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A NEW CUBIC TRANSMUTED POWER FUNCTION DISTRIBUTION: PROPERTIES, INFERENCE AND APPLICATION

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Abstract. A new cubic transmuted power function distribution has been proposed by using the cubic transmuted family of distributions, proposed by [1]. The proposed distribution provides transmuted power function distribution as a special case. The properties of the proposed distribution are studied that include shape, moments, quantiles, entropy, random number generation and order statistics. The maximum likelihood estimation of the parameters of the proposed distribution is discussed. A simulation study has been conducted to observe the performance of the estimation procedure. The proposed distribution has been applied to real data sets to compare the suitability of the model.

1. Introduction

Probability distributions have widespread applications in many areas of life. The need for probability distributions has always been there to model the phenomenon in different fields of science and engineering. The standard probability distributions can be used to model the data originated from different domains but the need has always been there to extend the standard probability distributions for much wider applicability. The work on families of distributions has been done by various authors to extend the standard probability distributions by introducing new parameters to base distribution such that more flexible distributions can be obtained which increase the reliability to study the behavior of real-life data. A new family of distributions, named as the beta-G family of distributions, has been proposed by [2] by using the logit of the beta distribution. The proposed family of distributions extends the

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distribution of order statistics. The Kumaraswamy-G family of distributions has been proposed by [3] by using the distribution function of the Kumaraswamy distribution given by [4]. A more general method to extend the standard probability distributions has been proposed by [5] by using a combination of two probability distributions. The family of distributions proposed by [5] is named as the *T-X* family of distributions. The cumulative distribution function (*cdf*) of this family of distributions is

$$F_{T-X}(x) = \int_{a}^{W[G(x)]} r(t)\, dt \tag{1.1}$$

where $r(t)$is the density of any random variable defined on $[a, b]$, where a can be -∞ and b can be $+\infty$ and $W[G(x)]$ is any function of $G(x)$ such that $W(0) = a$ and $W(1) = b$. The *T-X* family of distributions provide beta-G and Kumaraswamy-G families of distributions turned out to be a special case of the *T-X* family of distribution.

A simple method to extend any baseline distribution has been proposed by [6] by adding one new parameter. The proposed family of distributions is named as the transmuted family of distributions. The *cdf* of this family of distributions is

$$F(x) = G(x) + \lambda G(x)[1 - G(x)] \quad -1 \leq \lambda \leq 1, \tag{1.2}$$

where λ is the transmutation parameter. The transmuted distribution reduces to the baseline distribution for $\lambda = 0$. The transmuted family of distributions has been extended by [1] and [7] by adding one more parameter and have named the proposed family of distributions as the cubic transmuted family of distributions. The *cdf* of the cubic transmuted family of distributions proposed by [1] is

$$F(x) = G(x) + \lambda_1 G(x)[1 - G(x)] + \lambda_2 G^2(x)[1 - G(x)] \quad x \in R, \tag{1.3}$$

where λ_1 and λ_2 are transmutation parameters such that $(\lambda_1, \lambda_2) \in [-1, 1]$and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. Also, $g(x)$ and $G(x)$ are, respectively, the density and distribution functions of any baseline distribution. The density function corresponding to (1.3) is

$$f(x) = g(x)\left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1) G(x) - 3\lambda_2 G^2(x)\right] \quad x \in R. \tag{1.4}$$

The cubic transmuted family of distributions, given in (1.3), reduces to the transmuted family of distributions, given in (1.2), for $\lambda_2 = 0$. Also, the cubic transmuted family of distributions, (1.3), reduced to the baseline distribution for $\lambda_1 = \lambda_2 = 0$. The cubic transmuted family of distributions, (1.3) has not been much explored, and in this paper, we have used this family of distributions with the baseline power function distribution to propose a new cubic transmuted power function distribution. The organization of the paper is given below.

A new cubic transmuted power function distribution has been introduced in section 2. In section 3 some useful properties of the proposed cubic transmuted power function distribution have been discussed. The maximum likelihood estimation of the parameters is given in Section 4. Some numerical studies are given in Section 5. The conclusions and recommendations are given in Section 6.

