,\ v)]. \tag{7.9}$$

By using the inversion formula of 2D-QFrMT on (7.9), the solution of (7.5) is obtained as

$$\zeta = \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty} \int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{-s_\alpha} \left(e^{-ASt} + \frac{1}{AS}\sinh(AS)t \right) \mathcal{M}_{\alpha,\ \beta}\left[f(u,\ v)\right] v^{-s_\beta} ds_\alpha ds_\beta.$$

c.) Consider the partial differential equation with non constant coefficient as in [7],

$$u^2 f_{uu}(u,v) - 2uv f_{uv}(u,v) + v^2 f_{vv}(u,v) = e^{-(u+v)}. \tag{7.10}$$

Applying 2D-QFrMT on both sides of (7.10), we get

$$(1+s_\alpha)(s_\alpha)\mathcal{M}_{\alpha,\ \beta}[f] - 2s_\alpha s_\beta \mathcal{M}_{\alpha,\ \beta}[f] + \mathcal{M}_{\alpha,\ \beta}[f](1+s_\beta)(s_\beta) = \Gamma(s_\alpha)\Gamma(s_\beta). \tag{7.11}$$

After simplifying we obtain

$$\mathcal{M}_{\alpha,\ \beta}[f] = \frac{\Gamma(s_\alpha)\Gamma(s_\beta)}{(s_\alpha - s_\beta)^2 + (s_\alpha + s_\beta)}. \tag{7.12}$$

By using the inversion formula of 2D-QfrMT as given in (3.3), the solution is obtained as

$$f = \frac{1}{(2\pi)^2\mathbf{ij}} \int_{b-\mathbf{j}\infty}^{b+\mathbf{j}\infty} \int_{a-\mathbf{i}\infty}^{a+\mathbf{i}\infty} u^{-s_\alpha} \frac{\Gamma(s_\alpha)\Gamma(s_\beta)}{(s_\alpha - s_\beta)^2 + (s_\alpha + s_\beta)} v^{-s_\beta} ds_\alpha ds_\beta. \tag{7.13}$$

8. Conclusion

In this paper, two-dimensional quaternionic fractional Mellin transform for fractional order is proposed and its inversion formula is also obtained. Different characteristics and mathematical properties of two-dimensional quaternionic fractional Mellin transform are derived. Convolution and Parseval's theorem are proved. Some pplications of two-dimensional quaternionic fractional Mellin transform are also demonstrated.

Acknowledgement

The authors would like to appreciate the efforts of editors and reviewers. This study received no specific grant from the government, commercial, or non-profit funding organizations.

References

[1] O. Akay, and G.F. Boudreaux-Bartels, Fractional Mellin transform: an extension of fractional frequency concept for scale, *Proceedings 8th IEEE Digital Signal Processing Workshop* pages CD–ROM. IEEE, 1998.

[2] S. K. Q. Al-Omari and K. Adem, On Modified Mellin Transform of Generalized Functions, *Abstract and Applied Analysis* Article ID 539240, 2013.

[3] L.C. Andrews, *Special functions of mathematics for engineers*, New York: McGraw-Hill, 1992.

[4] E. Biner, and O. Akay, Digital computation of the fractional Mellin transform, *13th European Signal Processing Conference* (2005), 1–4.

[5] L. Debnath, D. Bhatta, *Integral Transforms and Their Applications*, third edition, CRC Press Book, 2015.

[6] T.A. Ell, Quaternion-Fourier Transforms for Analysis of Two-Dimensional Linear Time-Invariant Partial Differential Systems, *In Proc. of the 32nd Conf. on Decision and Control* IEEE (1993), 1830–1841.

[7] H. Eltayeb and A. Kiliçman, A note on Mellin transform and partial differential equations, *International Journal of Pure and Applied Mathematics* **34(4)** (2007), 457–468.

[8] R. R. Goldberg, *Methods of Real Analysis*, Oxford and IBH Publishing, 1970.

[9] E. Hitzer, Quaternion Fourier Transform on Quaternion Fields and Generalizations, *Advances in Applied Clifford Algebras* **17(3)** (2007), 497–517.

[10] E. Hitzer, Quaternionic Fourier-Mellin transform, *Proceedings of the 19^{th} ICFIDCAA Hiroshima 2011* Tohoku UP, 2012.

[11] B. Mawardi, E. Hitzer, A. Hayashi, and R. Ashino, An uncertainty principle for quaternion Fourier transform, *Computers & Mathematics with Applications* **56(9)** (2008), 2411–2417.

[12] A. M. Mathai, Mellin convolutions, statistical distributions and fractional calculus, *Fractional Calculus and Applied Analysis* **21(2)** (2018), 376–398.

[13] B. Nefzi, K. Brahim and A. Fitouhi, On the finite Mellin transform in quantum calculus and application, *Acta Mathematica Scientia* **38(4)** (2018), 1393–1410.

[14] M. Omran and A. Kilicman, On fractional order Mellin transform and some of its properties, *Tbilisi Mathematical Journal* **10(1)** (2017), 315–324.

[15] K. Parmar and V. R. Lakshmi Gorty, One-Dimensional Quaternion Mellin Transform and its applications, *Proceedings of the Jangjeon Mathematical Society* **24(1)** (2021), 99–112.

[16] K. Parmar and V. R. Lakshmi Gorty, Application and graphical interpretation of a new two-dimensional quaternion fractional Fourier transform, *International Journal of Analysis and Applications* **19(4)** (2021), 561–575.

[17] K. Parmar and V. R. Lakshmi Gorty, Quaternion Stieltjes Transform and Quaternion Laplace-Stieltjes Transform, *Communications in Mathematics and Applications* **12(3)** (2021), 633–643.

[18] K. Parmar and V. R. Lakshmi Gorty, Numerical Computation of Finite Quaternion Mellin Transform Using a New Algorithm, *In: Singh, M., Tyagi, V., Gupta, P.K., Flusser, J., Ören, T., Sonawane, V.R. (eds) Advances in Computing and Data Sciences. ICACDS 2021. Communications in Computer and Information Science* **1441** (2021), 172–182.

[19] V. D. Sharma and P. B. Deshmukh, Generalized two-dimensional fractional Mellin transform, *IEEE. Second International Conference on Emerging Trends in Engineering & Technology* (2009), 900–903.

[20] S. H. Zheng, Y. S. Chen and D. H. Li, Realization of two-dimensional Mellin transform, *Chinese physics* **11(1)** (1991), 58–62.

(1) Department of Basic Science and Humanities, NMIMS, MPSTME, Vile-Parle (West), Mumbai-400056, Maharashtra, India.

Email address: Khinal.Parmar@nmims.edu

(2) Department of Basic Science and Humanities, NMIMS, MPSTME, Vile-Parle (West), Mumbai-400056, Maharashtra, India.

Email address: VR.Lakshmigorty@nmims.edu