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NEW NOTIONS OF ROUGH STATISTICAL CONVERGENCE OF TRIPLE SEQUENCES IN GRADUAL NORMED LINEAR SPACES

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Abstract. In the present article, we introduce and investigate the concept of statistical convergence for triple sequences in gradual normed linear spaces. We prove some of its fundamental properties and a few implication relations. We then concentrate on rough statistical convergence for triple sequences in gradual normed linear spaces and established some of its features based on the limit set $st^3-LIM_x^r(\mathcal{G})$.

1. Introduction

The notion of statistical convergence was first presented by Fast [19] and Steinhaus [31] independently in 1951. The main idea behind statistical convergence was the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is denoted and defined by

$$\delta(A) = \lim_{n} \frac{1}{n} \left|\{k \in A : k \leq n\}\right|,$$

where the vertical bars indicate the cardinality of the enclosed set. A real-valued sequence $x = (x_k)$ is said to be statistically convergent to the real number x_0 if for each $\eta > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - x_0| \geq \eta\}) = 0.$$

Subsequently, the idea was further investigated from the sequence space point of view by Fridy [21], Šalát [29], Connor [12], Tripathy [34, 35], and many researchers to provide deeper insights into the summability theory. Mursaleen and Edely [22]

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extended this concept over double sequences and mainly worked on the relationship between statistical convergence and statistical Cauchy double sequences, statistical convergence, and strong Cesaro summable double sequences. Besides this, Tripathy [33] examined various properties of the sequence spaces formed by statistical convergent double sequences and proved a decomposition theorem. Also, in 2007, Sahiner et. al. [28] introduced and investigated statistical convergence for triple sequences. Later on, Savaş and Esi [30] developed and investigated it for probabilistic normed spaces. Statistical convergence got attention from a vast class of researchers due to its wide applicability in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

On the other hand, the concept of rough convergence was first examined by Phu [24] for finite dimensional normed linear spaces. Although a similar investigation was carried out in fuzzy settings by Burgin [9] but the results in this paper will emphasize those of Phu. Let r be a non-negative real number. A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be rough convergent to $x_0 \in X$ with roughness degree r, provided that for all $\eta > 0$, there exists a $N \in \mathbb{N}$ so that for all $k \geq N$,

$$\|x_k - x_0\| < r + \eta.$$

Symbolically, it is represented as $x_k \xrightarrow{r-\|\cdot\|} x_0$. It is clear from the above definition that for $r = 0$, the above definition turns to the definition of usual convergence in normed linear spaces. The prime results of Phu [24] are mainly based on the limit set $LIM^r x$ which is closed, convex, and having no smaller bound of its diameter. It should be noted that the idea of rough convergence occurs quite naturally in numerical analysis and has significant applications there. Phu [25], further investigated the notion of rough convergence for infinite dimensional normed spaces and proved some interesting results therein. Combining the notions of rough convergence and statistical convergence, in 2008, Aytar [6] developed rough statistical convergence.

A sequence $x = (x_k)$ in a normed linear space $(X, \|\cdot\|)$ is said to be rough statistically convergent to $x_0 \in X$ with roughness degree r (≥ 0), if for each $\eta > 0$,

$$\delta(\{k \in \mathbb{N} : \|x_k - x_0\| \geq r + \eta\}) = 0.$$

Symbolically, it is denoted as $x_k \xrightarrow{st_r-\|\cdot\|} x_0$. Since the natural density of a finite set is zero, so it is clear from the above definition that if a sequence is rough convergent, then it is also rough statistically convergent.

For an extensive study on rough convergence and its recent progress, [2, 3, 4, 5, 7, 8, 13, 15, 16, 23] can be addressed, where many more references can be found.

In another direction, the notion of fuzzy sets was introduced by Zadeh [36] in 1965. These days, it has wide applications in different branches of science and engineering. The term "fuzzy number (FN)" is significant in the study of fuzzy set theory. FNs were essentially the generalization of intervals, not numbers. Indeed FNs do not obey a couple of algebraic features of classical numbers. So the term "FN" is debatable to many researchers due to its different behavior. The term "fuzzy intervals" is often utilized by several authors in place of FNs. To overcome the confusion among the researchers, in 2008, Fortin et al. [20] put forward the notion of gradual real numbers (GRN) as elements of fuzzy intervals. GRN are mainly known by their respective assignment function whose domain is the interval $(0, 1]$. So, all real numbers can be thought of as gradual numbers (GN) with a constant assignment function. The GRN also obey all the algebraic features of the classical real numbers and have been utilized in optimization problems and computation.

Sadeqi and Azari [27] were the first to present the notion of GNLS. They studied various properties from both the algebraic and topological points of view. Further improvement in this direction has been taken place due to Ettefagh et al. [17, 18], Choudhury and Debnath [10, 11], and many others. For more details, one may refer to [1, 14, 32].

2. Definitions and Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers respectively and by the convergence of a triple sequence we mean the convergence in Pringsheim's [26] sense.

Definition 2.1. [26] A real valued triple sequence $x = (x_{ijk})$ is said to be convergent to a real number x_0, if for any $\eta > 0$, there exists a positive integer $k_0 = k_0(\eta)$ such that for all $i, j, k \geq k_0$,

$$|x_{ijk} - x_0| < \eta.$$

Definition 2.2. [28] Let $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $K_{l,m,n}$ denote the set

$$\{(i,j,k) \in K : i \leq l, j \leq m, k \leq n\}.$$

The triple natural density of K is denoted and defined by

$$\delta^3(K) = \lim_{l,m,n\to\infty} \frac{|K_{l,m,n}|}{lmn},$$

provided that the limit exists.

Definition 2.3. [28] A real valued triple sequence $x = (x_{ijk})$ is said to be statistical convergent to a real number x_0 if for each $\eta > 0$,

$$\delta^3 \left(\{(i,j,k) \in \mathbb{N}^3 : |x_{ijk} - x_0| \geq \eta\}\right) = 0.$$

In this case, l is called the statistical limit of the triple sequence x and symbolically it is expressed as $x_{ijk} \xrightarrow{st} l$.

Definition 2.4. [28] A real valued sequence $x = (x_{ijk})$ is called to be statistical Cauchy provided that for all $\eta > 0$, there are three positive integers $M = M(\eta)$, $N = N(\eta)$ and $P = P(\eta)$ so that

$$\delta^3 \left(\{(i,j,k) \in \mathbb{N}^3 : |x_{ijk} - x_{MNP}| \geq \eta\}\right) = 0.$$

Definition 2.5. [20] A GRN $\tilde{g}$ is described by an assignment function $\mathcal{R}_{\tilde{g}} : (0,1] \to \mathbb{R}$. The set of all GRN is indicated by $\mathcal{G}(\mathbb{R})$. A GRN $\tilde{g}$ is named to be non-negative provided that for all $\psi \in (0,1]$, $\mathcal{R}_{\tilde{g}}(\psi) \geq 0$. $\mathcal{G}^*(\mathbb{R})$ is utilized to denote the set of all non-negative GRNs.