2. A New Cubic Transmuted Power Function Distribution

The power function distribution is a simple yet very useful distribution. The density and distribution function of the power function distribution are

$$g(x) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha} \quad 0 < x < \theta \quad \& \quad G(x) = \frac{x^\alpha}{\theta^\alpha} \quad 0 < x < \theta, \tag{2.1}$$

where $\alpha > 0$ is the shape parameter and $\theta > 0$ is the scale parameter. The power function distribution given in (2.1), reduces to the uniform distribution over $[0, \theta]$ for $\alpha = 1$. The power function distribution has been studied by various authors in different contexts. The relations for moments of lower generalized order statistics for the distribution have been obtained by [8]. Some characterizations of the distribution by using lower record values have been given by [9]. The transmuted power function distribution has been proposed by [10] by using *cdf* of the distribution in the transmuted family of distributions, (1.2). In the following, we have proposed the cubic transmuted power function distribution by using the density and distribution function of the power function distribution in (1.3) and (1.4). The *cdf* of the new cubic transmuted power function (*NCTPF*) distribution is

$$F(x) = (1 + \lambda_1)\frac{x^\alpha}{\theta^\alpha} + (\lambda_2 - \lambda_1)\frac{x^{2\alpha}}{\theta^{2\alpha}} - \lambda_2\frac{x^{3\alpha}}{\theta^{3\alpha}} \quad 0 < x < \theta, \tag{2.2}$$

where $(\lambda_1, \lambda_2) \in [-1, 1]$and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. The density function corresponding to (2.2) is

$$f(x) = \frac{\alpha x^{\alpha-1}}{\theta^{\alpha}} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2 \frac{x^{2\alpha}}{\theta^{2\alpha}}\right] \quad 0 < x < \theta. \tag{2.3}$$

The proposed Cubic Transmuted Power Function distribution reduces to the Transmuted Power Function distribution, proposed by [10], for $\lambda_2 = 0$. Also for $\lambda_1 = \lambda_2 = \lambda$, the proposed cubic transmuted power function distribution provides the cubic transmuted power function distribution of [11] as a special case.
The hazard rate function of the distribution is immediately written from (2.2) and (2.3) as

$$h(x) = \frac{f(x)}{1 - F(x)} = \alpha x^{\alpha-1}\left(\frac{1}{\theta^{\alpha} - x^{\alpha}} + \frac{\lambda_1\theta^{\alpha} + 2\lambda_2 x^{\alpha}}{\theta^{2\alpha} - \lambda_1\theta^{\alpha}x^{\alpha} - \lambda_2 x^{2\alpha}}\right) \quad 0 < x < \theta. \tag{2.4}$$

The mode of the distribution is obtained by solving $\partial \ln f(x)/\partial x = 0$ for x. Now

$$\ln f(x) = \ln\alpha + (\alpha - 1)\ln x - \alpha\ln\theta + \ln\left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right]$$

and $\frac{\partial}{\partial x}\ln f(x) = \frac{\alpha-1}{x} + \frac{2\alpha(\lambda_2-\lambda_1)\left(x^{\alpha-1}/\theta^{\alpha}\right) - 6\alpha\lambda_2\left(x^{2\alpha-1}/\theta^{2\alpha}\right)}{1+\lambda_1+2(\lambda_2-\lambda_1)(x^{\alpha}/\theta^{\alpha}) - 3\lambda_2\left(x^{2\alpha}/\theta^{2\alpha}\right)}$.

Also $\frac{\partial}{\partial x}\ln f(x)\big|_{x=0} = \infty$ *and* $\frac{\partial}{\partial x}\ln f(x)\big|_{x=\theta} = \frac{\alpha+\lambda_1+\lambda_2-3\alpha\lambda_1-5\alpha\lambda_2-1}{(1-\lambda_1-\lambda_2)\theta}$. We can see that $(\ln f)'(\theta)$ changes sign and also attains infinity and hence there is no mode of the distribution. The plots of density, distribution and hazard rate functions of the proposed cubic transmuted power function distribution are given in Figure 1 below. Although, the plots show some peaks but these represents the local maximum of the density function.

