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CLOSED BALLS INCLUDED IN THE INVERTED MULTIBROT AND MULTICORN SETS

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Abstract. The aim of this article is to compute a radius of a closed ball included in the Inverted Multibrot and Multicorn sets. More exactly, for $w \in \mathbb{C}^*$ a complex solution of the equation $z^{k-1} = -1$ we compute $r > 0$ such that $\overline{B}(w, r)$ is included in the Inverted Multibrot set $\mathcal{N}_k$ of the the function $z^k + \frac{1}{c}$, $c \in \mathbb{C}^*$ for every $k \geq 2$ and for $w \in \mathbb{C}^*$ a complex solution of the equation $z^{k+1} = -1$ we compute $r > 0$ such that $\overline{B}(w, r)$ is included in the Inverted Multicorn set $\mathcal{N}_k^*$ of the function $\overline{z}^k + \frac{1}{c}$, $c \in \mathbb{C}^*$ for every $k \geq 2$.

1. Introduction

Dynamical systems generated by the iterations of the quadratic polynomial $z^2 + c$ were studied in [2] where it is proved that the well-known Mandelbrot set is connected. Mandelbrot set was naturally generalized, on one hand, to the Multibrot sets given by the iteration of the polynomial $z^d + c$, $d \geq 2$ and, on the other hand, to the Multicorn sets given by the iteration of the polynomials $\overline{z}^d + c$, $d \geq 2$. The intersections of the Multibrot set of $z^d + c$ with the rays $\mathbb{R}_+ w$ where $w^{d-1} = \pm 1$, $d \geq 2$, were given in [1], the exact intervals of the cross section of the Multibrot set of $z^d + c$, $d \geq 3$, d odd, were given in ([6], [7]) and the exact intervals of the cross section of the Multibrot set of $z^d + c$, $d \geq 2$, d even, were given in ([8]). About the Multicorn sets, we can say that the intersections of the Multicorn set of $\overline{z}^d + c$ with the rays $\mathbb{R}_+ w$ where $w^{d+1} = \pm 1$, $d \geq 2$ were given in [11]. The connectedness of the Tricorn (particular case of Multicorn) given by the functions $\overline{z}^2 + c$ was proven in [5].

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For $\omega \in \mathbb{C}^*$ a complex solution of the equation $z^{k-1} = -1$ we calculate in [3] a radius $r > 0$ such that $\overline{B}(\omega, r)$ is included in the Multibrot of $z^k + c$ for every $k \geq 2$. Moreover, for $\omega \in \mathbb{C}^*$ a complex solution of the equation $z^{k+1} = -1$ we also calculate in [3] a radius $r > 0$ such that $\overline{B}(\omega, r)$ is included in the Multicorn of $\overline{z}^k + c$ for every $k \geq 2$.

In this paper we continue the work from [3] for the Inverted Multibrot and Multicorn sets. More exactly, for $\omega \in \mathbb{C}^*$ a complex solution of the equation $z^{k-1} = -1$ we calculate a radius $r > 0$ such that $\overline{B}(\omega, r)$ is included in the Inverted Multibrot of $z^k + \frac{1}{c}$, $c \in \mathbb{C}$ for every $k \geq 2$ and for $\omega \in \mathbb{C}^*$ a complex solution of the equation $z^{k+1} = -1$ we calculate a radius $r > 0$ such that $\overline{B}(\omega, r)$ is included in the Inverted Multicorn of $\overline{z}^k + \frac{1}{c}$, $c \in \mathbb{C}$ for every $k \geq 2$.

We recall now that the Inverted Mandelbrot sets were also studied in [9] and we have the following well-known definitions.

Definition 1.1. Let $f_c(z) = z^k + \frac{1}{c}$ with $c \in \mathbb{C}^*$ and $k \in \mathbb{N}$, $k \geq 2$. The *Inverted Multibrot set* is defined by

$$\mathcal{N}_k = \{c \in \mathbb{C}^* \mid \{f_c^k(0)\}_{k\geq 1} \text{ is bounded } \} = \{c \in \mathbb{C}^* \mid \{f_c^k(0)\}_k \overset{k\to\infty}{\not\to} \infty\}$$

which is equivalent to

$$\mathcal{N}_k = \{c \in \mathbb{C}^* \mid \{R_n\}_{n\geq 1} \text{ is bounded}\} = \{c \in \mathbb{C}^* \mid \{R_n\}_{n\geq 1} \overset{n\to\infty}{\not\to} \infty\}$$

where the sequence of complex numbers $(R_n)_{n\geq 1}$ from above is satisfying the reccurence $R_{n+1} = R_n^k + \frac{1}{c}$ for every $n \geq 1$ with $R_1 = \frac{1}{c}$ (for $k = 2$ we obtain the Inverted Mandelbrot set $\mathcal{N}_2$).

Definition 1.2. Let $f_c(z) = \overline{z}^k + \frac{1}{c}$ with $c \in \mathbb{C}^*$ and $k \in \mathbb{N}$, $k \geq 2$. The *Inverted Multicorn set* is defined by

$$\mathcal{N}_k^* = \{c \in \mathbb{C}^* \mid \{f_c^k(0)\}_{k\geq 1} \text{ is bounded}\} = \{c \in \mathbb{C}^* \mid \{f_c^k(0)\}_k \overset{k\to\infty}{\not\to} \infty\}$$

which is equivalent to

$$\mathcal{N}_k^* = \{c \in \mathbb{C}^* \mid \{R_n\}_{n\geq 1} \text{ is bounded}\} = \{c \in \mathbb{C}^* \mid \{R_n\}_{n\geq 1} \overset{n\to\infty}{\not\to} \infty\}$$

where the sequence of complex numbers $(R_n)_{n\geq 1}$ from above is satisfying the reccurence $R_{n+1} = \overline{R}_n^k + \frac{1}{c}$ for every $n \geq 1$ with $R_1 = \frac{1}{c}$.

2. Preliminary lemmas

In this section we prove some preliminary lemmas.