The gradual operations between the elements of $\mathcal{G}(\mathbb{R})$ was itendifed as follows:

Definition 2.6. [20] Assume $*$ be any operation in $\mathbb{R}$ and take $\tilde{g}_1, \tilde{g}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{R}_{\tilde{g}_1}$ and $\mathcal{R}_{\tilde{g}_2}$ respectively. At that time, $\tilde{g}_1 * \tilde{g}_2 \in \mathcal{G}(\mathbb{R})$ is determined with the assignment function $\mathcal{R}_{\tilde{g}_1 * \tilde{g}_2}$ given by

$$\mathcal{R}_{\tilde{g}_1 * \tilde{g}_2}(\psi) = \mathcal{R}_{\tilde{g}_1}(\psi) * \mathcal{R}_{\tilde{g}_2}(\psi),\ \forall \psi \in (0,1].$$

Especially, the gradual addition $\tilde{g}_1 + \tilde{g}_2$ and the gradual scalar multiplication $c\tilde{g} (c \in \mathbb{R})$ are itendifed by

$$\mathcal{R}_{\tilde{g}_1 + \tilde{g}_2}(\psi) = \mathcal{R}_{\tilde{g}_1}(\psi) + \mathcal{R}_{\tilde{g}_2}(\psi) \quad \textit{and} \quad \mathcal{R}_{c\tilde{g}}(\psi) = c\mathcal{R}_{\tilde{g}}(\psi),\ \ \forall \psi \in (0,1].$$

The constant GRN $\tilde{p}$ is itendified by the constant assignment function $\mathcal{R}_{\tilde{p}}(\psi) = p$ for any $\psi \in (0, 1]$, for any $p \in \mathbb{R}$. Especially, $\tilde{0}$ and $\tilde{1}$ are the constant GNs itendifed by $\mathcal{R}_{\tilde{0}}(\psi) = 0$ and $\mathcal{R}_{\tilde{1}}(\psi) = 1$ respectively. One can simply confirm that $\mathcal{G}(\mathbb{R})$ with the gradual multiplication and addition forms a real vector space.

Definition 2.7. [27] Suppose X be a real vector space. The function $\|\cdot\|_{\mathcal{G}} : X \to \mathcal{G}^*(\mathbb{R})$ is named to be a gradual norm (GN) on X, provided that for all $\psi \in (0, 1]$, the subsequent three situations supply for any $x, y \in X$:
($\mathcal{G}_1$) $\mathcal{R}_{\|x\|_{\mathcal{G}}}(\psi) = \mathcal{R}_{\tilde{0}}(\psi)$ iff $x = 0$;
($\mathcal{G}_2$) $\mathcal{R}_{\|\rho x\|_{\mathcal{G}}}(\psi) = |\rho|\mathcal{R}_{\|x\|_{\mathcal{G}}}(\psi)$ for any $\rho \in \mathbb{R}$;
($\mathcal{G}_3$) $\mathcal{R}_{\|x+y\|_{\mathcal{G}}}(\psi) \le \mathcal{R}_{\|x\|_{\mathcal{G}}}(\psi) + \mathcal{R}_{\|y\|_{\mathcal{G}}}(\psi)$.

The pair $(X, \|\cdot\|_{\mathcal{G}})$ is named as GNLS.

Example 2.1. [27] *Suppose $X = \mathbb{R}^t$ and for $x = (x_1, x_2, ..., x_t) \in \mathbb{R}^t, \psi \in (0, 1]$, identify $\|\cdot\|_{\mathcal{G}}$ by*

$$\mathcal{R}_{\|x\|_{\mathcal{G}}}(\psi) = e^{\psi} \sum_{i=1}^{t} |x_i|.$$

At that time, $\|\cdot\|_{\mathcal{G}}$ is an GN on $\mathbb{R}^t$ and $(\mathbb{R}^t, \|\cdot\|_{\mathcal{G}})$ is an GNLS.

Definition 2.8. [27] Suppose $x = (x_k) \in (X, \|\cdot\|_{\mathcal{G}})$. At that time, x is named to be gradually convergent to $x_0 \in X$ provided that for all $\psi \in (0, 1]$ and $\eta > 0$, there is a natural number $N(= N_\eta(\psi))$ so that for all $k \ge N$,

$$\mathcal{R}_{\|x_k - x_0\|_{\mathcal{G}}}(\psi) < \eta.$$

Symbolically, $x_k \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$.

Definition 2.9. [27] A sequence x is named to be gradually Cauchy provided that for all $\psi \in (0, 1]$ and $\eta > 0$, there exists a natural number $N(= N_\eta(\psi))$ so that

$$\mathcal{R}_{\|x_p - x_q\|_{\mathcal{G}}}(\psi) < \eta$$

holds for all $p, q \ge N$.

Definition 2.10. A triple sequence $x = (x_{ijk})$ in $(X, \|\cdot\|_{\mathcal{G}})$ is called to be gradually rough convergent to $x_0 \in X$, provided that for each $\psi \in (0, 1]$ and $\eta > 0$, there is a positive integer $k_0 = k_0(\psi, \eta)$ so that for all $i, j, k \ge k_0$,

$$\mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi) < r+\eta.$$

Symbolically, it is denoted as $x_{ijk} \xrightarrow{r-\|\cdot\|_{\mathcal{G}}} x_0$.

For $r=0$, the above definition reduces to the definition of gradual convergence of the triple sequence x to x_0, which is represented as $x_{ijk} \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$.

Throughout the following sections, r indicates a non-negative real number and $\mathbf{0}$ denotes the zero element of $\mathbb{R}^t$.

3. Statistical Convergence of Triple Sequences In GNLS

In this section, we present our findings related to statistical convergence for triple sequences in GNLS.

Definition 3.1. Assume $x=(x_{ijk}) \in (X, \|\cdot\|_{\mathcal{G}})$. Then, x is called to be gradually statistical convergent (in short $st^3(\mathcal{G})-$convergent) to $x_0 \in X$ provided that for all $\psi \in (0,1]$ and $\eta > 0$,

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right) = 0.$$

Symbolically we write, $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$.

Theorem 3.1. *When a triple sequence $x=(x_{ijk})$ is gradual convergent to $x_0 \in X$, then x is $st^3(\mathcal{G})-$convergent to $x_0 \in X$.*

Proof. The proof is easy so omitted. □

Hovewer the converse of the above theorem is not true. The next example demonstrates the fact.

Example 3.1. *Suppose $X=\mathbb{R}^t$ and $\|\cdot\|_{\mathcal{G}}$ be the GN itendified in Example 2.1. Contemplate the sequence $x=(x_{ijk})$ in $\mathbb{R}^t$ determined as*

$$x_{ijk} = \begin{cases} (0,0,...,0,t) & if\ i=u^2, j=v^2, k=w^2\ for\ some\, u,v,w \in \mathbb{N} \\ \mathbf{0} & otherwise. \end{cases}$$

Then, for any $\eta > 0$ and $\psi \in (0,1]$,

$$\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}-\mathbf{0}\right\|_{\mathcal{G}}}(\psi) \geq \eta\right\} \subseteq \{1,4,9,..\} \times \{1,4,9,..\} \times \{1,4,9,..\}.$$

Hence, $x_{ijk} \xrightarrow{st^3(\mathcal{G})} \mathbf{0}$ *in* $\mathbb{R}^t$*. Hovewer, it is obvious from the definition that,* x *is not gradual convergent to* $\mathbf{0}$.