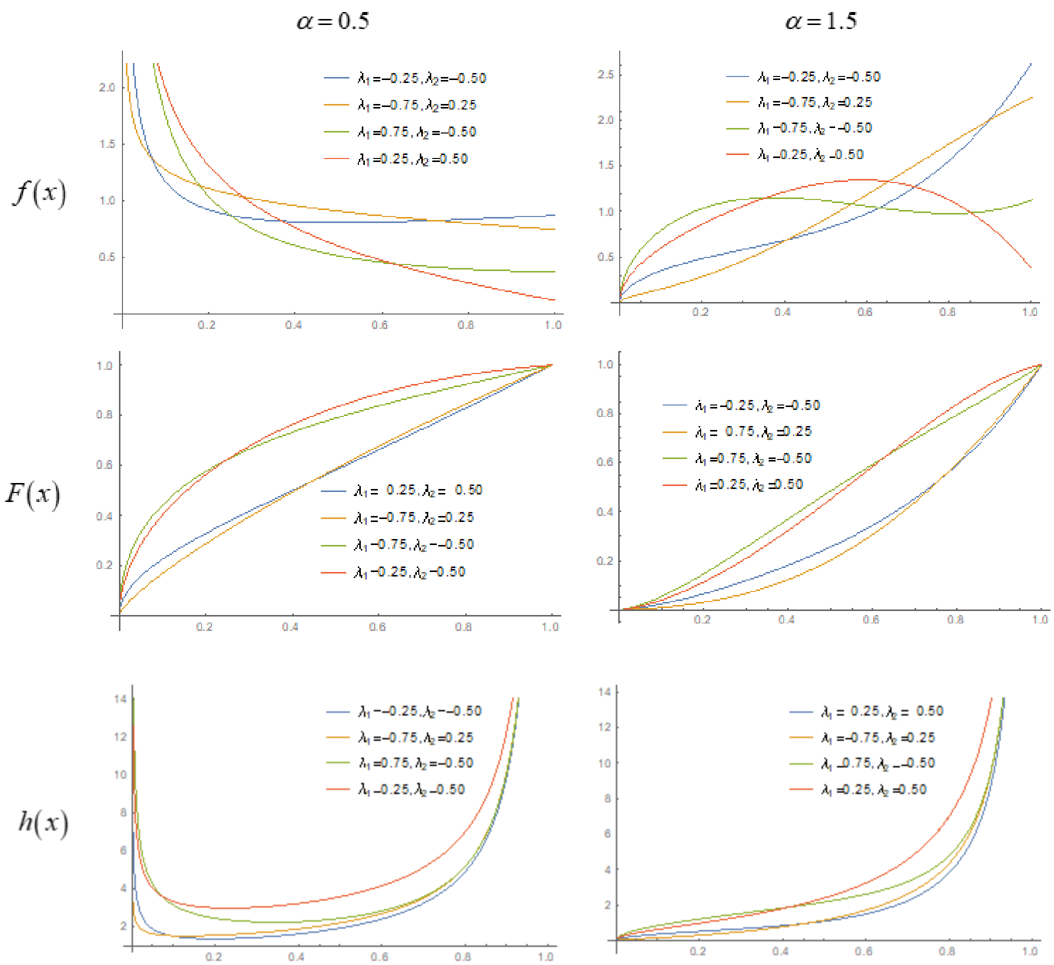

FIGURE 1. Plots of Density, Distribution and Hazard Rate functions

3. PROPERTIES OF THE NEW CUBIC TRANSMUTED POWER FUNCTION DISTRIBUTION

The properties of any probability distribution extensively help to study the behavior of the distribution. Some useful properties of the proposed *CTPF* distribution are discussed in the following subsections.

3.1. **Moments.** The moments are useful to study some useful properties of a distribution. In the following, we have obtained the rth moment for the proposed *CTPF* distribution. Now, the rth moment of the *CTPF* distribution is

$$\mu_r' = E\left(X^r\right) = \int_0^\theta x^r f\left(x\right) dx = \int_0^\theta x^r \left[\frac{\alpha x^{\alpha-1}}{\theta^\alpha}\left\{1+\lambda_1+2\left(\lambda_2-\lambda_1\right)\frac{x^\alpha}{\theta^\alpha}-3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]dx$$
$$= \left(1+\lambda_1\right)\int_0^\theta \frac{\alpha x^{\alpha+r-1}}{\theta^\alpha}dx+2\left(\lambda_2-\lambda_1\right)\int_0^\theta \frac{\alpha x^{2\alpha+r-1}}{\theta^{2\alpha}}dx-3\lambda_2\int_0^\theta \frac{\alpha x^{3\alpha+r-1}}{\theta^{3\alpha}}dx.$$