Lemma 2.1. *Let $k \geq 4$ and the recurrence $R_{n+1} = R_n^k + \frac{1}{c}$ be satisfied for every $n \geq 1$ where $R_1 = \frac{1}{c}$ with $c \in \mathbb{C}^*$. We suppose that $|c - w| \leq \alpha$ with $\alpha \in (0, 1)$ where w is a solution of the equation $z^{k-1} = -1$. Then $|R_2| \leq \frac{(\alpha+1)^{k-1} - 1}{(1-\alpha)^k}$.*

Proof. From the hypothesis we obtain that $|c| \leq \alpha + 1$ and $\frac{1}{|c|} \leq \frac{1}{1-\alpha}$. Let also $\{w, w_1, ..., w_{k-2}\} \subset \mathbb{C}$ be the set of all the solutions of the equation $z^{k-1} = -1$. We have that $|w| = |w_i| = 1$ for every $i \in \{1, ..., k-2\}$ and from Viète's relations ([10]) we obtain:

$$\begin{cases} w_1 + w_2 + ...+ w_{k-2} = -w \\ w_1w_2 + w_1w_3 + ...+ w_{k-3}w_{k-2} = w^2 \\ w_1w_2w_3 + w_1w_2w_4 + ...+ w_{k-4}w_{k-3}w_{k-2} = -w^3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ w_1w_2...w_{k-3} + ...+ w_2w_3...w_{k-2} = (-1)^{k-3}w^{k-3} \\ w_1w_2...w_{k-2} = \frac{(-1)^{k-2}}{w} \end{cases}$$

Then we have

$$|R_2| = \left|R_1^k + \tfrac{1}{c}\right| = \left|\tfrac{1}{c^k} + \tfrac{1}{c}\right| = \frac{\left|c^{k-1}+1\right|}{|c|^k} =$$
$$\frac{1}{|c|^k}|(c-w)(c-w_1)...(c-w_{k-2})| = \frac{|c-w|}{|c|^k}|(c-w_1)(c-w_2)...(c-w_{k-2})| =$$
$$\frac{|c-w|}{|c|^k}\left|c^{k-2} + (-1)^1wc^{k-3} + (-1)^2w^2c^{k-4} + ... + (-1)^{k-3}w^{k-3}c + \frac{(-1)^{k-2}}{w}\right| \leq$$
$$\frac{|c-w|}{|c|^k}\left(|c|^{k-2} + |c|^{k-1} + ... + |c| + 1\right) \leq$$
$$\frac{\alpha}{(1-\alpha)^k}\left[(\alpha+1)^{k-2} + (\alpha+1)^{k-1} + ... + (\alpha+1) + 1\right] =$$
$$\left[\frac{\alpha}{(1-\alpha)^k}\right]\left[\frac{(\alpha+1)^{k-1}-1}{(\alpha+1)-1}\right] = \frac{(\alpha+1)^{k-1}-1}{(1-\alpha)^k}$$

□

Lemma 2.2. *Let $k \geq 4$ and the recurrence $R_{n+1} = R_n^k + \frac{1}{c}$ be satisfied for every $n \geq 1$ where $R_1 = \frac{1}{c}$ with $c \in \mathbb{C}^*$. We suppose that $|c - w| \leq \alpha$ with $\alpha \in (0, 1)$ where w is a solution of the equation $z^{k-1} = -1$. In the hypothesis of Lemma 2.1 we suppose that there exists $R > 1$ such that $|R_2| \leq \frac{(\alpha+1)^{k-1} - 1}{(1-\alpha)^k} < \frac{1}{R}$. Then:*

a). $|R_4| \leq \left(\frac{1}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^{k-1} - 2}{(1-\alpha)^k}$.

b). If $|R_4| \le \left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)^k+\frac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}<\frac{1}{R}$ *then* $|R_{2n}| \le \left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)^k+\frac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}<\frac{1}{R}$ *for every* $n \ge 2$.

Proof. a). From the hypothesis we obtain that $|c| \le \alpha+1$ and $\frac{1}{|c|} \le \frac{1}{1-\alpha}$. We denote by $C_n^k = \frac{n!}{k!(n-k)!}$ for every $k = 0, 1, ..., n$ and we have

$$|R_4| = \left|R_3^k + \tfrac{1}{c}\right| = \left|\left(R_2^k+\tfrac{1}{c}\right)^k+\tfrac{1}{c}\right| =$$
$$\left|\left(R_2^k\right)^k + C_k^1\left(R_2^k\right)^{k-1}\left(\tfrac{1}{c}\right)+C_k^2\left(R_2^k\right)^{k-2}\left(\tfrac{1}{c}\right)^2+...+C_k^{k-1}R_2^k\left(\tfrac{1}{c}\right)^{k-1}+\tfrac{1}{c^k}+\tfrac{1}{c}\right| \le$$
$$\left|R_2^k\right|^k + C_k^1\left|R_2^k\right|^{k-1}\tfrac{1}{|c|}+C_k^2\left|R_2^k\right|^{k-2}\tfrac{1}{|c|^2}+...+C_k^{k-1}\left|R_2^k\right|\tfrac{1}{|c|^{k-1}}+\left|\tfrac{1}{c^k}+\tfrac{1}{c}\right| =$$
$$\left(\left|R_2^k\right|+\tfrac{1}{|c|}\right)^k - \tfrac{1}{|c|^k}+\left|\tfrac{1}{c^k}+\tfrac{1}{c}\right| =$$
$$\left(\left|R_2^k\right|\right)\left[\left(\left|R_2^k\right|+\tfrac{1}{|c|}\right)^{k-1}+...+\left(\left|R_2^k\right|+\tfrac{1}{|c|}\right)\tfrac{1}{|c|^{k-2}}+\tfrac{1}{|c|^{k-1}}\right]+\left|\tfrac{1}{c^k}+\tfrac{1}{c}\right| <$$
$$\left(\tfrac{1}{R^k}\right)\left[\left(\tfrac{1}{R^k}+\tfrac{1}{1-\alpha}\right)^{k-1}+...+\left(\tfrac{1}{R^k}+\tfrac{1}{1-\alpha}\right)\tfrac{1}{(1-\alpha)^{k-2}}+\tfrac{1}{(1-\alpha)^{k-1}}\right]+\tfrac{(\alpha+1)^{k-1}-1}{(1-\alpha)^k} =$$
$$\left(\tfrac{1}{R^k}\right)\left[\frac{\left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)^k-\frac{1}{(1-\alpha)^k}}{\left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)-\frac{1}{1-\alpha}}\right]+\tfrac{(\alpha+1)^{k-1}-1}{(1-\alpha)^k} = \left(\tfrac{1}{1-\alpha}+\tfrac{1}{R^k}\right)^k+\tfrac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}$$