Theorem 3.2. *Assume* $x = (x_{ijk}) \in (X, \|\cdot\|_{\mathcal{G}})$*. Then,* $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$ *iff there is a set*

$$M = \{(l_i, m_j, n_k) : l_1 < l_2 < ... < l_i < ...; m_1 < m_2 < ... < m_j < ...; n_1 < n_2 < ... < n_k < ...\} \subset \mathbb{N}^3$$

such that $\delta^3(M) = 1$ *and* $x_{l_i m_j n_k} \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$.

Proof. Firstly, we assume that there exists a set $M = \{(l_i, m_j, n_k) : l_1 < l_2 < ... < l_i < ...; m_1 < m_2 < ... < m_j < ...; n_1 < n_2 < ... < n_k < ...\} \subset \mathbb{N}^3$ satisfying

$$\delta^3(M) = 1 \text{ and } x_{l_i m_j n_k} \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0.$$

Then, for all $\psi \in (0, 1]$ and $\eta > 0$, there exists $N(= N_\eta(\psi)) \in \mathbb{N}$ so that

$$\mathcal{R}_{\left\|x_{l_i m_j n_k} - x_0\right\|_{\mathcal{G}}}(\psi) < \eta,\ \forall i, j, k \geq N.$$

Let $B(\psi, \eta) = \left\{(i, j, k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk} - x_0\right\|_{\mathcal{G}}}(\psi) \geq \eta\right\}$. Then, the inclusion

$$B(\psi, \eta) \subset (\mathbb{N}^3) \setminus (\{l_{N+1}, l_{N+2}, ...\} \times \{m_{N+1}, m_{N+2}, ...\} \times \{n_{N+1}, n_{N+2}, ...\})$$

holds and as a consequence we have $\delta^3(B(\psi, \eta)) = 0$. Hence, $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$.

For the converse part, assume that $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$ holds. Then, for every $\psi \in (0, 1]$ and $s \in \mathbb{N}$, $\delta^3(M_s) = 1$, where

$$M_s = \left\{(i, j, k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk} - x_0\right\|_{\mathcal{G}}}(\psi) < \tfrac{1}{s}\right\}.$$

From the construction of M_s's, it is clear that

$$M_1 \supset M_2 \supset ... \supset M_s \supset M_{s+1} \supset ... \tag{3.1}$$

Let us choose $(u_1, v_1, w_1) \in M_1$ be an arbitrary element. Then, there exists $(u_2, v_2, w_2) \in M_2$ such that for all $l \geq u_2, m \geq v_2, n \geq w_2$,

$$\tfrac{1}{lmn} \left|\{i \leq l, j \leq m, k \leq n : (i, j, k) \in M_2\}\right| > \tfrac{1}{2},$$

holds. In a similar way, there exists $(u_3, v_3, w_3) \in M_3$ such that for all $l \geq u_3, m \geq v_3, n \geq w_3$,

$$\tfrac{1}{lmn} \left|\{i \leq l, j \leq m, k \leq n : (i, j, k) \in M_3\}\right| > \tfrac{2}{3},$$

satisfies. Proceeding like this, we can construct three sequences $(u_s), (v_s)$, and (w_s) of positive integers such that $(u_s, v_s, w_s) \in M_s$ and for all $l \geq u_s, m \geq v_s, n \geq w_s$,

$$\frac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : (i,j,k) \in M_s\}| > 1 - \frac{1}{s}, \tag{3.2}$$

is true. Let us construct M as follows: each element of the set $[1, u_1] \times [1, v_1] \times [1, w_1]$ belong to M and any element of the set $[u_s, u_{s+1}] \times [v_s, v_{s+1}] \times [w_s, w_{s+1}]$ belongs to M if and only if it belongs to M_s $(s \in \mathbb{N})$.

From (3.1) and (3.2), we have for each $u_s \leq l < u_{s+1}, v_s \leq m < v_{s+1}, w_s \leq n < w_{s+1}$,

$$\frac{|\{i \leq l, j \leq m, k \leq n : (i,j,k) \in M\}|}{lmn} \geq \frac{|\{i \leq l, j \leq m, k \leq n : (i,j,k) \in M_s\}|}{lmn} > 1 - \frac{1}{s}.$$

Consequently, $\delta^3(M) = 1$. Let $\eta > 0$ be given. By Archimedean property, choose $s \in \mathbb{N}$ such that $\frac{1}{s} < \eta$. Further, let $(i,j,k) \in M$ be such that $i \geq u_s, j \geq v_s$, and $k \geq w_s$. Then, there exists $t \geq s$ such that $u_t \leq i \leq u_{t+1}, v_t \leq j \leq v_{t+1}, w_t \leq k \leq w_{t+1}$. But by the definition of M, $(i,j,k) \in M_t$.

Therefore,

$$\mathcal{R}_{\|x_{ijk} - x_0\|_{\mathcal{G}}}(\psi) < \tfrac{1}{t} \leq \tfrac{1}{s} < \eta.$$

Hence, $x_{l_i m_j n_k} \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$ holds. □

Definition 3.2. Take $x = (x_{ijk})$ as a triple sequence in $(X, \|\cdot\|_{\mathcal{G}})$. Then, x is named to be gradually statistical Cauchy (in short $st^3(\mathcal{G})-$Cauchy) provided that for all $\eta > 0$ and $\psi \in (0,1]$, there are $N_1, N_2, N_3 \in \mathbb{N}$ so that

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk} - x_{N_1 N_2 N_3}\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right) = 0.$$

Theorem 3.3. *Let $x = (x_{ijk})$ be a triple sequence in the GNLS $(X, \|\cdot\|_{\mathcal{G}})$. Then, the following statements are equivalent:*

(i) $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$;

(ii) x is a $st^3(\mathcal{G})-$Cauchy sequence;

(iii) x is a sequence for which there is a gradual convergent triple sequence (y_{ijk}) so that

$$\delta^3(\{(i,j,k) \in \mathbb{N}^3 : x_{ijk} \neq y_{ijk}\}) = 0.$$

Proof. $(i) \Rightarrow (ii)$ Take $x = (x_{ijk}) \in X$ and assume $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$. At that time, for all $\eta > 0$ and $\psi \in (0,1]$,

$$\delta^3 \left(C(\psi,\eta) \right) = 0, \text{ where } C(\psi,\eta) = \left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\| x_{ijk} - x_0 \right\|_{\mathcal{G}}}(\psi) \geq \eta \right\}.$$

Clearly, $\delta^3 \left((\mathbb{N}^3 \setminus C(\psi,\eta) \right) = 1$ and therefore, is non-empty. Select $(N_1,N_2,N_3) \in \mathbb{N}^3 \setminus C(\psi,\eta)$. Afterwards, we obtain $\mathcal{R}_{\left\| x_{ijk} - x_{N_1N_2N_3} \right\|_{\mathcal{G}}}(\psi) < \eta$.

