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THE CYCLE-COMPLETE GRAPH RAMSEY NUMBERS $R(C_n, K_8)$, FOR $10 \leq n \leq 15$

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ABSTRACT. Given two graphs H_1 and H_2, the Ramsey number $R(H_1, H_2)$ is the smallest natural number n such that each graph of order n contains a copy of H_1 or its complement contains a copy of H_2. In this paper, we find the exact Ramsey number $R(C_n, K_8)$ for $10 \leq n \leq 15$, where C_n is the cycle on n vertices and K_8 is the complete graph of order 8.

1. Introduction

In this paper, all graphs are simple graphs. Given two graphs H_1 and H_2, the Ramsey number $R(H_1, H_2)$ is the smallest natural number n such that each graph of order n contains a copy of H_1 or its complement contains a copy of H_2. Note that, if the complement of a graph G contains a complete graph K_s, then G contains s independent vertices. The existence of Ramsey numbers is known from 1930, by Frank Ramsey [10]. The Ramsey numbers attracted a lot of researches and established the whole field of Ramsey theory. The existence of these numbers has been known since 1930 but their quantitative behavior is still not well understood. There are many open problems and conjectures about Ramsey numbers for different kind of graphs. In this paper we focus on the $R(C_n, K_m)$, where C_n is a cycle on n vertices and K_m is the complete graph of order m.

Erdős et al. [7] conjectured that, for all integers $3 \leq m \leq n$, we have

$$R(C_n, K_m) = (n-1)(m-1) + 1.$$

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The exception for the conjecture is when $n = m = 3$ since $R(3,3) = 6$. Bondy and Erdős [5] proved the conjecture for $n \geq m^2 - 2$. Also, the conjecture was confirmed for $m = 3$ [8, 11]. Yang et al. [13] and Bollobás et al. [4] proved the conjecture for $m = 4$ and $m = 5$, respectively. Schiermeyer [12] confirmed the conjecture for $m = 6$. Baniabedalruhman [1] and Baniabedalruhman and Jaradat [2] confirmed the conjecture for $n = 7, 8$ and $m = 7$. Furthermore, Cheng et al. [6] proved that the conjecture is true for $m = 7$ and showed that $R(C_7, K_8) = 43$. Jaradat and Alzaleq [9] confirmed the conjecture for $n = m = 8$. Moreover, Bataineh, Jaradat and Al-Zaleq [3] proved the conjecture for $n = 9$ and $m = 8$. In this paper, we prove the Conjecture for $m = 8$ and $10 \leq n \leq 15$. That is we prove that for $10 \leq n \leq 15$, we have $R(C_n, K_8) = 7(n-1) + 1$.

1.1. **Notations and Terminology.** The terminologies used through out the paper are presented here. For a given graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The order of G which is the number of vertices in G is denoted by $|G|$. The graph that consists of n disjoint copies of G is denoted by nG. The path and the cycle of order n are denoted by P_n and C_n, respectively. A graph $K_1 + H$ is obtained by connecting each vertex in H to the vertex of K_1. The neighborhood of a vertex u which is the sub-graph induced by the set of all vertices adjacent to u is denoted by $N(u)$ and $N[u] = N(u) \cup \{u\}$. Moreover, for a sub-graph H we define the neighborhood of H as the sub-graph induced by the set of vertices $W = \{w : w \in N(v) \text{ for some } v \in H\}$ and $N[H] = N(H) \cup H$. Let $u \in V(G)$, $G - \{u\}$ denotes the graph that obtained by deleting u and all edges adjacent to it. The graph $G - H$ is the graph obtained by deleting all vertices in H and all edges that has an end vertex in H. The degree of a vertex u is the number of edges adjacent to it and is denoted by $d(u)$. Also, $\delta(G)$ denotes the minimum degree of the graph G. The order of the largest independent set in the graph G is denoted by $\alpha(G)$. Moreover, $\alpha(N(u))$ denotes the order of the largest independent set in the neighborhood set of a vertex u.

2. Preliminary Lemmas

In this section, we prove a sequence of preliminary lemmas that are necessary in the proof of the main result. Note that, since $R(C_n, K_8) = 7(n-1) + 1$, $n = 8, 9$, then in what follows we assume that G is a graph such that $|G| \geq 7(n-1)+1$ and $n \geq 10$.

Lemma 2.1. *Let G be a graph that contains neither a copy of C_n nor an 8-element independent set. If $\{u_1, u_2, \cdots, u_r\}$, $1 \leq r \leq 7$, is an independent set, then* $|N(\{u_1, u_2, \cdots, u_r\})| \geq r(n-2)+1$.

Proof. Let $\{u_1, u_2, \cdots, u_r\}$ be an independent set and let $|N(\{u_1, u_2, \cdots, u_r\})| < r(n-2)+1$. Then $|G-N[\{u_1, u_2, \cdots, u_r\}]| \geq (7-r)(n-1)+1$. Since $R(C_n, K_{8-r}) = (7-r)(n-1)+1$ and G does not contain a copy of C_n, then $G-N[\{u_1, u_2, \cdots, u_r\}]$ contains an $(8-r)$-element independent set. This set with $\{u_1, u_2, \cdots, u_r\}$ is an 8-element independent set, which is a contradiction. Therefore, we have $|N(\{u_1, u_2, \cdots, u_r\})| \geq r(n-2)+1$. Especially for $r=1$ we have $\delta(G) \geq n-1$. Thus, the proof is completed. □