b). We have that $|R_4| \le \left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)^k+\frac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}<\frac{1}{R}$ is true by the assumption in the hypothesis. We suppose that $|R_{2n}| \le \left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)^k+\frac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}<\frac{1}{R}$ and we will prove by mathematical induction that $|R_{2n+2}| \le \left(\frac{1}{1-\alpha}+\frac{1}{R^k}\right)^k+\frac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}<\frac{1}{R}$. We have that

$$|R_{2n+2}| = \left|R_{2n+1}^k+\tfrac{1}{c}\right| = \left|\left(R_{2n}^k+\tfrac{1}{c}\right)^k+\tfrac{1}{c}\right| < \tfrac{1}{R}$$

taking into consideration that $\left|R_{2n}^k\right| < \frac{1}{R^k}$ and making the same computations as in point a) from above. Hence, by mathematical induction, we have that

$$|R_{2n}| \le \left(\tfrac{1}{1-\alpha}+\tfrac{1}{R^k}\right)^k+\tfrac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k}<\tfrac{1}{R}$$

for every $n \ge 2$. □

Lemma 2.3. *For every* $n \in \mathbb{N}$, $n \ge 2$ *and* $x, a \in \mathbb{C}$ *we have* $x^n = (x-a)^n + C_n^1 a(x-a)^{n-1} + C_n^2 a^2(x-a)^{n-2} + ... + C_n^{n-1}a^{n-1}(x-a)+a^n$ *where* $C_n^k = \frac{n!}{k!(n-k)!}$ *for every* $k = 0, 1, ..., n$.

Proof. For $n = 2$ we have $x^2 = (x-a)^2+2a(x-a)+a^2 = x^2-2ax+a^2+2ax-2a^2+a^2 = x^2$. We suppose now that

$$x^n = (x-a)^n + C_n^1 a(x-a)^{n-1} + C_n^2 a^2(x-a)^{n-2} + ... + C_n^{n-1}a^{n-1}(x-a) + a^n$$

and we prove, by mathematical induction, that

$$x^{n+1} = (x-a)^{n+1} + C_{n+1}^{1} a(x-a)^{n} + C_{n+1}^{2} a^{2}(x-a)^{n-1} + ... + C_{n+1}^{n} a^{n}(x-a) + a^{n+1}$$

We have that

$$\begin{gathered} x^{n+1} = x \cdot x^{n} = x\left[(x-a)^{n} + C_{n}^{1} a(x-a)^{n-1} + ... + C_{n}^{n-1} a^{n-1}(x-a) + a^{n}\right] = \\ (x-a+a)\left[(x-a)^{n} + C_{n}^{1} a(x-a)^{n-1} + ... + C_{n}^{n-1} a^{n-1}(x-a) + a^{n}\right] = \\ (x-a)^{n+1} + C_{n}^{1} a(x-a)^{n} + ... + C_{n}^{n-1} a^{n-1}(x-a)^{2} + a^{n}(x-a) + \\ a(x-a)^{n} + C_{n}^{1} a^{2}(x-a)^{n-1} + ... + C_{n}^{n-1} a^{n}(x-a) + a^{n+1} = \\ (x-a)^{n+1} + \left(C_{n}^{1} + C_{n}^{0}\right) a(x-a)^{n} + \left(C_{n}^{2} + C_{n}^{1}\right) a^{2}(x-a)^{n-1} + ... + \\ \left(C_{n}^{n-1} + C_{n}^{n-2}\right) a^{n-1}(x-a)^{2} + \left(C_{n}^{n} + C_{n}^{n-1}\right) a^{n}(x-a) + a^{n+1} = \\ (x-a)^{n+1} + C_{n+1}^{1} a(x-a)^{n} + ... + C_{n+1}^{n} a^{n}(x-a) + a^{n+1} \end{gathered}$$

□

Lemma 2.4. *Let $k \geq 4$ and the recurrence $R_{n+1} = \overline{R_n}^{k} + \frac{1}{c}$ be satisfied for every $n \geq 1$ where $R_1 = \frac{1}{c}$ with $c \in \mathbb{C}^{*}$. We suppose that $|c-w| \leq \alpha$ with $\alpha \in (0,1)$ where w is a solution of the equation $z^{k+1} = -1$. Then $|R_2| \leq \frac{(\alpha+1)^{k} - 1 + \alpha}{(1-\alpha)^{k+1}}$.*

Proof. i). Let $k = 2p \geq 4$, w be a solution of the equation $z^{k+1} = z^{2p+1} = -1$ and $|c-w| \leq \alpha$ with $\alpha \in (0,1)$. Then $|c| \leq \alpha+1$, $\frac{1}{|c|} \leq \frac{1}{1-\alpha}$, $|w| = 1$, $w^{k^2} = \left(w^{2p}\right)^{2p} = \left(-\frac{1}{w}\right)^{2p} = \frac{1}{w^{2p}} = -w$, $\overline{c-w} = \overline{c} - \overline{w} = \overline{c} - (-w^{2p}) = \overline{c} + w^{2p}$ and thus $|c-w| = |\overline{c} - \overline{w}|$ $= |\overline{c} + w^{2p}| \leq \alpha$. Hence we obtain the following