Let $B(\psi,\eta) = \left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\| x_{ijk} - x_{N_1N_2N_3} \right\|_{\mathcal{G}}}(\psi) \geq 2\eta \right\}$. At this time we demonstrate that the subsequent inclusion is true

$$B(\psi,\eta) \subseteq C(\psi,\eta).$$

For if $(u,v,w) \in B(\psi,\eta)$ we have

$$2\eta \leq \mathcal{R}_{\left\| x_{uvw} - x_{N_1N_2N_3} \right\|_{\mathcal{G}}}(\psi) \leq \mathcal{R}_{\| x_{uvw} - x_0 \|_{\mathcal{G}}}(\psi) + \mathcal{R}_{\left\| x_0 - x_{N_1N_2N_3} \right\|_{\mathcal{G}}}(\psi) < \mathcal{R}_{\| x_{uvw} - x_0 \|_{\mathcal{G}}}(\psi) + \eta,$$

which gives $(u,v,w) \in C(\psi,\eta)$. Thus, we conclude that $\delta^3 \left(B(\psi,\eta) \right) = 0$, which means x is $st^3(\mathcal{G})-$Cauchy sequence.

$(ii) \Rightarrow (iii)$ Let $x = (x_{ijk})$ be a $st^3(\mathcal{G})-$Cauchy sequence. Select $N_1,N_2,N_3 \in \mathbb{N}$ so that

$$\delta^3 \left(\left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\| x_{ijk} \right\|_{\mathcal{G}}}(\psi) \notin I \right\} \right) = 0,$$

where

$$I = \left[\mathcal{R}_{\left\| x_{N_1N_2N_3} \right\|_{\mathcal{G}}}(\psi) - 1, \mathcal{R}_{\left\| x_{N_1N_2N_3} \right\|_{\mathcal{G}}}(\psi) + 1 \right].$$

Again select $M_1,M_2,M_3 \in \mathbb{N}$ so that $\delta^3 \left(\left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\| x_{ijk} \right\|_{\mathcal{G}}}(\psi) \notin I' \right\} \right) = 0$, where

$$I' = \left[\mathcal{R}_{\left\| x_{M_1M_2M_3} \right\|_{\mathcal{G}}}(\psi) - \frac{1}{2}, \mathcal{R}_{\left\| x_{M_1M_2M_3} \right\|_{\mathcal{G}}}(\psi) + \frac{1}{2} \right].$$

Now as the equality

$$\left\{ i \leq l, j \leq m, k \leq n : \mathcal{R}_{\left\| x_{ijk} \right\|_{\mathcal{G}}}(\psi) \notin I \cap I' \right\}$$
$$= \left\{ i \leq l, j \leq m, k \leq n : \mathcal{R}_{\left\| x_{ijk} \right\|_{\mathcal{G}}}(\psi) \notin I \right\} \cup \left\{ i \leq l, j \leq m, k \leq n : \mathcal{R}_{\left\| x_{ijk} \right\|_{\mathcal{G}}}(\psi) \notin I' \right\}$$

holds, so we have to have

$$\delta^3 \left(\left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\| x_{ijk} \right\|_{\mathcal{G}}}(\psi) \notin I \cap I' \right\} \right) = 0.$$

Denote $I \cap I'$ by I_1. Then, it is obvious that I_1 is a closed interval with $diam(I_1) \leq 1$, where $diam(I_1)$ represents the length of the interval I_1. Proceeding like this, we choose $N_1(2), N_2(2), N_3(2) \in \mathbb{N}$ so that

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \notin I''\right\}\right) = 0,$$

where

$$I'' = \left[\mathcal{R}_{\|x_{N_1(2)N_2(2)N_3(2)}\|_{\mathcal{G}}}(\psi) - \frac{1}{4}, \mathcal{R}_{\|x_{N_1(2)N_2(2)N_3(2)}\|_{\mathcal{G}}}(\psi) + \frac{1}{4}\right].$$

Let us denote $I_1 \cap I''$ by I_2. Then, by the previous argument we can say that I_2 is a closed interval with $diam(I_2) \leq \frac{1}{2}$ satisfying

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \notin I_2\right\}\right) = 0.$$

Continuing in this way, we obtain a sequence (I_t) of closed intervals such that

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_t \supseteq I_{t+1} \supseteq ...$$

and

$$diam(I_t) \leq 2^{1-t}.$$

By Nested Interval Theorem, there exists a $\lambda \in \mathbb{R}$ such that

$$\bigcap_{t=1}^{\infty} I_t = \{\lambda\}.$$

Now we choose an increasing sequence of natural numbers (T_t) such that

$$\frac{1}{lmn}\left|\left\{i \leq l, j \leq m, k \leq n : \mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \notin I_t\right\}\right| < \frac{1}{t} \tag{3.3}$$

if $l, m, n > T_t$. Define a triple subsequence (z_{ijk}) of (x_{ijk}) consisting of all terms x_{ijk} such that $i, j, k > T_1$ and if $T_m < i, j, k \leq T_{m+1}$ then $\mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \notin I_m$. Define the triple sequence (y_{ijk}) as follows:

$$y_{ijk} = \begin{cases} \tilde{\lambda}, \text{ if } x_{ijk} \text{ is a term of } (z_{ijk}) \\ x_{ijk}, \text{ otherwise,} \end{cases}$$

where $\mathcal{R}_{\tilde{\lambda}}(\psi) = \lambda$, $\forall\, \psi \in (0,1]$. Then, $y_{ijk} \to \tilde{\lambda}$; for, if $\eta > \frac{1}{t} > 0$ and $i, j, k > T_t$ then either x_{ijk} is a term of (z_{ijk}), which means $y_{ijk} = \tilde{\lambda}$ or $y_{ijk} = x_{ijk}$, $\mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \in I_t$ and $\mathcal{R}_{\|y_{ijk}-\tilde{\lambda}\|_{\mathcal{G}}}(\psi) \leq diam(I_t) \leq 2^{1-t}$.

We also claim that $\delta^3(\{k \in \mathbb{N} : x_{ijk} \neq y_{ijk}\}) = 0$. Because if $T_t < l, m, n < T_{t+1}$, then the inclusion

$$\{i \leq l, j \leq m, k \leq n : x_{ijk} \neq y_{ijk}\} \subseteq \left\{i \leq l, j \leq m, k \leq n : \mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \notin I_t\right\}$$

holds and consequently by (3.3),

$$\tfrac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : x_{ijk} \neq y_{ijk}\}| < \tfrac{1}{lmn}.$$

Letting $l, m, n \to \infty$ on both sides of the above inequation, we obtain

$$\lim_{l,m,n\to\infty} \frac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : x_{ijk} \neq y_{ijk}\}| = 0,$$

i.e.,

$$\delta^3(\big\{(i,j,k) \in \mathbb{N}^3 : x_{ijk} \neq y_{ijk}\big\}) = 0.$$

(iii)$\Rightarrow$ (i) Finally we assume that $\delta^3(\{(i,j,k) \in \mathbb{N}^3 : x_{ijk} \neq y_{ijk}\}) = 0$ and $y_{ijk} \xrightarrow{\|\cdot\|_{\mathcal{G}}} x_0$.