Solving the above integrals, the rth moment of the *CTPF* distribution is

$$\mu_r^{'} = \frac{\alpha\theta^r\left[6\alpha^2 + \alpha r\left(5 - 3\lambda_1 - \lambda_2\right) + r^2\left(1 - \lambda_1 - \lambda_2\right)\right]}{\left(\alpha + r\right)\left(2\alpha + r\right)\left(3\alpha + r\right)}. \tag{3.1}$$

The moments can be used to obtain the mean, variance, skewness and kurtosis of the distribution. Specifically, the mean and variance of the distribution are

$$\mu = E\left(X\right) = \frac{\alpha\theta\left[6\alpha^2 + \alpha\left(5 - 3\lambda_1 - \lambda_2\right) + \left(1 - \lambda_1 - \lambda_2\right)\right]}{\left(\alpha + 1\right)\left(2\alpha + 1\right)\left(3\alpha + 1\right)}$$

and

$$\begin{aligned}\sigma^2 \quad &= \quad \frac{\alpha\theta^2\left[6\alpha^2 + 2\alpha\left(5 - 3\lambda_1 - \lambda_2\right) + 4\left(1 - \lambda_1 - \lambda_2\right)\right]}{\left(\alpha + 2\right)\left(2\alpha + 2\right)\left(3\alpha + 2\right)} \\ &\quad - \left[\frac{\alpha\theta\left[6\alpha^2 + \alpha\left(5 - 3\lambda_1 - \lambda_2\right) + \left(1 - \lambda_1 - \lambda_2\right)\right]}{\left(\alpha + 1\right)\left(2\alpha + 1\right)\left(3\alpha + 1\right)}\right]\end{aligned}$$

The mean, variance, skewness and kurtosis for the proposed *NCTPF* distribution are given in Table 1 and Table 2. From these tables, we can see that for a fixed value of α, and for different combinations of λ_1 and λ_2, the mean increases with increase in θ. Also, for fixed θ, the mean increases with an increase in α. It can also be observed that for fixed α the variance increases with an increase in θ and for fixed θ, the variance decreases with an increase in α. The variance exhibits this behavior for both combinations of $\lambda_1$andλ_2. The table 2 for skewness shows that if both λ_1 and λ_2 are negative then the distribution is positively skewed for $\alpha < 1$ and negatively skewed otherwise. Also, if both λ_1 and λ_2 are positive then the distribution is positively skewed for $\alpha \leq 1$ and negatively skewed otherwise. The kurtosis of the distribution shows interesting behavior. When both λ_1 and λ_2 are negative then the distribution is platy-kurtic for $\alpha \leq 1$and is lepto-kurtic otherwise. When both λ_1 and λ_2 are positive then the kurtosis changes behavior with an increase in α. It can also be seen that the parameter θ does not affect skewness and kurtosis.

3.2. **Moment Generating and Characteristic Functions.** The moment generating function is useful to obtain the moments. The moment generating function of a random variable is defined as

$$M_X\left(t\right) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tX} f\left(x\right) dx.$$

The moment generating function for the *CTPF* distribution is readily written as

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\theta e^{tx} f(x)\,dx = \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\theta x^r f(x)\,dx$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\theta x^r \left[\frac{\alpha x^{\alpha-1}}{\theta^\alpha} \left\{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right] dx,$$

or

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!} \frac{\alpha\theta^r \left[6\alpha^2 + \alpha r(5 - 3\lambda_1 - \lambda_2) + r^2(1 - \lambda_1 - \lambda_2)\right]}{(\alpha + r)(2\alpha + r)(3\alpha + r)} \tag{3.2}$$

The rth moment can be easily obtained as the coefficient of $t^r/r!$ in (3.2). The characteristic function of the distribution is immediately written from (3.2) as

(3.3)

$$\phi_X(t) = \int_0^\theta e^{itx} f(x)\,dx = \sum_{r=0}^\infty \frac{(it)^r}{r!} \frac{\alpha\theta^r \left[6\alpha^2 + \alpha r(5 - 3\lambda_1 - \lambda_2) + r^2(1 - \lambda_1 - \lambda_2)\right]}{(\alpha + r)(2\alpha + r)(3\alpha + r)},$$

where $i = \sqrt{-1}$is the imaginary number. The rth moment can also be obtained as the coefficient of $(it)^r/r!$ in the expansion of $\phi_X(t)$.