$$\begin{gathered} |R_2| = \left|\overline{R_1}^{k} + \frac{1}{c}\right| = \left|\frac{1}{\overline{c}^{2p}} + \frac{1}{c}\right| = \frac{\left|\overline{c}^{2p} + c\right|}{|c|^{2p+1}} = \\ \left(\frac{1}{|c|^{2p+1}}\right)\left|\left(\overline{c} + w^{2p}\right)^{2p} + ... + C_{2p}^{2p-1}(-1)^{2p-1}\left(w^{2p}\right)^{2p-1}\left(\overline{c} + w^{2p}\right) - w + c\right| \leq \\ \left(\frac{1}{|c|^{2p+1}}\right)\left[\left|\overline{c} + w^{2p}\right|^{2p} + C_{2p}^{1}\left|\overline{c} + w^{2p}\right|^{2p-1} + ... + C_{2p}^{2p-1}\left|\overline{c} + w^{2p}\right| + |c-w|\right] = \\ \left(\frac{1}{|c|^{2p+1}}\right)\left[\left(\left|\overline{c} + w^{2p}\right| + 1\right)^{2p} - 1 + |c-w|\right] = \\ \left(\frac{1}{|c|^{2p+1}}\right)\left\{\left(\left|\overline{c} + w^{2p}\right|\right)\left[\left(\left|\overline{c} + w^{2p}\right| + 1\right)^{2p-1} + ... + \left(\left|\overline{c} + w^{2p}\right| + 1\right) + 1\right] + |c-w|\right\} \leq \\ \frac{1}{(1-\alpha)^{2p+1}}\left\{\alpha\left[(\alpha+1)^{2p-1} + ... + (\alpha+1) + 1\right] + \alpha\right\} = \\ \frac{1}{(1-\alpha)^{2p+1}}\left[\alpha\frac{(\alpha+1)^{2p} - 1}{(\alpha+1) - 1} + \alpha\right] = \frac{(\alpha+1)^{2p} - 1 + \alpha}{(1-\alpha)^{2p+1}} = \frac{(\alpha+1)^{k} - 1 + \alpha}{(1-\alpha)^{k+1}} \end{gathered}$$

ii). Let $k = 2p+1 \geq 5$, w be a solution of the equation $z^{k+1} = z^{2p+2} = -1$ and $|c-w| \leq \alpha$ with $\alpha \in (0,1)$. Then $|c| \leq \alpha+1$, $\frac{1}{|c|} \leq \frac{1}{1-\alpha}$, $|w| = 1$, $w^{k^2} = \left(w^{2p+1}\right)^{2p+1} = \left(-\frac{1}{w}\right)^{2p+1} = -\frac{1}{w^{2p}} = -(-w) = w$, $\overline{c-w} = \overline{c} - \overline{w} = \overline{c} - (-w^{2p+1}) = \overline{c} + w^{2p+1}$ and thus $|c-w| = |\overline{c} - \overline{w}| = |\overline{c} + w^{2p+1}| \leq \alpha$. Hence we obtain the following

$$|R_2| = \left|\overline{R_1}^k + \frac{1}{c}\right| = \left|\frac{1}{\overline{c}^{2p+1}} + \frac{1}{c}\right| = \frac{\left|\overline{c}^{2p+1}+c\right|}{|c|^{2p+2}} =$$
$$\left(\frac{1}{|c|^{2p+2}}\right)\left|(\overline{c}+w^{2p+1})^{2p+1} + ... + C_{2p+1}^{2p}(-1)^{2p}\left(w^{2p+1}\right)^{2p}\left(\overline{c}+w^{2p+1}\right) - w + c\right| \leq$$
$$\left(\frac{1}{|c|^{2p+2}}\right)\left[\left|\overline{c}+w^{2p+1}\right|^{2p+1} + C_{2p+1}^{1}\left|\overline{c}+w^{2p+1}\right|^{2p} + ... + C_{2p+1}^{2p}\left|\overline{c}+w^{2p+1}\right| + |c-w|\right] =$$
$$\left(\frac{1}{|c|^{2p+2}}\right)\left[\left(\left|\overline{c}+w^{2p+1}\right|+1\right)^{2p+1} - 1 + |c-w|\right] =$$
$$\left(\frac{1}{|c|^{2p+2}}\right)\left\{\left(\left|\overline{c}+w^{2p+1}\right|\right)\left[\left(\left|\overline{c}+w^{2p+1}\right|+1\right)^{2p} + ... + 1\right] + |c-w|\right\} \leq$$
$$\frac{1}{(1-\alpha)^{2p+2}}\left\{\alpha\left[(\alpha+1)^{2p} + ... + (\alpha+1) + 1\right] + \alpha\right\} =$$
$$\frac{1}{(1-\alpha)^{2p+2}}\left[\alpha\frac{(\alpha+1)^{2p+1}-1}{(\alpha+1)-1} + \alpha\right] = \frac{(\alpha+1)^{2p+1} - 1+\alpha}{(1-\alpha)^{2p+2}} = \frac{(\alpha+1)^{k} - 1+\alpha}{(1-\alpha)^{k+1}}$$

Hence we conclude that if w is a solution of the equation $z^{k+1} = -1$ and $|c-w| \leq \alpha \in (0,1)$ then $|R_2| \leq \frac{(\alpha+1)^k - 1+\alpha}{(1-\alpha)^{k+1}}$ for every $k \geq 4$. □

Lemma 2.5. *Let $k \geq 4$ and the recurrence $R_{n+1} = \overline{R_n}^k + \frac{1}{c}$ be satisfied for every $n \geq 1$ where $R_1 = c$* with $c \in \mathbb{C}^*$. *We suppose that $|c-w| \leq \alpha$ with $\alpha \in (0,1)$ where w is a solution of the equation $z^{k+1} = -1$. In the hypothesis of Lemma 2.4 we suppose that there exists $R > 1$ such that $|R_2| \leq \frac{(\alpha+1)^k -1+\alpha}{(1-\alpha)^{k+1}} < \frac{1}{R}$. Then:*

a). $|R_4| \leq \left(\frac{1}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^k -2+2\alpha}{(1-\alpha)^{k+1}}$.

b). If $|R_4| \leq \left(\frac{1}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^k -2+2\alpha}{(1-\alpha)^{k+1}} < \frac{1}{R}$ then $|R_{2n}| \leq \left(\frac{1}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^k -2+2\alpha}{(1-\alpha)^{k+1}} < \frac{1}{R}$ for every $n \geq 2$.