Then, by definition for any $\eta > 0$ and $\psi \in (0, 1]$, the set

$$\left\{i \leq l, j \leq m, k \leq n : \mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}$$

contains a finite number of elements say N_0. Now as the inclusion

$$\begin{aligned}\left\{i \leq l, j \leq m, k \leq n : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\} \\ \subseteq \{i \leq l, j \leq m, k \leq n : x_{ijk} \neq y_{ijk}\} \\ \cup \left\{i \leq l, j \leq m, k \leq n : \mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\end{aligned}$$

holds, so we must have,

$$\begin{aligned}\frac{1}{lmn} \left|\left\{i \leq l, j \leq m, k \leq n : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right| \\ \leq \frac{1}{lmn} |\{i \leq l, j \leq m, k \leq n : x_{ijk} \neq y_{ijk}\}| + \frac{N_0}{lmn}.\end{aligned}$$

Letting $l, m, n \to \infty$ on both sides of the above inequality and utilizing the fact that

$$\delta^3\left(\big\{(i,j,k) \in \mathbb{N}^3 : x_{ijk} \neq y_{ijk}\big\}\right) = 0$$

we obtain $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$. This completes the proof. □

4. Rough Statistical Convergence of Triple Sequences In GNLS

A triple sequence that is not statistically convergent, may be rough statistically convergent for some roughness degree r in a normed space. However, since every GNLS is not necessarily a normed linear space (Example 3.18 of [27]), so it is quite natural to investigate the above properties of triple sequences in GNLS setting. In this section, we put forward our findings regarding the rough statistical convergence of triple sequences in GNLS. We begin with the following definitions:

Definition 4.1. A triple sequence $x = (x_{ijk})$ is named to be gradually rough statistical convergent (briefly $st_r^3(\mathcal{G})-$convergent) to $x_0 \in Y$, provided that for all $\psi \in (0, 1]$ and $\eta > 0$,

$$\delta^3 \left(\left\{ (i, j, k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk} - x_0\|_{\mathcal{G}}}(\psi) \geq r + \eta \right\} \right) = 0.$$

Symbolically we write, $x_{ijk} \xrightarrow{st_r^3(\mathcal{G})} x_0$.

Here r is named the degree of roughness. For $r = 0$, the above definition reduces to Definition 3.1. But our main intention is to deal with the case $r > 0$. There are some reasons for such interest. Since a gradually statistical convergent sequence $y = (y_{ijk})$ with $y_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$ often cannot be measured or calculated accurately, one has to deal with an approximated triple sequence $x = (x_{ijk})$ satisfying

$$\delta^3 \left(\left\{ (i, j, k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk} - y_{ijk}\|_{\mathcal{G}}}(\psi) > r \right\} \right) = 0.$$

Then, no one can guarantee the gradually statistical convergence of x, but since for any $\eta > 0$, the following inclusion

$$\left\{ (i, j, k) \in \mathbb{N}^3 : \mathcal{R}_{\|y_{ijk} - x_0\|_{\mathcal{G}}}(\psi) \geq \eta \right\} \supseteq \left\{ (i, j, k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk} - x_0\|_{\mathcal{G}}}(\psi) \geq r + \eta \right\}.$$

holds, one can certainly assure the $st_r^3(\mathcal{G})-$convergence of x. We serve the subsequent example to illustrate the above fact more preciously:

Example 4.1. *Assume $X = \mathbb{R}^t$ and let $\|\cdot\|_{\mathcal{G}}$ be the GN itendified in Example 2.1. Contemplate the triple sequence (y_{ijk}) in X defined as*

$$y_{ijk} = \begin{cases} \left(0, 0, ..., 0, 0.5 + 2 \cdot \frac{(-1)^{i+j+k}}{i+j+k}\right), & \text{when } i, j, k \text{ are perfect squares} \\ (0, 0, ..., 0, 0.5), & \text{otherwise.} \end{cases}$$

Then, we get

$$\mathcal{R}_{\left\|y_{ijk}-(0,0,...,0,0.5)\right\|_{\mathcal{G}}}(\psi)=\begin{cases}\frac{2e^{\psi}}{i+j+k}, & \text{when } i,j,k \text{ are perfect squares}\\ 0, & \text{otherwise.}\end{cases}$$

So, for any $\eta>0$, the following inclusion

$$\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\|q_{uv}-(0,0,...,0,0.5)\|_{\mathcal{G}}}(\psi)\geq\eta\right\}\subseteq\{(1,1,1),(4,4,4),(9,9,9),...\}$$

supplies and eventually $y_{ijk}\xrightarrow{st^3(\mathcal{G})}(0,0,...,0,0.5)$. But for sufficiently large i,j, and k, it is not possible to compute y_{ijk} exactly by computer however it is rounded to the nearest one. So, in the interest of simplicity, we approximate y_{ijk} by $x_{ijk}=(0,0,...,0,z)$ at the perfect square values of i,j and k, where z is the integer satisfying $z-0.5<y_{ijk}<z+0.5$. At that time, the triple sequence (x_{ijk}) does not $st^3(\mathcal{G})-$converge anymore. On the other hand according to the definition $x_{ijk}\xrightarrow{st^3_{0.5}(\mathcal{G})}(0,0,...,0,0.5)$.

It is obvious that for $r>0$, the $st^3_r(\mathcal{G})-$limit of a triple sequence is not necessarily unique. As a result, our fundamental interest is to deal with the case $r>0$. Therefore, we construct $st^3_r(\mathcal{G})-$limit set of a triple sequence $x=(x_{ijk})$, determined as follows:

$$st^3-LIM^r_x(\mathcal{G})=\left\{x_0\in X:x_{ijk}\xrightarrow{st^3_r(\mathcal{G})}x_0\right\}.$$

Theorem 4.1. *Assume (x_{ijk}) and (y_{ijk}) be two triple sequences in $(X,\|\cdot\|_{\mathcal{G}})$ so that $x_{ijk}\xrightarrow{st^3_{r_1}(\mathcal{G})}x_0$ and $y_{ijk}\xrightarrow{st^3_{r_2}(\mathcal{G})}y_0$. Then,*
(i) $x_{ijk}+y_{ijk}\xrightarrow{st^3_{(r_1+r_2)}(\mathcal{G})}x_0+y_0$ and (ii) $\mu x_{ijk}\xrightarrow{st^3_{|\mu|r_1}(\mathcal{G})}\mu x_0$ for any $\mu\in\mathbb{R}$.