Table 1: **Mean and Variance of Cubic Transmuted Power Function Distribution**

Mean										
α	$\lambda_1 = -0.75$ *and* $\lambda_2 = -0.25$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	0.483	0.967	1.450	1.933	2.417	2.900	3.383	3.867	4.350	4.833
1.0	0.646	1.292	1.938	2.583	3.229	3.875	4.521	5.167	5.813	6.458
1.5	0.730	1.459	2.189	2.918	3.648	4.377	5.107	5.836	6.566	7.295
2.0	0.781	1.562	2.343	3.124	3.905	4.686	5.467	6.248	7.029	7.810
2.5	0.816	1.632	2.447	3.263	4.079	4.895	5.711	6.527	7.342	8.158
3.0	0.841	1.682	2.523	3.364	4.205	5.046	5.888	6.729	7.570	8.411
3.5	0.860	1.720	2.581	3.441	4.301	5.161	6.021	6.882	7.742	8.602
4.0	0.875	1.750	2.626	3.501	4.376	5.251	6.126	7.002	7.877	8.752
4.5	0.887	1.775	2.662	3.549	4.437	5.324	6.211	7.098	7.986	8.873
5.0	0.897	1.795	2.692	3.589	4.486	5.384	6.281	7.178	8.075	8.973
α	$\lambda_1 = 0.25$ *and* $\lambda_2 = 0.50$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	0.242	0.483	0.725	0.967	1.208	1.450	1.692	1.933	2.175	2.417
1.0	0.417	0.833	1.250	1.667	2.083	2.500	2.917	3.333	3.750	4.167
1.5	0.528	1.057	1.585	2.114	2.642	3.170	3.699	4.227	4.756	5.284
2.0	0.605	1.210	1.814	2.419	3.024	3.629	4.233	4.838	5.443	6.048
2.5	0.660	1.320	1.980	2.640	3.300	3.960	4.620	5.280	5.940	6.600
3.0	0.702	1.404	2.105	2.807	3.509	4.211	4.913	5.614	6.316	7.018
3.5	0.734	1.469	2.203	2.938	3.672	4.407	5.141	5.876	6.610	7.345
4.0	0.761	1.521	2.282	3.043	3.803	4.564	5.325	6.085	6.846	7.607
4.5	0.782	1.564	2.347	3.129	3.911	4.693	5.475	6.258	7.040	7.822

5.0	0.800	1.600	2.401	3.201	4.001	4.801	5.601	6.402	7.202	8.002
Variance										
α	$\lambda_1 = -0.75$ *and* $\lambda_2 = -0.25$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	0.090	0.361	0.812	1.443	2.255	3.247	4.420	5.773	7.306	9.020
1.0	0.066	0.265	0.596	1.060	1.656	2.384	3.245	4.239	5.365	6.623
1.5	0.048	0.192	0.432	0.768	1.200	1.727	2.351	3.071	3.887	4.798
2.0	0.036	0.144	0.324	0.575	0.899	1.294	1.761	2.301	2.912	3.595
2.5	0.028	0.111	0.250	0.445	0.696	1.002	1.364	1.781	2.254	2.783
3.0	0.022	0.089	0.199	0.354	0.554	0.797	1.085	1.417	1.794	2.214
3.5	0.018	0.072	0.162	0.288	0.451	0.649	0.883	1.154	1.460	1.803
4.0	0.015	0.060	0.135	0.239	0.374	0.538	0.733	0.957	1.211	1.495
4.5	0.013	0.050	0.113	0.202	0.315	0.454	0.617	0.806	1.021	1.260
5.0	0.011	0.043	0.097	0.172	0.269	0.387	0.527	0.689	0.872	1.076
α	$\lambda_1 = 0.25$ *and* $\lambda_2 = 0.50$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	0.061	0.243	0.546	0.970	1.516	2.183	2.972	3.881	4.912	6.064
1.0	0.068	0.272	0.613	1.089	1.701	2.450	3.335	4.356	5.513	6.806
1.5	0.060	0.241	0.543	0.966	1.509	2.172	2.957	3.862	4.888	6.034
2.0	0.051	0.204	0.458	0.815	1.273	1.833	2.496	3.260	4.125	5.093
2.5	0.043	0.171	0.384	0.683	1.067	1.536	2.090	2.730	3.456	4.266
3.0	0.036	0.144	0.323	0.574	0.898	1.293	1.759	2.298	2.908	3.591
3.5	0.030	0.122	0.274	0.488	0.762	1.097	1.494	1.951	2.469	3.048
4.0	0.026	0.104	0.235	0.418	0.653	0.940	1.280	1.672	2.116	2.612
4.5	0.023	0.090	0.203	0.361	0.565	0.813	1.107	1.446	1.830	2.259
5.0	0.020	0.079	0.177	0.315	0.493	0.710	0.966	1.262	1.597	1.971