Proof. a). We have that

$$|R_4| = \left|\overline{R_3}^k + \frac{1}{c}\right| = \left|\left(R_2^k + \frac{1}{\overline{c}}\right)^k + \frac{1}{c}\right| =$$
$$\left|\left(R_2^k\right)^k + C_k^1\left(R_2^k\right)^{k-1}\left(\frac{1}{\overline{c}}\right) + ... + C_k^{k-1}\left(R_2^k\right)\left(\frac{1}{\overline{c}}\right)^{k-1} + \frac{1}{\overline{c}^k} + \frac{1}{c}\right| \leq$$
$$\left|R_2^k\right|^k + C_k^1\left|R_2^k\right|^{k-1}\frac{1}{|\overline{c}|} + C_k^2\left|R_2^k\right|^{k-2}\frac{1}{|\overline{c}|^2} + ... + C_k^{k-1}\left|R_2^k\right|\frac{1}{|\overline{c}|^{k-1}} + \left|\frac{1}{\overline{c}^k} + \frac{1}{c}\right| =$$
$$\left(\left|R_2^k\right| + \frac{1}{|\overline{c}|}\right)^k - \frac{1}{|\overline{c}|^k} + \left|\frac{1}{\overline{c}^k} + \frac{1}{c}\right| =$$
$$\left|R_2^k\right|\left[\left(\left|R_2^k\right| + \frac{1}{|\overline{c}|}\right)^{k-1} + ... + \left(\left|R_2^k\right| + \frac{1}{|\overline{c}|}\right)\frac{1}{|\overline{c}|^{k-2}} + \frac{1}{|\overline{c}|^{k-1}}\right] + \left|\frac{1}{\overline{c}^k} + \frac{1}{c}\right| \leq$$
$$\left(\frac{1}{R^k}\right)\left[\left(\frac{1}{R^k} + \frac{1}{1-\alpha}\right)^{k-1} + ... + \left(\frac{1}{R^k} + \frac{1}{1-\alpha}\right)\left(\frac{1}{1-\alpha}\right)^{k-2} + \left(\frac{1}{1-\alpha}\right)^{k-1}\right] + \frac{(\alpha+1)^k -1+\alpha}{(1-\alpha)^{k+1}} =$$
$$\left(\frac{1}{R^k}\right)\left[\frac{\left(\frac{1}{R^k}+\frac{1}{1-\alpha}\right)^k - \frac{1}{(1-\alpha)^k}}{\left(\frac{1}{R^k}+\frac{1}{1-\alpha}\right) - \left(\frac{1}{1-\alpha}\right)}\right] + \frac{(\alpha+1)^k -1+\alpha}{(1-\alpha)^{k+1}} = \left(\frac{1}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^k -2+2\alpha}{(1-\alpha)^{k+1}}$$

b). The proof is similar to point b) from Lemma 2.2. □

3. Main results

In this section we will denote by $\overline{B}(a,r)=\{z\in\mathbb{C}\mid |z-a|\le r\}$ where $a\in\mathbb{C}$ and $r>0$ a closed ball (disk) in the complex plane. In Theorem 3.1 we consider $w\in\mathbb{C}$ a solution of the equation $z^{k-1}=-1$, $k\ge 2$ and we compute $r>0$ such that $\overline{B}(w,r)$ is included in the Inverted Multibrot set $\mathcal{N}_k$ for every $k\ge 2$ and in Theorem 3.2 we consider $w\in\mathbb{C}$ a solution of the equation $z^{k+1}=-1$, $k\ge 2$ and we compute $r>0$ such that $\overline{B}(w,r)$ is included in the Inverted Multicorn set $\mathcal{N}_k^*$ for every $k\ge 2$.

Theorem 3.1. *Let $\mathcal{N}_k$ be the Inverted Multibrot set of $f_c(z)=z^k+\frac{1}{c}$, $c\in\mathbb{C}^*$ for every $k\ge 2$. Then*
a). $\overline{B}\left(-1,\frac{1}{11}\right)\subset\mathcal{N}_2$
b). $\overline{B}\left(w,\frac{1}{14}\right)\subset\mathcal{N}_3$ where w is a solution of the equation $z^2=-1$
c). $\overline{B}\left(w,\frac{1}{5k}\right)\subset\mathcal{N}_k$ where w is a solution of the equation $z^{k-1}=-1$ for every $k\ge 4$.

Proof. For this proof we will use the recurrence $R_{n+1}=R_n^k+\frac{1}{c}$ for every $n\ge 1$ where $R_1=\frac{1}{c}$ with $c\in\mathbb{C}^*$ and $k\ge 2$.

a). For $k=2$ we have $R_{n+1}=R_n^2+\frac{1}{c}$ where $R_1=\frac{1}{c}$ and let $w=-1$. For $|c-w|$ $=|c+1|\le\frac{1}{11}$ we have $|R_1|=\frac{1}{|c|}\le\frac{11}{10}$. Thus

$$|R_2|=\left|R_1^2+\frac{1}{c}\right|=\left|\frac{1}{c^2}+\frac{1}{c}\right|=\frac{|c+1|}{|c|^2}\le\frac{1}{11}\left(\frac{11}{10}\right)^2=\frac{11}{100}<0.2=\frac{1}{5}$$

Moreover

$$|R_4|=\left|\left(R_2^2+\frac{1}{c}\right)^2+\frac{1}{c}\right|=\left|R_2^4+2\left(\frac{1}{c}\right)R_2^2+\frac{1}{c^2}+\frac{1}{c}\right|\le$$
$$|R_2|^4+2\left(\frac{1}{|c|}\right)|R_2|^2+\left|\frac{1}{c^2}+\frac{1}{c}\right|<\left(\frac{1}{5}\right)^4+2\left(\frac{11}{10}\right)\left(\frac{1}{5}\right)^2+\frac{11}{100}\cong 0.1996<0.2=\frac{1}{5}$$

Hence by Lemma 2.2, point b), we have $|R_{2n}|<\frac{1}{5}$ for every $n\ge 1$ and thus $R_n\not\to\infty$ when $n\longrightarrow\infty$. Thus $\overline{B}\left(-1,\frac{1}{11}\right)\subset\mathcal{N}_2$.

b). For $k=3$ we have $R_{n+1}=R_n^3+\frac{1}{c}$ where $R_1=\frac{1}{c}$ and $w^2=-1$ implies $w\in\{i,-i\}$. Let $w=-i$ and suppose $|c-w|=|c+i|\le\frac{1}{14}$. Then $|R_1|=\frac{1}{|c|}\le\frac{14}{13}$ and

$$|R_2|=\left|R_1^3+\frac{1}{c}\right|=\left|\frac{1}{c^3}+\frac{1}{c}\right|=\left|\frac{c^2+1}{c^3}\right|=\left|\frac{(c+i)^2-2i(c+i)}{c^3}\right|\le\frac{|c+i|^2+2|c+i|}{|c|^3}\le$$
$$\left(\frac{14}{13}\right)^3\left[\left(\frac{1}{14}\right)+2\left(\frac{1}{14}\right)\right]=\frac{406}{2197}\cong 0.18479745<0.25=\frac{1}{4}$$