Proof. (i) Since, $x_{ijk}\xrightarrow{st^3_{r_1}(\mathcal{G})}x_0$ and $y_{ijk}\xrightarrow{st^3_{r_2}(\mathcal{G})}q_0$, so for any $\psi\in(0,1]$ and $\eta>0$, $\delta^3(P)=\delta^3(Q)=0$, where

$$P=\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi)\geq r_1+\tfrac{\eta}{2}\right\}\text{ and}$$
$$Q=\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|y_{ijk}-y_0\right\|_{\mathcal{G}}}(\psi)\geq r_2+\tfrac{\eta}{2}\right\}.$$

As the inclusion

$$(\mathbb{N}^3\setminus P)\cap(\mathbb{N}^3\setminus Q)\subseteq\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|(x_{ijk}+y_{ijk})-(x_0+y_0)\right\|_{\mathcal{G}}}(\psi)<r_1+r_2+\eta\right\}$$

holds, so we obtain

$$\delta^3\left(\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|(x_{ijk}+y_{ijk})-(x_0+y_0)\right\|_{\mathcal{G}}}(\psi)\geq r_1+r_2+\eta\right\}\right)=0.$$

Hence, $x_{ijk}+y_{ijk}\xrightarrow{st^3_{(r_1+r_2)}(\mathcal{G})}x_0+y_0$.

(ii) When $\mu = 0$, then there is nothing to prove. So let us presume that $\mu \neq 0$. Now as the situations

$$\mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi) \leq r_1 \text{ and } \mathcal{R}_{\left\|\mu x_{ijk}-\mu x_0\right\|_{\mathcal{G}}}(\psi) \leq |\mu| r_1$$

are equivalent in gradual normed algebras, so the result follows. □

Now for $r_1 = r_2 = 0$, the above theorem reduces to the following result:

Corollary 4.1. *Presume (x_{ijk}) and (y_{ijk}) be two triple sequences in $(X, \|\cdot\|_{\mathcal{G}})$ so that $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$ and $y_{ijk} \xrightarrow{st^3(\mathcal{G})} y_0$. Then,*
(i) $x_{ijk} + y_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0 + y_0$ and (ii) $\mu x_{ijk} \xrightarrow{st^3(\mathcal{G})} \mu x_0$ for any $\mu \in \mathbb{R}$.

Theorem 4.2. *Take $x = (x_{ijk}) \in (X, \|\cdot\|_{\mathcal{G}})$. Then,*

$$diam(st^3 - LIM_x^r(\mathcal{G})) = \sup\left\{\mathcal{R}_{\|q-t\|_{\mathcal{G}}}(\psi) : q, t \in st^3 - LIM_x^r(\mathcal{G}), \psi \in [0,1)\right\} \leq 2r.$$

In general, $diam(st^3 - LIM_x^r(\mathcal{G}))$ has no smaller bound.

Proof. If possible, let us suppose that $diam(st^3 - LIM_x^r(\mathcal{G})) > 2r$. Afterwards, there are $q_0, t_0 \in st^3 - LIM_x^r(\mathcal{G})$ and $\psi_0 \in [0,1)$ so that $\mathcal{R}_{\|q_0-t_0\|_{\mathcal{G}}}(\psi_0) > 2r$. Select $\eta > 0$ in such a manner that

$$\eta < \frac{\mathcal{R}_{\|q_0-t_0\|_{\mathcal{G}}}(\psi_0)}{2} - r. \tag{4.1}$$

Since, $q_0, t_0 \in st^3 - LIM_x^r(\mathcal{G})$, so for any $\psi \in (0,1]$ and $\eta > 0$, $\delta^3(A) = \delta^3(B) = 0$, where

$$A = \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}-q_0\right\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\} \text{ and}$$
$$B = \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}-t_0\right\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}.$$

As a result, $\delta^3((\mathbb{N}^3 \setminus A) \cap (\mathbb{N}^3 \setminus B)) = 1$ and eventually $(\mathbb{N}^3 \setminus A) \cap (\mathbb{N}^3 \setminus B)$ is non-empty. Take $(i_0, j_0, k_0) \in (\mathbb{N}^3 \setminus A) \cap (\mathbb{N}^3 \setminus B)$. At that time, we acquire

$$\mathcal{R}_{\|q_0-t_0\|_{\mathcal{G}}}(\psi_0) \leq \mathcal{R}_{\left\|x_{i_0j_0k_0}-q_0\right\|_{\mathcal{G}}}(\psi_0) + \mathcal{R}_{\left\|x_{i_0j_0k_0}-t_0\right\|_{\mathcal{G}}}(\psi_0) < 2(r+\eta),$$

which contradicts (4.1).

For the second part, presume (x_{ijk}) be a triple sequence in $(X, \|\cdot\|_{\mathcal{G}})$ so that $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$. As a result, for any $\psi \in (0,1]$ and $\eta > 0$,

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right) = 0.$$

Now for each $q_0 \in (x_0 + \bar{N}(r,\psi)) = \left\{\kappa \in X : \mathcal{R}_{\|x_0-\kappa\|_{\mathcal{G}}}(\psi) \le r\right\}$, the following inequation

$$\mathcal{R}_{\left\|x_{ijk}-q_0\right\|_{\mathcal{G}}}(\psi) \le \mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi) + \mathcal{R}_{\|x_0-q_0\|_{\mathcal{G}}}(\psi) < r + \eta,$$

supplies whenever $(i,j,k) \notin \{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi) \ge \eta\}$. This shows that $q_0 \in st^3 - LIM_x^r(\mathcal{G})$ and subsequently

$$st^3 - LIM_x^r(\mathcal{G}) = (x_0 + \bar{N}(r,\psi))$$

supplies. Since, $diam(x_0 + \bar{N}(r,\psi)) = 2r$, so in general upper bound $2r$ of the diameter of the set $st^3 - LIM_x^r(\mathcal{G})$ cannot be decreased anymore. □

Taking $r = 0$ in the above theorem, we can get the subsequent result:

Corollary 4.2. *Let $x = (x_{ijk})$ be a triple sequence in $(X, \|\cdot\|_{\mathcal{G}})$ so that $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$. Then, x_0 is uniquely determined.*

Definition 4.2. The triple sequence (x_{ijk}) is named to be $st^3(\mathcal{G})-$bounded provided that for all $\psi \in (0,1]$, there is an $M(= M(\psi)) > 0$ so that

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}\right\|_{\mathcal{G}}}(\psi) > M\right\}\right) = 0.$$

Theorem 4.3. *A triple sequence $x = (x_{ijk})$ in $(X, \|\cdot\|_{\mathcal{G}})$ is $st^3(\mathcal{G})-$bounded iff there exists some $r \ge 0$ so that $st^3 - LIM_x^r(\mathcal{G}) \ne \emptyset$.*

Proof. Let $x = (x_{ijk})$ be $st^3(\mathcal{G})-$bounded. Then, for each $\psi \in (0,1]$, there exists $M(= M(\psi)) > 0$ so that

$$\delta^3(A) = 0, \text{ where } A = \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\left\|x_{ijk}\right\|_{\mathcal{G}}}(\psi) > M\right\}.$$