Lemma 2.2. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of K_{n-1}.*

Proof. Suppose that G contains a copy of K_{n-1}. Let $U = \{u_1, u_2, \cdots, u_{n-1}\}$ be the vertices of K_{n-1} and let $T = G-U$. Since $\delta(G) \geq n-1$ and G does not contain a copy of C_n, then we have $N(u_i) \cap T \neq \phi$, for $i = 1, 2, \cdots, n-1$ and $N(u_i) \cap N(u_j) \cap T = \phi$, for $1 \leq i < j \leq n-1$. Moreover, for $1 \leq i < j \leq n-1$ and every $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$, we have $xy \notin E(G)$. Then the neighbors of $\{u_1, \cdots, u_{n-1}\}$ form an independent set. Therefore, G has an 8-element independent set, which is a contradiction. Hence, the proof is completed. □

Lemma 2.3. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of $K_1 + C_{n-2}$.*

Proof. Suppose that G contains a copy of $K_1 + C_{n-2}$. Let $U = \{u_1, u_2, \cdots, u_{n-2}\}$ and v be the vertices of C_{n-2} and K_1, respectively. Also, let $T = G - (U \cup \{v\})$. Since $\delta(G) \geq n-1$ and G does not contain a copy of C_n, then we have $N(u_i) \cap T \neq \phi$, for $i = 1, 2, \cdots, n-2$, and $N(u_i) \cap N(u_j) \cap T = \phi$, for $1 \leq i < j \leq n-2$. Moreover, for $1 \leq i < j \leq n-2$ and every $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$, we have $xy \notin E(G)$. Since $n-2 \geq 8$, then the neighbors of $\{u_1, \cdots, u_{n-2}\}$ form an 8-element independent set. Therefore, G has an 8-element independent set, which is a contradiction. So, the proof is completed. □

Lemma 2.4. *If G contains neither a copy of C_n nor an 8-element independent set, then $N(\{u_1, u_2\}) - \{v\}$ does not contain a copy of C_{n-2} for any $u_1, u_2 \in N(v)$ where $v \in V(G)$.*

Proof. Suppose that $N(\{u_1, u_2\}) - \{v\}$ contains $C_{n-2} = c_1c_2 \cdots c_{n-2}$. If all vertices of C_{n-2} are adjacent to exactly one of u_1 or u_2, then by Lemma 2.3, we have a contradiction. Therefore, the vertices of C_{n-2} are adjacent to u_1 and u_2. Now, if there is an integer $1 \leq i \leq n-2$ such that $u_1c_i, u_2c_{i+2} \in E(G)$, then $u_1vu_2c_{i+2}c_{i+3} \cdots c_iu_1$ is a copy of C_n, which is a contradiction. Thus, all vertices of C_{n-2} with odd indices are adjacent to one vertex of $\{u_1, u_2\}$ and the vertices with even indices are adjacent to the other vertex. Assume that, the vertices of C_{n-2} with odd indices are adjacent to u_1 and the vertices of C_{n-2} with even indices are adjacent to u_2. Then, $c_2u_2c_4c_5 \cdots c_1u_1c_3c_2$ is a copy of C_n, which is a contradiction. Therefore, the proof is completed. □

Lemma 2.5. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of $K_1 + P_{n-2}$.*

Proof. Suppose that G contains a copy of $K_1 + P_{n-2}$. Let $U = \{u_1, u_2, \cdots, u_{n-2}\}$ and v be the vertices of P_{n-2} and K_1, respectively. Also, let $T = G - (U \cup \{v\})$. Since $\delta(G) \geq n-1$, then for every $1 \leq i \leq n-2$, we have $N(u_i) \cap T \neq \phi$. Also G does not contain a copy of C_n, so for $i = 1, n-2$ and $1 \leq j \leq n-2$, we have $N(u_i) \cap N(u_j) \cap T = \phi$, See FIGURE 1. Moreover, we have $xy \notin E(G)$ for any $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$, for $i = 1, n-2$ and $1 \leq j \leq n-2$ and $i \neq j$. Now, let $w_1 \in N(u_1) \cap T$ and $w_2 \in N(u_{n-2}) \cap T$. Since G contains no copy of C_n, clearly we have $N[N[w_1]] \cap N[N[w_2]] \cap T = \phi$ and for any $x \in N(w_1) \cap T$ and $y \in N(w_2) \cap T$, we have $xy \notin E(G)$. Also, since G does not contain a copy of K_{n-1}, then for $i = 1, 2$, $N(w_i) \cap T$ has two independent vertices. Suppose that $r_1, r_2 \in N(w_1) \cap T$ and $r_3, r_4 \in N(w_2) \cap T$ are the independent vertices. By Lemma 2.1, $|(N(\{r_i, r_{i+1}\}) - \{w_i\}) \cap T| \geq 2(n-2) - 1 = 2(n-3) + 1 = R(C_{n-2}, K_3)$, for $i = 1, 3$. Note that, for $1 \leq i \leq 4$, we have $N(r_i) \cap (\{u_1, \cdots, u_{n-2}\} \cup \{v\}) = \phi$. By

Lemma 2.4, $(N(\{r_i, r_{i+1}\}) - \{w_i\}) \cap T$ does not contain a copy of C_{n-2} and hence $\alpha((N(\{r_i, r_{i+1}\}) - \{w_i\}) \cap T) \geq 3$, for $i = 1, 3$. Also, $N(v) \cap T \neq \phi$ since $\delta(G) \geq n-1$. Let $m \in N(v) \cap T$. Thus, $N(m)$ contains two independent vertices as otherwise G contains a copy of K_{n-1}. In addition, $N(m) \cap (N(\{r_i, r_{i+1}\}) \cap T) = \phi$ and $xy \notin E(G)$ for any $x \in N(m)$ and $y \in N(\{r_i, r_{i+1}\}) \cap T$, $i = 1, 3$, as otherwise G contains a copy of C_n. Therefore, $\alpha(G) \geq \alpha(N(\{r_1, r_2\}) \cap T) + \alpha(N(\{r_3, r_4\}) \cap T) + \alpha(N(m)) \geq 3 + 3 + 2 = 8$ and hence G contains an 8-element independent set, which is a contradiction. Therefore, the proof is completed. □