Moreover

$$|R_4| = \left|\left(R_2^3+\tfrac{1}{c}\right)^3+\tfrac{1}{c}\right| = \left|R_2^9+3\left(\tfrac{1}{c}\right)R_2^6+3\left(\tfrac{1}{c}\right)^2R_2^3+\tfrac{1}{c^3}+\tfrac{1}{c}\right| \leq$$
$$|R_2|^9+3\left(\tfrac{1}{|c|}\right)|R_2|^6+3\left(\tfrac{1}{|c|}\right)^2|R_2|^3+\left|\tfrac{1}{c^3}+\tfrac{1}{c}\right| <$$
$$\left(\tfrac{1}{4}\right)^9+3\left(\tfrac{14}{13}\right)\left(\tfrac{1}{4}\right)^6+3\left(\tfrac{14}{13}\right)^2\left(\tfrac{1}{4}\right)^3+\tfrac{406}{2197} \cong 0.2399539266 < 0.25 = \tfrac{1}{4}$$

Hence by Lemma 2.2, point b), we have $|R_{2n}| < \frac{1}{4}$ for every $n \geq 1$ and thus $R_n \not\to \infty$ when $n \longrightarrow \infty$. Thus $\overline{B}\left(-i, \frac{1}{14}\right) \subset \mathcal{N}_3$. Similar proof for $w = i$. Hence $\overline{B}\left(w, \frac{1}{14}\right) \subset \mathcal{N}_3$ where w is a solution of the equation $z^2 = -1$.

c). Let $k \geq 4$ and let us we first make the following five remarks:

i). For $R > 1.25$ the sequence $\varepsilon_k = \frac{k}{R^k}$ for every $k \geq 4$ is decreasing. Indeed, we have that $\frac{\varepsilon_{k+1}}{\varepsilon_k} = \left(\frac{k+1}{R^{k+1}}\right)\left(\frac{R^k}{k}\right) = \frac{k+1}{Rk} < 1 \Longleftrightarrow k+1 < Rk \Longleftrightarrow 1 < (R-1)k$ which is true since $R > 1.25$ and $k \geq 4$. Hence we obtain that $0 < \varepsilon_k \leq \varepsilon_4 = \frac{4}{R^4}$ for every $k \geq 4$.

ii). The sequence $a_k = \frac{k-1}{5k}$ for every $k \geq 4$ is increasing. Indeed, we have that $\frac{a_{k+1}}{a_k} = \frac{k^2}{k^2-1} > 1$ for every $k \geq 4$. Thus $a_k < \lim\limits_{n\longrightarrow\infty} a_n = \lim\limits_{n\longrightarrow\infty} \frac{n-1}{5n} = \frac{1}{5}$ for every $k \geq 4$.

iii). The sequence $c_k = \frac{k}{5k-1}$ for every $k \geq 4$ is decreasing. Indeed, we have that $\frac{c_{k+1}}{c_k} = \frac{5k^2+4k-1}{5k^2+4k} < 1$ for every $k \geq 4$. Thus $c_k \leq c_4 = \frac{4}{19}$ for every $k \geq 4$.

iv). According to ([8]) for every $r > 0$ and $x > -1$ we have a generalized version of Bernoulli's inequality which is $1 + rx \leq (1+x)^r \leq e^{rx}$.

v). For $\alpha \in (0,1)$ we have that $-\alpha \in (-1,0)$ and we can apply Bernoulli's inequality from above and we obtain $(1-\alpha)^k = (1+(-\alpha))^k \geq 1 + k(-\alpha) = 1 - k\alpha$. Moreover for $\alpha \in \left(0, \frac{1}{k}\right)$ we have $1 - k\alpha > 0$ and we obtain also $\frac{1}{(1-\alpha)^k} \leq \frac{1}{1-k\alpha}$.

We return now to the proof of point c). We suppose that w is a solution of the equation $z^{k-1} = -1$ and $|c - w| \leq \frac{1}{5k} = \alpha$. Then from Lemma 2.1 we have

$$|R_2| \leq \tfrac{(\alpha+1)^{k-1}-1}{(1-\alpha)^k} \leq \tfrac{e^{(k-1)\alpha}-1}{1-k\alpha} = \tfrac{e^{\frac{k-1}{5k}}-1}{1-k\left(\frac{1}{5k}\right)} = \tfrac{e^{a_k}-1}{1-\frac{1}{5}} =$$
$$\left(\tfrac{5}{4}\right)\left[e^{a_k}-1\right] < \left(\tfrac{5}{4}\right)\left[e^{1/5}-1\right] \cong 0.2767 < \tfrac{1}{3} = \tfrac{1}{R}$$

Thus we obtain $R = 3$ and the sequence $\varepsilon_k = \frac{k}{3^k}$ for every $k \geq 4$ verifies $0 < \varepsilon_k \leq \varepsilon_4 = \frac{4}{3^4} = \frac{4}{81}$. Then from Lemma 2.2 we have

$$|R_4| \leq \left(\tfrac{1}{1-\alpha}+\tfrac{1}{R^k}\right)^k + \tfrac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k} = \left(1+\tfrac{\alpha}{1-\alpha}+\tfrac{1}{R^k}\right)^k + \tfrac{(\alpha+1)^{k-1}-2}{(1-\alpha)^k} \leq$$
$$e^{k\left(\frac{\alpha}{1-\alpha}+\frac{1}{R^k}\right)} + \tfrac{e^{(k-1)\alpha}-2}{1-k\alpha} = e^{c_k+\frac{k}{R^k}} + \left(\tfrac{5}{4}\right)\left[e^{a_k}-2\right] \leq e^{c_4+\frac{4}{3^4}} + \left(\tfrac{5}{4}\right)\left[e^{1/5}-2\right] =$$
$$e^{\frac{4}{19}+\frac{4}{81}} + \left(\tfrac{5}{4}\right)\left[e^{1/5}-2\right] \cong (1.2968121) + (-0.9732465) = 0.3235655 < \tfrac{1}{3} = \tfrac{1}{R}$$

Hence by Lemma 2.2, point b), we have $|R_{2n}| < \frac{1}{3}$ for every $n \geq 1$ and thus $R_n \not\to \infty$ when $n \longrightarrow \infty$. Thus $\overline{B}\left(w, \frac{1}{5k}\right) \in \mathcal{N}_k$ where w is a solution of the equation $z^{k-1} = -1$ for every $k \geq 4$. □

Theorem 3.2. *Let $\mathcal{N}_k^*$ be the Inverted Multicorn set of $f_c(z) = \overline{z}^k + \frac{1}{c}$, $c \in \mathbb{C}^*$ for every $k \geq 2$. Then*

a). $\overline{B}\left(w, \frac{1}{29}\right) \subset \mathcal{N}_3^*$ where w is a solution of the equation $z^3 = -1$

b). $\overline{B}\left(w, \frac{1}{31}\right) \subset \mathcal{N}_3^*$ where w is a solution of the equation $z^4 = -1$

c).$\overline{B}\left(w, \frac{1}{7(k+1)}\right) \subset \mathcal{N}_k^*$ where w is a solution of the equation $z^{k+1} = -1$ for every $k \geq 4$.