Suppose

$$B = \sup\left\{\mathcal{R}_{\left\|x_{ijk}\right\|_{\mathcal{G}}}(\psi) : (i,j,k) \in (\mathbb{N}^3 \setminus A), \psi \in [0,1)\right\}.$$

Then, the set $st^3 - LIM_x^B(\mathcal{G})$ includes the zero vector of X and eventually

$$st^3 - LIM_x^B(\mathcal{G}) \ne \emptyset.$$

Conversely, presume that $st^3 - LIM_x^r(\mathcal{G}) \ne \emptyset$ for some $r \ge 0$. At that time, for $x_0 \in st^3 - LIM_x^r(\mathcal{G})$,

$$\delta^3\left(\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|x_{ijk}-x_0\right\|_{\mathcal{G}}}(\psi)\geq r+\eta\right\}\right)=0$$

holds for any $\psi\in(0,1]$ and $\eta>0$. This implies that x is $st^3(\mathcal{G})-$bounded. □

Theorem 4.4. *Let $x=(x_{ijk})\in(X,\|\cdot\|_{\mathcal{G}})$. When $q_0\in st^3-LIM_x^{r_0}(\mathcal{G})$ and $q_1\in st^3-LIM_x^{r_1}(\mathcal{G})$, then*

$$q_\lambda=(1-\lambda)q_0+\lambda q_1\in st^3-LIM_x^{(1-\lambda)r_0+\lambda r_1}(\mathcal{G}),\ \textit{where}\ \lambda\in[0,1].$$

Proof. Since $q_0\in st^3-LIM_x^{r_0}(\mathcal{G})$ and $q_1\in st^3-LIM_x^{r_1}(\mathcal{G})$, so for each $\psi\in(0,1]$ and $\eta>0$, $\delta^3(A)=\delta^3(B)=0$, where

$$A=\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|x_{ijk}-q_0\right\|_{\mathcal{G}}}(\psi)\geq r_0+\eta\right\}\text{ and}$$
$$B=\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|x_{ijk}-q_1\right\|_{\mathcal{G}}}(\psi)\geq r_1+\eta\right\}.$$

Subsequently, for any $(i,j,k)\in(\mathbb{N}^3\setminus A)\cap(\mathbb{N}^3\setminus B)$,

$$\begin{aligned}\mathcal{R}_{\left\|x_{ijk}-q_\lambda\right\|_{\mathcal{G}}}(\psi)&\leq(1-\lambda)\mathcal{R}_{\left\|x_{ijk}-q_0\right\|_{\mathcal{G}}}(\psi)+\lambda\mathcal{R}_{\left\|x_{ijk}-q_1\right\|_{\mathcal{G}}}(\psi)\\&<(1-\lambda)(r_0+\eta)+\lambda(r_1+\eta)\\&=(1-\lambda)r_0+\lambda r_1+\eta.\end{aligned}$$

This demonstrates that,

$$\left\{(i,j,k)\in\mathbb{N}^3:\mathcal{R}_{\left\|x_{ijk}-q_\lambda\right\|_{\mathcal{G}}}(\psi)\geq(1-\lambda)r_0+\lambda r_1+\eta\right\}\subseteq A\cup B.$$

Now since the set in the right-hand side has triple natural density zero, so the set in the left-hand side also has triple natural density zero. Hence, $q_\lambda\in st^3-LIM_x^{(1-\lambda)r_0+\lambda r_1}(\mathcal{G})$. □

Corollary 4.3. *Let $x=(x_{ijk})\in(X,\|\cdot\|_{\mathcal{G}})$. Then, the set $st^3-LIM_x^r(\mathcal{G})$ is convex.*

Theorem 4.5. *Let $x=(x_{ijk})\in(X,\|\cdot\|_{\mathcal{G}})$. Then, the set $st^3-LIM_x^r(\mathcal{G})$ is gradually closed.*

Proof. Let $y=(y_{ijk})\in st^3-LIM_x^r(\mathcal{G})$ be such that

$$y_{ijk}\xrightarrow{\|\cdot\|_{\mathcal{G}}}y_0.$$

Then, for each $\psi\in(0,1]$ and $\eta>0$, there is an $N(=N_\eta(\psi))\in\mathbb{N}$ so that for all $i,j,k\geq N$,

$$\mathcal{R}_{\|y_{ijk}-y_0\|_{\mathcal{G}}}(\psi) < \tfrac{\eta}{2}.$$

Select $i_0, j_0, k_0 \in \mathbb{N}$ so that $i_0 \geq N, j_0 \geq N, k_0 \geq N$. Then, $\mathcal{R}_{\|y_{i_0j_0k_0}-y_0\|_{\mathcal{G}}}(\psi) < \tfrac{\eta}{2}$. On the other hand, since $(y_{ijk}) \subseteq st^3 - LIM_x^r(\mathcal{G})$, we must have

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-y_{i_0j_0k_0}\|_{\mathcal{G}}}(\psi) \geq r + \frac{\eta}{2}\right\}\right) = 0. \tag{4.2}$$

Suppose $(u,v,w) \notin \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-y_{i_0j_0k_0}\|_{\mathcal{G}}}(\psi) \geq r + \tfrac{\eta}{2}\right\}$.

Then, $\mathcal{R}_{\|x_{uvw}-y_{i_0j_0k_0}\|_{\mathcal{G}}}(\psi) < r + \tfrac{\eta}{2}$ and eventually

$$\mathcal{R}_{\|x_{uvw}-y_0\|_{\mathcal{G}}}(\psi) \leq \mathcal{R}_{\|x_{uvw}-y_{i_0j_0k_0}\|_{\mathcal{G}}}(\psi) + \mathcal{R}_{\|y_{i_0j_0k_0}-y_0\|_{\mathcal{G}}}(\psi) < r + \eta.$$

This gives that $(u,v,w) \notin \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}$ and subsequently from (4.2) we acquire

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-y_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right) = 0.$$

Hence, $y_0 \in st^3 - LIM_x^r(\mathcal{G})$ and the proof ends. □

Theorem 4.6. *Let $r_1 \geq 0$ and $r_2 \geq 0$. A triple sequence $x = (x_{ijk})$ in a GNLS $(X, \|\cdot\|_{\mathcal{G}})$ is $st^3_{(r_1+r_2)}(\mathcal{G})-$convergent to x_0 iff there is a triple sequence $y = (y_{ijk})$ so that*

$$y_{ijk} \xrightarrow{st^3_{r_1}(\mathcal{G})} x_0 \text{ and } \mathcal{R}_{\|x_{ijk}-y_{ijk}\|_{\mathcal{G}}}(\psi) \leq r_2$$

for all $(i,j,k) \in \mathbb{N}^3$.