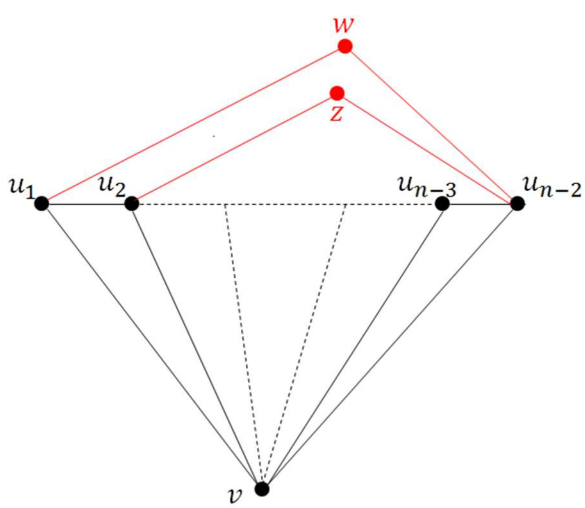

FIGURE 1. The graph $K_1 + P_{n-2}$ with $z \in (N(u_2) \cap N(u_{n-2}))$ and $w \in (N(u_1) \cap N(u_{n-2}))$.

Lemma 2.6. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of K_{n-2}.*

Proof. Suppose that G contains K_{n-2}. Let $U = \{u_1, u_2, \cdots, u_{n-2}\}$ be the vertices of K_{n-2} and let $T = G - U$. Since $\delta(G) \geq n-1$, G does not contain a copy of $K_1 + P_{n-2}$ and G does not contain a copy of C_n, then for $1 \leq k \leq n-2$ and $1 \leq i < j \leq n-2$, we have $N(u_k) \cap T \neq \phi$, $N(u_i) \cap N(u_j) \cap T = \phi$, and $xy \notin E(G)$ for any $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$. So, there are eight vertices in$N(U)$ such that they are mutually nonadjacent. Therefore G contains an 8-element independent set, which is a contradiction. Hence, the proof is completed. □

Lemma 2.7. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of $K_1 + C_{n-3}$.*

Proof. Suppose that G contains a copy of K_1+C_{n-3}. Let $U = \{u_1, u_2, \cdots, u_{n-3}\}$ and v be the vertices of C_{n-3} and K_1, respectively. Also, let $T = G - (U \cup \{v\})$. Now, let $\{u_1, u_2, u_3, u_4\}$ be a four consecutive vertices of $V(C_{n-3})$. We have the following observations:

(1) $xy \notin E(G)$ for any $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$, for $1 \le i < j \le 4$ and $x \ne y$, as otherwise a copy of C_n is produced.
(2) If there are $1 \le i' < j' \le 4$ such that $N(u_{i'}) \cap N(u_{j'}) \cap T \ne \phi$ for some $1 \le i' < j' \le 4$, then for all $1 \le i < j \le 4$ with $\{i, j\} \ne \{i', j'\}$, we have $N(u_i) \cap N(u_j) \cap T = \phi$. As otherwise a copy of C_n is produced.
(3) By Lemma 2.5, $xv \notin E(G)$ for any $x \in N(u_i) \cap T$, for $i = 1, 2, 3, 4$.
(4) Since $\delta(G) \ge n - 1$, then $|N(u_i) \cap T| \ge 2$, for $i = 1, 2, 3, 4$.
(5) By the above Observations, we conclude that there are four independent vertices $\{v_1, v_2, v_3, v_4\}$ such that $v_i \in N(u_i) \cap T$, $i = 1, 2, 3, 4$.
(6) Since G does not contain a copy of C_n and $\delta(G) \ge n - 1$, then there are four independent vertices $\{w_1, w_2, w_3, w_4\}$ such that $w_i \in N(v_i) \cap T$ and $w_iu_j \notin E(G)$, for $1 \le i \le 4$, $1 \le j \le n - 3$ and $i \ne j$. Note that, if $w_iu_j \in E(G)$ for some $1 \le i \le 4$, $1 \le j \le n - 3$ and $i < j$, then $w_iv_iu_iu_{i-1} \cdots u_1vu_{n-3}u_{n-4}u_{j+1}vu_{i+1}u_{i+2} \cdots u_jw_i$ is a cycle of order n, which is a contradiction.
(7) Since $\delta(G) \ge n - 1$, then $|N(w_i) \cap (T - \{v_i\})| \ge n - 3$, $1 \le i \le 4$. Hence, by Lemma 2.6 we have $\alpha(N(w_i) \cap (T - \{v_i\})) \ge 2$, $1 \le i \le 4$.
(8) Since G does not contain C_n, then $(N(w_i) \cap (T - \{v_i\})) \cap (N(w_j) \cap (T - \{v_j\})) = \phi$, for $1 \le i < j \le 4$. Thus, $\alpha(G) \ge \alpha(N(w_1) \cap (T - \{v_1\})) + \alpha(N(w_2) \cap (T - \{v_2\})) + \alpha(N(w_3) \cap (T - \{v_3\})) + \alpha(N(w_4) \cap (T - \{v_4\})) \ge 2 + 2 + 2 + 2 = 8$.