Proof. For this proof we will use the reccurence $R_{n+1} = \overline{R_n}^k + \frac{1}{c}$ for every $n \geq 1$ where $R_1 = \frac{1}{c}$ with $c \in \mathbb{C}^*$ and $k \geq 2$.

a). If $k = 2$ then the recurrence is $R_{n+1} = \overline{R}_n^2 + \frac{1}{c}$ with $R_1 = \frac{1}{c}$. Suppose w is a solution of the equation $z^3 = -1$ and $|c - w| \leq \frac{1}{29}$. Then $|\overline{c} + w^2| \leq \frac{1}{29}$ and $|R_1| = \frac{1}{|c|} \leq \frac{29}{28}$. Hence

$$|R_2| = \left|\overline{R_1}^2 + \frac{1}{c}\right| = \left|\frac{1}{\overline{c}^2} + \frac{1}{c}\right| = \frac{|\overline{c}^2 + c|}{|c|^3} = \left(\frac{1}{|c|^3}\right) |\left(\overline{c} + w^2\right)^2 - 2w^2\left(\overline{c} + w^2\right) + w^2 + c| =$$
$$\left(\frac{1}{|c|^3}\right) |\left(\overline{c} + w^2\right)^2 - 2w^2\left(\overline{c} + w^2\right) + c - w| \leq \left(\frac{1}{|c|^3}\right)\left[|\overline{c} + w^2|^2 + 2|\overline{c} + w^2| + |c - w|\right] \leq$$
$$\left(\frac{29}{28}\right)^3 \left[\left(\frac{1}{29}\right)^2 + \left(\frac{3}{29}\right)\right] \cong 0.114932 < 0.2 = \frac{1}{5}$$

Moreover

$$|R_4| = \left|\overline{R_3}^2 + \frac{1}{c}\right| = \left|(R_2^2 + \tfrac{1}{\overline{c}})^2 + \tfrac{1}{c}\right| = \left|R_2^4 + 2\left(\tfrac{1}{\overline{c}}\right) R_2^2 + \tfrac{1}{\overline{c}^2} + \tfrac{1}{c}\right| \leq$$
$$|R_2|^4 + 2\left(\frac{1}{|c|}\right)|R_2|^2 + \left|\frac{1}{\overline{c}^2} + \frac{1}{c}\right| \leq \left(\frac{1}{5}\right)^4 + 2\left(\frac{29}{28}\right)\left(\frac{1}{5}\right)^2 + 0.114932 \cong 0.1993897 < 0.2 = \frac{1}{5}$$

Hence by Lemma 2.5, point b), we obtain that $|R_{2n}| < \frac{1}{5}$ for every $n \geq 1$ and thus $R_n \not\to \infty$ when $n \longrightarrow \infty$. Hence $\overline{B}\left(w, \frac{1}{29}\right) \subset \mathcal{N}_2^*$ where w is a solution of the equation $z^3 = -1$.

b). If $k = 3$ then the recurrence is $R_{n+1} = \overline{R}_n^3 + \frac{1}{c}$ with $R_1 = \frac{1}{c}$. Suppose that w is a solution of the equation $z^4 = -1$ and $|c - w| \leq \frac{1}{31} = \alpha$. Then $|R_1| = |c| \leq \frac{31}{30}$ and $|\overline{c} + w^3| \leq \frac{1}{31}$. Hence we have

$$|R_2| = \left|\overline{R_1}^3 + \frac{1}{c}\right| = \left|\frac{1}{\overline{c}^3} + \frac{1}{c}\right| = \frac{|\overline{c}^3 + c|}{|c|^4} =$$
$$\left(\frac{1}{|c|^4}\right) |\left(\overline{c} + w^3\right)^3 - 3w^3\left(\overline{c} + w^3\right)^2 + 3w^6\left(\overline{c} + w^3\right) - w + c| \leq$$

$$\left(\frac{1}{|c|^4}\right)\left[|\overline{c}+w^3|^3+3|\overline{c}+w^3|^2+3|c+w^3|+|c-w|\right] \leq$$
$$\left(\frac{1}{|c|^4}\right)(\alpha^3+3\alpha^2+4\alpha) \leq \left(\frac{31}{30}\right)^4\left[\left(\frac{1}{31}\right)^3+3\left(\frac{1}{31}\right)^2+4\left(\frac{1}{31}\right)\right] \cong 0.1507135 < \frac{1}{4}$$

Moreover

$$|R_4| = \left|\overline{R_3}^3+\frac{1}{c}\right| = \left|(R_2^3+\frac{1}{\overline{c}})^3+\frac{1}{c}\right| = |R_2^9+3\left(\frac{1}{\overline{c}}\right)R_2^6+3\left(\frac{1}{\overline{c}}\right)^2R_2^3+\frac{1}{\overline{c}^3}+\frac{1}{c}| \leq$$
$$|R_2|^9+3\left(\frac{1}{|\overline{c}|}\right)|R_2|^6+3\left(\frac{1}{|\overline{c}|}\right)^2|R_2|^3+\left|\frac{1}{\overline{c}^3}+\frac{1}{c}\right| \leq$$
$$\left(\frac{1}{4}\right)^9+3\left(\frac{31}{30}\right)\left(\frac{1}{4}\right)^6+3\left(\frac{31}{30}\right)^2\left(\frac{1}{4}\right)^3+0.1507135 \cong 0.2015262 < 0.25 = \frac{1}{4}$$