Table 2: **Skewness and Kurtosis for the Cubic Transmuted Power Function Distribution**

Skewness										
α	$\lambda_1 = -0.75$ *and* $\lambda_2 = -0.25$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	0.013	0.013	0.013	0.013	0.013	0.013	0.013	0.013	0.013	0.013
1.0	-0.599	-0.599	-0.599	-0.599	-0.599	-0.599	-0.599	-0.599	-0.599	-0.599
1.5	-0.949	-0.949	-0.949	-0.949	-0.949	-0.949	-0.949	-0.949	-0.949	-0.949
2.0	-1.193	-1.193	-1.193	-1.193	-1.193	-1.193	-1.193	-1.193	-1.193	-1.193
2.5	-1.377	-1.377	-1.377	-1.377	-1.377	-1.377	-1.377	-1.377	-1.377	-1.377
3.0	-1.522	-1.522	-1.522	-1.522	-1.522	-1.522	-1.522	-1.522	-1.522	-1.522
3.5	-1.641	-1.641	-1.641	-1.641	-1.641	-1.641	-1.641	-1.641	-1.641	-1.641
4.0	-1.740	-1.740	-1.740	-1.740	-1.740	-1.740	-1.740	-1.740	-1.740	-1.740
4.5	-1.824	-1.824	-1.824	-1.824	-1.824	-1.824	-1.824	-1.824	-1.824	-1.824
5.0	-1.896	-1.896	-1.896	-1.896	-1.896	-1.896	-1.896	-1.896	-1.896	-1.896
α	$\lambda_1 = 0.25$ *and* $\lambda_2 = 0.50$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	1.093	1.093	1.093	1.093	1.093	1.093	1.093	1.093	1.093	1.093
1.0	0.287	0.287	0.287	0.287	0.287	0.287	0.287	0.287	0.287	0.287

1.5	-0.103	-0.103	-0.103	-0.103	-0.103	-0.103	-0.103	-0.103	-0.103	-0.103
2.0	-0.353	-0.353	-0.353	-0.353	-0.353	-0.353	-0.353	-0.353	-0.353	-0.353
2.5	-0.533	-0.533	-0.533	-0.533	-0.533	-0.533	-0.533	-0.533	-0.533	-0.533
3.0	-0.670	-0.670	-0.670	-0.670	-0.670	-0.670	-0.670	-0.670	-0.670	-0.670
3.5	-0.780	-0.780	-0.780	-0.780	-0.780	-0.780	-0.780	-0.780	-0.780	-0.780
4.0	-0.869	-0.869	-0.869	-0.869	-0.869	-0.869	-0.869	-0.869	-0.869	-0.869
4.5	-0.944	-0.944	-0.944	-0.944	-0.944	-0.944	-0.944	-0.944	-0.944	-0.944
5.0	-1.008	-1.008	-1.008	-1.008	-1.008	-1.008	-1.008	-1.008	-1.008	-1.008