Therefore, G contains an 8-element independent set, which is a contradiction. Thus, the proof is completed. □

Lemma 2.8. *If G contains neither a copy of C_n nor an 8-element independent set, then $N(\{u_1, u_2\}) - \{v\}$ does not contain a copy of C_{n-3} for any $u_1, u_2 \in N(v)$ where $v \in V(G)$.*

Proof. Suppose that $N(\{u_1, u_2\}) - \{v\}$ contains a copy of $C_{n-3} = c_1 c_2 \cdots c_{n-3}$. If all vertices of C_{n-3} are adjacent to exactly one of u_1 or u_2, then by Lemma 2.7, we have a contradiction. So there is an $1 \leq i \leq n-4$ so that $c_i u_1, c_{i+1} u_2 \in E(G)$. Thus, the vertices of C_{n-3}, u_1, u_2 and v form a copy of C_n, which is a contradiction. Therefore, the proof is completed. □

Lemma 2.9. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of $K_1 + P_{n-3}$.*

Proof. Suppose that G contains a copy of $K_1 + P_{n-3}$. Let $U = \{u_1, u_2, \cdots, u_{n-3}\}$ and v be the vertices of P_{n-3} and K_1, respectively. Also, let $T = G - (U \cup \{v\})$. We have the following observations:

(1) $xy \notin E(G)$ for any $x \in N(u_1) \cap T$ and $y \in N(u_{n-3}) \cap T$, with $x \neq y$, as otherwise a copy of C_n, with vertex set $\{x, y, u_{n-3}, v, u_{n-4}, \cdots, u_1, x\}$, is produced.

(2) Since $\delta(G) \geq n-1$, then there is $u_{n-2} \in N(v) \cap T$. By Lemma 2.5, we have $u_1 u_{n-2}, u_{n-3} u_{n-2} \notin E(G)$. Also, by Lemma 2.7 we have $u_1 u_{n-3} \notin E(G)$.

(3) $N(u_i) \cap N(u_{n-2}) \cap T = \phi$, for $i = 1, n-3$. Otherwise, if $l \in N(u_i) \cap N(u_{n-2}) \cap T$ for some $i = 1, n-3$, then a copy of C_n, with vertex set $\{l, u_{n-2}, v, u_{n-i-2}, \cdots, u_i, l\}$, is produced.

(4) There are two nonadjacent vertices $v_1 \in N(u_1) \cap T$ and $v_2 \in N(u_{n-3}) \cap T$.

(5) Since $\delta(G) \geq n-1$, then there are two nonadjacent vertices $w_1 \in N(v_1) \cap T$ and $w_2 \in N(v_2) \cap T$.

(6) Since G does not contain a copy of C_n, then $w_1 y \notin E(G)$ for any $y \in (N(v) - \{u_1\})$ and $w_2 z \notin E(G)$ for any $z \in (N(v) - \{u_{n-3}\})$.

(7) $|N(w_i) - (N[v] \cup \{v_i\})| \geq n-3$, for $i = 1, 2$, hence using Lemma 2.6, there are two nonadjacent vertices $s_1, s_2 \in N(w_1) - (N[v] \cup \{v_1\})$ and two nonadjacent vertices $s_3, s_4 \in N(w_2) - (N[v] \cup \{v_2\})$. Since G does not contain C_n, then

$(N(w_1)-(N[v]\cup\{v_1\}))\cap(N(w_2)-(N[v]\cup\{v_2\}))=\phi$ and hence $\{s_1,s_2,s_3,s_4\}$ is an independent set.

(8) By Lemma 2.1, $|N(\{s_1,s_2\})-(N[v]\cup\{w_1,v_1\})|\geq 2(n-4)+2\geq R(C_{n-3},K_3)$. Thus, by Lemma 2.8 there are three independent vertices $t_1,t_2,t_3\in N(\{s_1,s_2\})-(N[v]\cup\{w_1,v_1\})$. Moreover, $\{t_1,t_2,t_3,s_3,s_4\}$ is an independent set.

(9) $N(u_{n-2})-\{v\}$ has two independent vertices $\{z_1,z_2\}$. Otherwise, there will be a copy of K_{n-2} which is a contradiction with Lemma 2.6. If either z_1 or z_2 are adjacent to the vertices v_1 and v_2, then one can easily find a copy of C_n. Thus, z_1 and z_2 are adjacent to at most one vertex of $\{v_1,v_2\}$.

(10) By Lemma 2.1, we have $|N(\{z_1,z_2\})-(\{u_1,u_{n-3},u_{n-2},v_1,v_2\})|\geq 2(n-4)+1=R(C_{n-3},K_3)$. Thus, by Lemma 2.8 we have $N(\{z_1,z_2\})-(\{u_1,u_{n-3},u_{n-2},v_i\})$ has three independent vertices $\{h_1,h_2,h_3\}$.

(11) The set $\{t_1,t_2,t_3,s_3,s_4,h_1,h_2,h_3\}$ is an independent set.

Therefore, G contains an 8-element independent set, which is a contradiction. So, the proof is completed. □

Lemma 2.10. *If G contains neither a copy of C_n nor an 8-element independent set, then G does not contain a copy of K_{n-3}.*

Proof. Suppose that G contains a copy of K_{n-3}. Let $U=\{u_1,u_2,\cdots,u_{n-3}\}$ be the vertices of this copy of K_{n-3} and let $T=G-U$. Since $\delta(G)\geq n-1$ and G does not contain a copy of K_1+P_{n-3}, then for $1\leq k\leq n-3$ and $1\leq i<j\leq n-3$, we have $N(u_k)\cap T\neq\phi$ and $N(u_i)\cap N(u_j)\cap T=\phi$. Now, we consider two cases.

Case 2.0.1. For some $x\in N(u_i)\cap T$ and $y\in N(u_j)\cap T$ and $1\leq i<j\leq n-3$, we have $xy\in E(G)$.

Proof. Suppose that $xy\in E(G)$ for some $x\in N(u_i)\cap T$ and $y\in N(u_j)\cap T$ and $1\leq i\leq j\leq n-3$. Without loss of generality we may assume that $i=1$ and $j=2$. Then for $3\leq i<j\leq n-3$, we have $rs\notin E(G)$ for any $r\in N(u_i)\cap T$ and $s\in N(u_j)\cap T$. As otherwise one can find a copy of C_n in G. Since G does not contain a copy of C_n, then $N(w_i)\cap N(w_j)\cap T=\phi$, for $3\leq i<j\leq n-3$, $w_i\in N(u_i)\cap T$ and $w_j\in N(u_j)\cap T$. Thus, $|N(w_i)\cap T|\geq n-2$ and $\alpha(N(w_i)\cap T)\geq 2$ where $w_i\in N(u_i)\cap T$ and $3\leq i\leq n-3$. Since $n-5\geq 4$, then there are four vertices in

U such that each of them has at least two nonadjacent vertices. Therefore, G has an 8-element independent set, which is a contradiction.