Hence by Lemma 2.5, point b), we have $|R_{2n}| < \frac{1}{4}$ for every $n \geq 1$ and thus $R_n \not\to \infty$ when $n \longrightarrow \infty$. Hence $\overline{B}\left(w, \frac{1}{31}\right) \subset \mathcal{N}_3^*$ where w is a solution of the equation $z^4 = -1$.

c). Let $k \geq 4$ and let us first make the following five remarks:

i). For $R > 1.25$, the sequence $\varepsilon_k = \frac{k}{R^k}$ for every $k \geq 4$ is decreasing. Indeed, we have that $\frac{\varepsilon_{k+1}}{\varepsilon_k} = \left(\frac{k+1}{R^{k+1}}\right)\left(\frac{R^k}{k}\right) = \frac{k+1}{Rk} < 1 \Longleftrightarrow k+1 < Rk \Longleftrightarrow 1 < (R-1)k$ which is true since $R > 1.25$ and $k \geq 4$. Hence we obtain that $0 < \varepsilon_k \leq \varepsilon_4 = \frac{4}{R^4}$ for every $k \geq 4$.

ii). The sequence $a_k = \frac{k}{7(k+1)}$ for every $k \geq 4$ is increasing. Indeed, we have that $\frac{a_{k+1}}{a_k} = \frac{k^2+2k+1}{k^2+2k} > 1$ for every $k \geq 4$. Thus $a_k < \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} \frac{n}{7(n+1)} = \frac{1}{7}$ for every $k \geq 4$.

iii). The sequence $c_k = \frac{k}{7k+6}$ for every $k \geq 4$ is increasing. Indeed, we have that $\frac{c_{k+1}}{c_k} = \frac{7k^2+13k+6}{7k^2+13k} > 1$ for every $k \geq 4$. Thus $c_k < \lim_{n\longrightarrow\infty} c_n = \lim_{n\longrightarrow\infty} \frac{n}{7n+6} = \frac{1}{7}$ for every $k \geq 4$.

iv). According to ([8]) for every $r > 0$ and $x > -1$ we have a generalized version of Bernoulli's inequality which is $1+rx \leq (1+x)^r \leq e^{rx}$.

v). For $\alpha \in (0,1)$ we have that $-\alpha \in (-1,0)$ and we can apply Bernoulli's inequality from above and we obtain $(1-\alpha)^k = (1+(-\alpha))^k \geq 1+k(-\alpha) = 1-k\alpha$. Moreover for $\alpha \in \left(0, \frac{1}{k}\right)$ we have $1-k\alpha > 0$ and we obtain also $\frac{1}{(1-\alpha)^k} \leq \frac{1}{1-k\alpha}$.

We return now to the proof of point c). We suppose that w is a solution of the equation $z^{k+1} = -1$ and $|c-w| \leq \frac{1}{7(k+1)} = \alpha$. Then from Lemma 2.4 we have

$$|R_2| \leq \frac{(\alpha+1)^k-1+\alpha}{(1-\alpha)^{k+1}} \leq \frac{e^{k\alpha}-1+\alpha}{1-(k+1)\alpha} = \left(\frac{7}{6}\right)\left[e^{a_k}-1+\frac{1}{7(k+1)}\right] <$$
$$\left(\frac{7}{6}\right)\left[e^{1/7}-1+\frac{1}{35}\right] \cong 0.21249249 < \frac{1}{3} = \frac{1}{R}$$

Thus we obtain $R = 3$ and the sequence $\varepsilon_k = \frac{k}{3^k}$ for every $k \geq 4$ verifies $0 < \varepsilon_k \leq \varepsilon_4 = \frac{4}{3^4} = \frac{4}{81}$. Then from Lemma 2.5 we have

$$|R_4| \leq \left(\frac{1}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^k - 2 + 2\alpha}{(1-\alpha)^{k+1}} = \left(1 + \frac{\alpha}{1-\alpha} + \frac{1}{R^k}\right)^k + \frac{(\alpha+1)^k - 2 + 2\alpha}{(1-\alpha)^{k+1}} \leq$$
$$e^{k\left(\frac{\alpha}{1-\alpha} + \frac{1}{R^k}\right)} + \frac{e^{k\alpha} - 2 + 2\alpha}{1-(k+1)\alpha} \leq e^{c_k + \frac{k}{R^k}} + \left(\frac{7}{6}\right)\left[e^{a_k} - 2 + \frac{2}{7(k+1)}\right] \leq$$
$$e^{\frac{1}{7} + \frac{4}{81}} + \left(\frac{7}{6}\right)\left[e^{1/7} - 2 + \frac{2}{35}\right] \cong (1.2119611) + (-0.9208408) = 0.2911203 < \frac{1}{3}$$

Hence by Lemma 2.5, point b), we have $|R_{2n}| < \frac{1}{3}$ for every $n \geq 1$ and thus $R_n \not\to \infty$ when $n \longrightarrow \infty$. Thus $\overline{B}\left(w, \frac{1}{7(k+1)}\right) \subset \mathcal{N}_k^*$ where w is a solution of the equation $z^{k+1} = -1$ for every $k \geq 4$. $\square$

4. Conclusions

In this paper we have proved that $\overline{B}\left(-1, \frac{1}{11}\right) \subset \mathcal{N}_2$, $\overline{B}\left(w, \frac{1}{14}\right) \subset \mathcal{N}_3$ where w is a solution of the equation $z^2 = -1$ and $\overline{B}\left(w, \frac{1}{5k}\right) \subset \mathcal{N}_k$ where w is a solution of the equation $z^{k-1} = -1$ for every $k \geq 4$ and $\overline{B}\left(w, \frac{1}{29}\right) \subset \mathcal{N}_2^*$ where w is a solution of the equation $z^3 = -1$, $\overline{B}\left(w, \frac{1}{31}\right) \subset \mathcal{N}_3^*$ where w is a solution of the equation $z^4 = -1$ and $\overline{B}\left(w, \frac{1}{7(k+1)}\right) \subset \mathcal{N}_k^*$ where w is a solution of the equation $z^{k+1} = -1$ for every $k \geq 4$ where $\mathcal{N}_k$ and $\mathcal{N}_k^*$ are the Inverted Multibrot and Multicorn sets of the functions $z^k + \frac{1}{c}$ and $\overline{z}^k + \frac{1}{c}$, $c \in \mathbb{C}^*$, $k \geq 2$.

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