Proof. Let us assume that $y_{ijk} \xrightarrow{st^3_{r_1}(\mathcal{G})} x_0$. Afterwards, according to definition for any $\psi \in (0,1]$ and $\eta > 0$,

$$\delta^3(P) = 0, \text{ where } P = \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq r_1 + \eta\right\}.$$

Now since $\mathcal{R}_{\|x_{ijk}-y_{ijk}\|_{\mathcal{G}}}(\psi) \leq r_2$ supplies for all $(i,j,k) \in \mathbb{N}^3$, so for all $(i,j,k) \notin P$,

$$\mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \leq \mathcal{R}_{\|x_{ijk}-y_{ijk}\|_{\mathcal{G}}}(\psi) + \mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) < r_1 + r_2 + \eta.$$

This implies that

$$\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq r_1 + r_2 + \eta\right\} \subseteq P$$

and eventually by the property of triple natural density,

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq r_1 + r_2 + \eta\right\}\right) = 0.$$

Hence, $x_{ijk} \xrightarrow{st^3_{(r_1+r_2)}(\mathcal{G})} x_0$.

For the converse part, let us suppose that

$$x_{ijk} \xrightarrow{st^3_{(r_1+r_2)}(\mathcal{G})} x_0. \tag{4.3}$$

Define $y = (y_{ijk})$ by

$$y_{ijk} = \begin{cases} x_0, & if\ \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \le r_2 \\ x_{ijk} + r_2 \frac{x_0 - x_{ijk}}{\mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi)}, & otherwise. \end{cases}$$

Then, it is easy to observe that $\mathcal{R}_{\|x_{ijk}-y_{ijk}\|_{\mathcal{G}}}(\psi) \le r_2$ for all $(i,j,k) \in \mathbb{N}^3$.

Moreover,

$$\mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) = \begin{cases} 0, & if\ \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \le r_2 \\ \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) - r_2, & otherwise. \end{cases}$$

By (4.3), for each $\psi \in (0,1]$ and $\eta > 0$,

$$\delta^3 \left(\left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \ge r_1 + r_2 + \eta \right\} \right) = 0.$$

Now as the inclusion

$$\left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \ge r_1 + r_2 + \eta \right\} \supseteq \left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \ge r_1 + \eta \right\}$$

holds, so we obtain

$$\delta^3 \left(\left\{ (i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|y_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \ge r_1 + \eta \right\} \right) = 0.$$

Hence, $y_{ijk} \xrightarrow{st^3_{r_1}(\mathcal{G})} x_0$ and the proof ends. □

Corollary 4.4. *A triple sequence $(x_{ijk}) \in (X, \|\cdot\|_{\mathcal{G}})$ is $st^3_r(\mathcal{G})-$convergent to $x_0 \in X$ with roughness degree $r \ge 0$ iff there is a triple sequence $y = (y_{ijk})$ in X so that $x_{ijk} \xrightarrow{st^3(\mathcal{G})} x_0$ and $\mathcal{R}_{\|x_{ijk}-y_{ijk}\|} \le r$ for all $(i,j,k) \in \mathbb{N}^3$.*

Theorem 4.7. *Presume $(X, \|\cdot\|)$ be a normed linear space and suppose $f : (0,1] \to \mathbb{R}^+$ be a non-zero function. In [17], it was demonstrated that the map $\mathcal{R}_{\|x\|_{\mathcal{G}}} : (0,1] \to \mathbb{R}^+$ determined by*

$$\mathcal{R}_{\|x\|_{\mathcal{G}}}(\psi) = f(\psi) \|x\|, x \in X$$

determines a gradual norm on X. For any triple sequence (x_{ijk}) in X,
(i) When $x_{ijk} \xrightarrow{st_r^3-\|\cdot\|} x_0$, *then* $x_{ijk} \xrightarrow{st_r^3(\mathcal{G})} x_0$.
(ii) When $x_{ijk} \xrightarrow{st_r^3(\mathcal{G})} x_0$ *and there exists a* $\psi_0 \in (0,1]$ *so that* $f(\psi_0) = 1$, *then* $x_{ijk} \xrightarrow{st_{r'}^3-\|\cdot\|} x_0$ *for some* $r' \geq 0$.

Proof. (i) Since $x_{ijk} \xrightarrow{st_r^3-\|\cdot\|} x_0$, so for any $\eta > 0$ and $\psi_0 \in (0,1]$,

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \|x_{ijk} - x_0\| \geq r + \frac{\eta}{f(\psi_0)}\right\}\right) = 0.$$

So, the following inequation

$$\mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi_0) = f(\psi_0)\,\|x_{ijk} - x_0\| < rf(\psi_0) + \eta$$

supplies for any $(i,j,k) \notin \left\{(i,j,k) \in \mathbb{N}^3 : \|x_{ijk} - x_0\| \geq r + \frac{\eta}{f(\psi_0)}\right\}$ and the result follows by taking $r' = rf(\psi_0)$.
(ii) Since $x_{ijk} \xrightarrow{st_r^3(\mathcal{G})} x_0$, so for any $\eta > 0$ and $\psi \in (0,1]$,

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}\right) = 0.$$

Especially, for $\psi = \psi_0$, the following inequation

$$\|x_{ijk} - x_0\| = f(\psi_0)\,\|x_{ijk} - x_0\| = \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi_0) < r + \eta$$

supplies for any $(i,j,k) \notin \left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq r + \eta\right\}$ and the rest follows from the property of triple natural density. □

5. Concluding Remarks

In this paper, we have investigated the notion of statistical and rough statistical convergence in GNLS for triple sequences. Theorem 3.2 and Corollary 4.4 gives a necessary and sufficient condition for the respective convergences of a triple sequence in a GNLS. Theorem 3.3 relates a gradually statistical convergent triple sequence with a gradually statistical Cauchy triple sequence in a GNLS. Furthermore, Theorem 4.2, Theorem 4.4 and Theorem 4.5 established the several properties of the set $st^3 - LIM_x^r(\mathcal{G})$. Finally, Theorem 4.7 is established for a comparative study of rough statistical convergence of triple sequences in normed linear spaces and in gradual normed linear spaces. In future, as a continuation of this research, one can form the following sequence spaces

$$c_{\theta}^{st^3}(\mathcal{G}) = \{x = (x_{ijk}) : \text{there exists } x_0 \in X \text{ such that for all } \psi \in (0,1] \text{ and } \eta > 0, \quad \delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right) = 0\right\},$$

$$c^{st^3}(\mathcal{G}) = \{x = (x_{ijk}) : \text{there exists } x_0 \in X \text{ such that for all } \psi \in (0,1] \text{ and } \eta > 0, \quad \delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}-x_0\|_{\mathcal{G}}}(\psi) \geq \eta\right\}\right) = 0\right\},$$

and

$$l_{\infty}^{st^3}(\mathcal{G}) = \{x = (x_{ijk}) : \text{for all } \psi \in (0,1] \text{ there exists } M(=M(\psi)) > 0 \text{ such that } \delta^3\left(\left\{(i,j,k) \in \mathbb{N}^3 : \mathcal{R}_{\|x_{ijk}\|_{\mathcal{G}}}(\psi) > M\right\}\right) = 0\right\}$$

and utilize this study to investigate several important properties such as solidity, monotonicity, symmetric properties etc.

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