Case 2.0.2. For any $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$ and $1 \leq i < j \leq n-3$, we have $xy \notin E(G)$.

Proof. Suppose that for every $1 \leq i < j \leq n-3$, we have $xy \notin E(G)$ for any $x \in N(u_i) \cap T$ and $y \in N(u_j) \cap T$. For every $1 \leq i \leq j$ and $w_i \in N(u_i) \cap T$, we have $|N(w_i) \cap T| \geq n-2$ and so by Lemma 2.6, we have $\alpha(N(w_i) \cap T) \geq 2$. Moreover, since G does not contain any copy of C_n, then for every $1 \leq i < j \leq n-3$, and $w_i \in N(u_i) \cap T$ and $w_j \in N(u_j) \cap T$, we have $N(w_i) \cap N(w_j) \cap T = \phi$. Since $n-3 \geq 4$, using a similar argument as in the proof of Case 2.0.1, one can find an 8-element independent set. Which is a contradiction.

Therefore, G contains no copy of K_{n-3} and we are done. □

Lemma 2.11. *Let G be a graph that contains neither a copy of C_n nor an 8-element independent set. If H is a sub-graph of G with $|H| \geq 4(n-3)+3$, then $\alpha(H) \geq 5$.*

Proof. Suppose that $|H| \geq 4(n-3)+3$ and $\alpha(H) \leq 4$. We have the following observations:

(1) If there is a vertex $u \in V(H)$ such that $\alpha(N_H(u)) = 4$, let $\{u_1, u_2, u_3, u_4\}$ be the independent set in $N_H(u)$. Note that, every vertex $x \in V(H) \backslash N_H[u]$ has at least one neighbor in $\{u_1, u_2, u_3, u_4\}$. Otherwise, there is a vertex $u_5 \in V(H)$ such that $u_i u_5 \notin E(H)$, for $1 \leq i \leq 4$, and hence $\{u_1, u_2, u_3, u_4, u_5\}$ is a 5-element independent set in H, which is a contradiction. Thus, $|N_H(\{u_1, u_2, u_3, u_4\}) - \{u\}| \geq 4(n-4)+1 = R(C_{n-3}, K_5)$. Now, since $|N_H(\{u_1, u_2, u_3, u_4\}) - \{u\}| \geq R(C_{n-3}, K_5)$, then using a similar argument as in the proof of Lemma 2.8, $N_H(\{u_1, u_2, u_3, u_4\}) - \{u\}$ does not contain a copy of C_{n-3}. Therefore, the graph H has a 5-element independent set, which is a contradiction.

(2) If there is a vertex $u \in V(H)$ such that $\alpha(N_H(u)) = 3$, let $\{u_1, u_2, u_3\}$ be the independent set in $N_H(u)$. If, $|H - N_H[\{u_1, u_2, u_3\}]| \geq n-1$, then by Lemma 2.2, we have $H - N_H[\{u_1, u_2, u_3\}]$ contains two independent vertices $\{w_1, w_2\}$. The set of $\{u_1, u_2, u_3, w_1, w_2\}$ is a 5-element independent set in H, which is a contradiction. Thus, $|N_H(\{u_1, u_2, u_3\}) - \{u\}| \geq 3(n-4)+1 = R(C_{n-3}, K_4)$.

Therefore, using a similar arguments of the proof of Lemma 2.8, we conclude that $N_H(\{u_1, u_2, u_3\}) - \{u\}$ does not contain a copy of C_{n-3}. Thus, $N_H(\{u_1, u_2, u_3\}) - \{u\}$ contains a 4-element independent set $\{v_1, v_2, v_3, v_4\}$ such that $u_iv_i \in E(H)$, for $i = 1, 2, 3$, and $u_3v_4 \in E(H)$. Now, we have the following observations:

(a) If $u_iv_j \notin E(H)$ for all $i = 1, 2$ and $j = 3, 4$, then $\{u_1, u_2, v_3, v_4\}$ is an independent set. Note that, every vertex in $V(H) - \{u_1, u_2, v_3, v_4\}$ has at least one neighbor in $\{u_1, u_2, v_3, v_4\}$. As otherwise we have a 5-element independent set. Thus, $|N_H(\{u_1, u_2, v_3, v_4\}) - \{u_3, u\}| \geq 4(n - 3) - 3 = 4(n - 4) + 1 = R(C_{n-3}, K_5)$. If $N_H(\{u_1, u_2, v_3, v_4\}) - \{u_3, u\}$ does not contain a copy of C_{n-3}, then it contains a 5-element independent set, which is a contradiction. If $N(\{u_1, u_2, v_3, v_4\}) - \{u_3, u\}$ contains $C_{n-3} = c_1c_2 \cdots c_{n-3}c_1$, then by Lemmas 2.7 and 2.8, the vertices of C_{n-3} are adjacent to u_i and v_j for some $i = 1, 2$ and $j = 3, 4$. Let $1 \leq i' \leq n - 3$ be the index for which we have $u_ic_{i'}, v_jc_{i'+1} \in E(H)$ for some $i = 1, 2$ and $j = 3, 4$. Now, since for $i = 1, 2$ and $j = 3, 4$, we have a path of order 4 between u_i and v_j and H does not contain a copy of C_n, then $u_ic_{i'+2k}, v_jc_{i'+2k+1} \in E(H)$ for some $i = 1, 2$ and $j = 3, 4$ and for all $k = 0, 1, \cdots, \lfloor \frac{n-4}{2} \rfloor$. Moreover, for some non-negative integers r and w with $r < w \leq \lfloor \frac{n-4}{2} \rfloor$, if $c_{i'+2r}c_{i'+2w} \in E(H)$, then $c_{i'+2r}c_{i'+2w}c_{i'+2w+1} \cdots c_{i'+2r-1}v_ju_3uu_ic_{i'+2w-2}c_{i'+2w-3} \cdots c_{i'+2r}$ is a cycle of order n, for some $i = 1, 2$ and $j = 3, 4$. Thus, $c_{i'}, c_{i'+2}, c_{i'+4}$ are independent vertices. Therefore, $\{v_3, v_4, c_{i'}, c_{i'+2}, c_{i'+4}\}$ is a 5-element independent set, which is a contradiction.

(b) If $u_iv_j \in E(H)$ for some $i = 1, 2$ and $j = 3, 4$, let $u_1v_3 \in E(H)$, then by a similar argument as in the beginning of Observation 2a above, we have $|N(\{v_1, v_2, v_3, v_4\}) - \{u_1, u_2, u_3, u\}| \geq 4(n - 3) - 5 = 4(n - 5) + 3 \geq R(C_{n-4}, K_5)$. Thus, if $N(\{v_1, v_2, v_3, v_4\}) - \{u_1, u_2, u_3, u\}$ does not contain a copy of C_{n-4}, then it contains a 5-element independent set, which is a contradiction. Now, let $N(\{v_1, v_2, v_3, v_4\}) - \{u_1, u_2, u_3, u\}$ contains a copy of $C_{n-4} = c_1c_2 \cdots c_{n-4}c_1$. If the vertices of C_{n-4} are adjacent to one vertex of $\{v_1, \cdots, v_4\}$, say v_1, then $V(C_{n-4})$ contains a two independent vertices $\{c_{i'}, c_{j'}\}$ as otherwise K_{n-3} is produced. Hence, the set of $\{v_2, v_3, v_4, c_{i'}, c_{j'}\}$ is a 5-element independent set in H, which is a contradiction. If the vertices of C_{n-4} are adjacent to at least two vertices of $\{v_1, v_2, v_3, v_4\}$, then this case is similar to Observation (2a). In this Observation, we have a cycle of order $n-4$ and there is a path of order 5 between any two vertices of $\{v_1, v_2, v_3, v_4\}$, while in observation (2a), we had a cycle of order $n-3$ and there is a path of order 4 between u_i and v_j, for all $i = 1, 2$ and $j = 3, 4$. Fore example, if $v_1c_1, v_2c_2 \in E(H)$, then with a similar argument as in the proof of 2a, we have $v_1c_3, v_1c_5 \in E(H)$ and $v_2c_4, v_2c_6 \in E(H)$. Moreover, one can verify that the set $\{c_1, c_3, c_5\}$ is an independent set. Therefore, $\{v_2, v_3, c_1, c_3, c_5\}$ is a 5-element independent set.

(3) $\alpha(N_H(u)) \leq 2$ for any $u \in V(H)$. Since $|H| \geq 4(n-3)+3 = 3(n-1)+1+n-7 \geq R(C_n, K_4)$ and H does not contain a copy of C_n, then H contains a set of four independent vertices $\{u_1, u_2, u_3, u_4\}$. Note that, $|N_H(\{u_1, u_2, u_3, u_4\})| \geq 4(n-3)-1$ as otherwise, H contains a set of five independent vertices, which is a contradiction. Hence, there is a vertex in $\{u_1, u_2, u_3, u_4\}$, say u_1 such that $|N_H(u_1)| \geq n-3$. Since G does not contain a copy of K_{n-3} and $K_1 + P_{n-3}$ and $\alpha(N_H(u_1)) \leq 2$, then $N_H(u_1)$ is a union of two complete sub-graphs H_1 and H_2. Note that, since H does not contain a copy of K_{n-3}, then $|H_i| \geq 2$, for $i = 1, 2$. Since H does not contain a copy of K_{n-3} and $\alpha(H) \leq 4$, then $|H - N_H[H_1 \cup H_2]| \leq 2(n-4)$ and hence $|N_H[H_1 \cup H_2]| \geq 2(n-4) + 7 = 5 + R(C_{n-3}, K_3)$. Moreover, since H does not contain a copy of C_n and K_{n-3}, then $\alpha(N_H(H_1 \cup H_2) - \{u_1\}) \geq 3$ and hence $\alpha(N_H(H_1 \cup H_2)) \geq 4$, let $\{v_1, v_2, v_3, u_1\}$ be the independent set in $N_H(H_1 \cup H_2)$. Now, since $\alpha(H) \leq 4$, then the vertices v_1, v_2 and v_3 are adjacent to at least three vertices in $H_1 \cup H_2$. Without loss of generality suppose that $z_1, z_2, z_3, z_4 \in V(N_H(u_1))$ are the vertices such that $v_1z_1, v_2z_2, v_3z_3, z_1z_2, z_3z_4 \in E(H)$. Since $\alpha(H) \leq 4$, then every vertex in $H - (\{v_1, v_2, v_3, u_1\} \cup \{z_1, z_2, z_3, z_4\})$ has at least one neighbor in $\{v_1, v_2, v_3, u_1\}$. Therefore, $|N(\{v_1, v_2, v_3, u_1\}) - \{z_1, \cdots, z_4\}| \geq 4(n-3) - 5 =$

$4(n-5)+3 \geq R(C_{n-4}, K_5)$. Thus, if $N(\{v_1, v_2, v_3, u_1\}) - \{z_1, \cdots, z_4\}$ does not contain a copy of C_{n-4}, then it contains a 5-element independent set, which is a contradiction. Therefore, let $N(\{v_1, v_2, v_3, u_1\}) - \{z_1, \cdots, z_4\}$ contains a copy of $C_{n-4} = c_1c_2 \cdots c_{n-4}c_1$. Since, for $1 \leq i \leq 3$ there is a path of order 4 between u_1 and v_i, then the vertices of C_{n-4} are not adjacent to u_1 and v_i. Otherwise, a copy of C_n is produced, which is a contradiction. Moreover, there is a path of order 5 between any two vertices of $\{v_1, v_2, v_3\}$. Now, the rest of the proof is similar to Observation (2b).

Therefore, H has a 5-element independent set. Thus, the proof is completed. □

3. Main Result

In this section, we prove that $R(C_n, K_8) = 7(n-1)+1$, for $10 \leq n \leq 15$. The graph $7K_{n-1}$ contains neither a copy of C_n nor an 8-element independent set. Thus, $R(C_n, K_8) \geq 7(n-1)+1$. Therefore, it is sufficient to prove that $R(C_n, K_8) \leq 7(n-1)+1$, for $10 \leq n \leq 15$.

Theorem 3.1. *$R(C_n, K_8) = 7n-6$, for $10 \leq n \leq 15$.*

Proof. The proof is by induction. Note that, $R(C_r, K_8) = 7(r-1)+1$, for $r = 8, 9$. Now, let G be a graph of order at least $7(n-1)+1$ and let $R(C_m, K_8) = 7(m-1)+1$ for any $9 \leq m < n$ and $10 \leq n \leq 15$. If G contains a copy of C_n or an 8-element independent set, then we are done. Therefore, assume that G contains neither a copy of C_n nor an 8-element independent set. Since, by Lemma 2.1, $\delta(G) \geq n-1$ and by Lemma 2.2, G does not contain a copy of K_{n-1}, then $\alpha(N(u)) \geq 2$ for any vertex $u \in V(G)$. To finish the proof, we consider two cases.

Case 3.1.1. There is a vertex $u \in V(G)$ such that $\alpha(N(u)) \geq 3$.

Proof. Let u be a vertex in $V(G)$ such that $\alpha(N(u)) \geq 3$. Without loss of generality, assume that $\alpha(N(u)) = 3$ and let $\{u_1, u_2, u_3\}$ be the independent set in $N(u)$. We have the following observations:

(1) If u_1 and u_2 are isolated vertices in $N(u)$, then by Lemma 2.1, $|N(u) - \{u_1, u_2\}| \geq n-3$. Hence, by Lemma 2.10, $N(u) - \{u_1, u_2\}$ contains two independent vertices. Those two independent vertices with $\{u_1, u_2\}$ is a 4-element independent set in $N(u)$, which is a contradiction. Therefore, at most one vertex of $\{u_1, u_2, u_3\}$ is isolated in $N(u)$.

(2) If $|G\backslash(N(\{u_1, u_2, u_3\}) - \{u\})| \geq 4(n-3)+3$, then by Lemma 2.11 $G\backslash(N(\{u_1, u_2, u_3\}) - \{u\})$ contains five independent vertices. Those five independent vertices with $\{u_1, u_2, u_3\}$ is an 8-element independent set, which is a contradiction. Therefore, $|N(\{u_1, u_2, u_3\}) - \{u\}| \geq 3(n-4)+12 \geq 4(n-4)+1 = R(C_{n-3}, K_5)$, for $10 \leq n \leq 15$.

(3) By Observation 2 and Lemma 2.8, $N(\{u_1, u_2, u_3\}) - \{u\}$ does not contain C_{n-3}. Thus, $N(\{u_1, u_2, u_3\}) - \{u\}$ contains a 5-element independent set, say $\{v_1, v_2, v_3, v_4, v_5\}$. Moreover, using Observation 1, assume that $v_1u_1, v_2u_2, v_3u_2, v_4u_3, v_5u_3 \in E(G)$.

(4) Since $\alpha(G) \leq 7$ and G does not contain a copy of K_{n-3}, then $|G - N[\{v_1, v_2, v_3, v_4, v_5\}]| \leq 2(n-4)$. Therefore, $|N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3\}| \geq 5(n-4) + 13 \geq R(C_{n-3}, K_7)$, for $10 \leq n \leq 15$.

(5) If $u_1u_4 \in E(G)$ for some $u_4 \in V(N(u))$, then $N(\{v_1, v_2\}) - \{u, u_1, u_2, u_4\}$ does not contain C_{n-3}. Otherwise, if it contains $C_{n-3} = c_1c_2 \cdots c_{n-3}c_1$, then by Lemma 2.7, we have $v_1c_1, v_2c_2 \in E(G)$. Therefore, $v_1c_4, v_2c_5 \in E(G)$, as otherwise a copy of $C_n = c_1v_1u_1uu_2v_2c_4c_5 \cdots c_{n-3}c_1$ is produced. Moreover, $c_1v_1u_1u_4uu_2v_2c_5c_6 \cdots c_{n-3}c_1$ is a copy of a cycle of order n, which is a contradiction. Thus, $N(\{v_1, v_2\}) - \{u, u_1, u_2, u_4\}$ does not contain a copy of C_{n-3}.

(6) Let $u_4, u_5 \in N(u)$ be the vertices such that $u_1u_4, u_2u_5, uu_5 \in E(G)$. Then, we have $|N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3, u_4, u_5\}| \geq R(C_{n-3}, K_7)$, for $10 \leq n \leq 14$. Thus, using the details explained in Observation 5, we conclude that $N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3, u_4, u_5\}$ contains a 7-element independent set. Let $\{w_1, \cdots, w_7\}$ be the independent set in $N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3, u_4, u_5\}$.

(7) One can easily check that $|N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3, u_5\}| \geq R(C_{n-3}, K_7)$, for $n = 15$, where $u_2u_5, uu_5 \in E(G)$. If $N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3, u_4\}$ contains a copy of $C_{n-3} = c_1c_2 \cdots c_{n-3}c_1$, then its vertices are adjacent to the vertices of $\{v_1, v_4, v_5\}$. Without loss of generality suppose that $c_1v_1, c_1v_i \in E(G)$ for some $i = 4, 5$. Then, $v_1c_{3k+1}, v_ic_{3k+2} \in E(G)$, for some $i = 4, 5$ and for all $k = 0, 1, 2, 3$, as otherwise a copy of C_n is produced. Moreover, $\{c_1, c_4, c_7, c_{10}\}$ is an independent set as otherwise a copy of C_n is produced. For example, if $c_1c_{10} \in E(G)$, then $c_{10}c_1v_1u_1uu_3v_4c_2 \cdots c_9c_{10}$ is a cycle of order 15. Also, if $c_1c_7 \in E(G)$, then $c_1c_7c_6c_5 \cdots c_2v_4u_3uu_1v_1c_{10}c_{11}c_{12}c_1$ is a cycle of order 15. Therefore, $\{c_1, c_4, c_7, c_{10}, v_2, v_3, v_4, v_5\}$ is an 8-element independent set, which is a contradiction. Thus, let $\{w_1, \cdots, w_7\}$ be the independent set in $N(\{v_1, v_2, v_3, v_4, v_5\}) - \{u, u_1, u_2, u_3, u_5\}$.

(8) Let $v_6, v_7 \in V(G)$ be the vertices such that $u_1u_4, u_2u_5, uu_4, uu_5, v_2v_6, u_2v_6, v_4v_7, u_3v_7 \in E(G)$. Since $\alpha(G) \leq 7$, then $|N(\{w_1, \cdots, w_7\}) - \{u, u_1, \cdots, u_5, v_1, v_2, \cdots, v_7\}| \geq 7(n-4) + 1 \geq R(C_{n-3}, K_8)$, for $10 \leq n \leq 15$. Now, suppose $N(\{w_1, \cdots, w_7\}) - \{u, u_1, \cdots, u_5, v_1, \cdots, v_7\}$ contains a copy of C_{n-3}. Then, by Lemmas 2.7 and 2.8, the vertices of C_{n-3} are adjacent to vertices in $N(v_i) - \{u, u_1, \cdots, u_5\}$ and vertices in $N(v_j) - \{u, u_1, \cdots, u_5\}$, for some $1 \leq i < j \leq 5$. Moreover, by using a similar arguments as in Observation 5, we have the graph G contains a copy of C_n, which is a contradiction. For example, let $c_1w_1, c_2w_2, v_1w_1, v_2w_2 \in E(G)$. If $c_{n-7}w_1 \notin E(G)$, then for some $i = 1, 2$, $2 \leq j \leq 5$ and $2 \leq k \leq 7$, we have a copy of $C_n = c_1w_1v_1u_1uu_iv_jw_kc_{n-7}c_{n-6} \cdots c_1$. And, if $c_{n-7}w_1 \in E(G)$, then $c_2c_3 \cdots c_{n-7}w_1v_1u_1u_4uu_2v_2w_2c_2$ is a cycle of order n. Therefore, $N(\{w_1, \cdots, w_7\}) - \{u, u_1, \cdots, u_5, v_1, \cdots, v_7\}$ does not contain a copy of C_{n-3}. Hence, $N(\{w_1, \cdots, w_7\}) - \{u, u_1, \cdots, u_5, v_1, \cdots, v_7\}$ contains an 8-element independent set, which is a contradiction.

As a result, if there is a vertex u in $V(G)$ such that $\alpha(N(u)) \geq 3$, then G contains a copy of C_n or an 8-element independent set, which is a contradiction.

Case 3.1.2. $\alpha(N(u)) \leq 2$ for any vertex $u \in V(G)$.

Proof. Suppose that $\alpha(N(u)) \leq 2$ for any $u \in V(G)$. Since $\delta(G) \geq n-1$ and G does not contain a copy of K_n, then $\alpha(N(u)) = 2$ for any $u \in V(G)$. Now, let $u \in G$, then by Lemmas 2.9 and 2.10, $N(u)$ is a union of two complete sub-graphs, H_1 and H_2. Since G does not contain C_n, then $(N(H_1) - \{u\}) \cap (N(H_2) - \{u\}) = \phi$. By Lemma 2.1, $(N(w_i) \cap (G - N[u])) \neq \phi$ for any $w_i \in V(H_i)$, for $i = 1, 2$. Since $\alpha(N(v)) = 2$ for any $v \in V(G)$, then $N(w_i) \cap (G - N[u])$ is a complete graph for any $w_i \in V(H_i)$, for $i = 1, 2$. Otherwise, if there is a vertex w_i in $V(H_i)$, for $i = 1, 2$, such that $N(w_i) \cap (G - N[u])$ contains two independent vertices $\{z_1, z_2\}$, then $\{z_1, z_2, u\}$ is a 3-element independent set in $N(w_i)$, which is a contradiction. Moreover, since $(N(x) - (H_i \cup \{u\}))$ is a complete graph, for $i = 1, 2$, and G does not contain a copy of C_n, then $(N(x) - (H_i \cup \{u\})) \cap (N(y) - (H_i \cup \{u\})) = \phi$ for any $x, y \in V(H_i)$, for $i = 1, 2$. Therefore, $\alpha((N(H_1) - \{u\}) \cup (N(H_2) - \{u\})) = |H_1| + |H_2| \geq n - 1 \geq 8$, which is a contradiction.

Therefore, $R(C_n, K_8) = 7(n-1) + 1$, for $10 \leq n \leq 15$. So, the proof is completed.

□

4. Conclusion

In this paper, we find the exact Ramsey number $R(C_n, K_8)$ for $10 \leq n \leq 15$. The proofs of the lemmas have some new ideas which will help the researchers on this topic. Also, this result will help to find the Ramsey number $R(C_n, K_m)$ for all $n \geq m$.

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