Kurtosis										
α	$\lambda_1 = -0.75$ *and* $\lambda_2 = -0.25$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	-1.244	-1.244	-1.244	-1.244	-1.244	-1.244	-1.244	-1.244	-1.244	-1.244
1.0	-0.620	-0.620	-0.620	-0.620	-0.620	-0.620	-0.620	-0.620	-0.620	-0.620
1.5	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224	0.224
2.0	1.044	1.044	1.044	1.044	1.044	1.044	1.044	1.044	1.044	1.044
2.5	1.800	1.800	1.800	1.800	1.800	1.800	1.800	1.800	1.800	1.800
3.0	2.487	2.487	2.487	2.487	2.487	2.487	2.487	2.487	2.487	2.487
3.5	3.108	3.108	3.108	3.108	3.108	3.108	3.108	3.108	3.108	3.108
4.0	3.669	3.669	3.669	3.669	3.669	3.669	3.669	3.669	3.669	3.669
4.5	4.178	4.178	4.178	4.178	4.178	4.178	4.178	4.178	4.178	4.178
5.0	4.641	4.641	4.641	4.641	4.641	4.641	4.641	4.641	4.641	4.641
α	$\lambda_1 = 0.25$ *and* $\lambda_2 = 0.50$									
	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$	$\theta = 8$	$\theta = 9$	$\theta = 10$
0.5	0.286	0.286	0.286	0.286	0.286	0.286	0.286	0.286	0.286	0.286
1.0	-0.943	-0.943	-0.943	-0.943	-0.943	-0.943	-0.943	-0.943	-0.943	-0.943
1.5	-0.932	-0.932	-0.932	-0.932	-0.932	-0.932	-0.932	-0.932	-0.932	-0.932
2.0	-0.700	-0.700	-0.700	-0.700	-0.700	-0.700	-0.700	-0.700	-0.700	-0.700
2.5	-0.417	-0.417	-0.417	-0.417	-0.417	-0.417	-0.417	-0.417	-0.417	-0.417
3.0	-0.131	-0.131	-0.131	-0.131	-0.131	-0.131	-0.131	-0.131	-0.131	-0.131
3.5	0.142	0.142	0.142	0.142	0.142	0.142	0.142	0.142	0.142	0.142
4.0	0.397	0.397	0.397	0.397	0.397	0.397	0.397	0.397	0.397	0.397
4.5	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632	0.632
5.0	0.850	0.850	0.850	0.850	0.850	0.850	0.850	0.850	0.850	0.850

3.3. **Geometric and Harmonic Means.** The geometric mean is a useful measure in finance and is used to see average growth rate in stocks or market value. The harmonic mean is another useful measure to see the average rate of change in speed or stocks, etc. The geometric mean (GM) and harmonic mean (HM) for a continuous random variable are defined as

$$\ln GM = E\left[\ln X\right] \ \ and \ \ \left(HM\right)^{-1} = E\left(X^{-1}\right).$$

Now, for $CTPF$ distribution the geometric mean is given as

$$\ln GM = \int_0^\theta (\ln x) f(x)\, dx = \int_0^\theta (\ln x) \frac{\alpha x^{\alpha-1}}{\theta^\alpha} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \frac{x^{2\alpha}}{\theta^{2\alpha}}\right] dx.$$

Solving the integral we have

$$\ln GM = \ln\theta - \frac{6 + 3\lambda_1 + \lambda_2}{6\alpha} \Rightarrow GM = \theta \exp\left(-\frac{6 + 3\lambda_1 + \lambda_2}{6\alpha}\right). \tag{3.4}$$

Again, the harmonic mean for the $CTPF$ distribution is

$$(HM)^{-1} = \int_0^\theta x^{-1} \frac{\alpha x^{\alpha-1}}{\theta^\alpha} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \frac{x^{2\alpha}}{\theta^{2\alpha}}\right] dx.$$

Solving the integral the result is obtained as

$$(HM)^{-1} = \frac{\alpha[(3\alpha-1)(2\alpha+\lambda_1-1)+\lambda_2(\alpha-1)]}{\theta(\alpha-1)(2\alpha-1)(3\alpha-1)} \Rightarrow HM = \frac{\theta(\alpha-1)(2\alpha-1)(3\alpha-1)}{\alpha[(3\alpha-1)(2\alpha+\lambda_1-1)+\lambda_2(\alpha-1)]}. \tag{3.5}$$

The geometric mean and harmonic mean can be computed for different values of the parameters.

3.4. **The Conditional Moments.** The conditional moments are useful when the random variable is truncated below a specific point. Such moments are useful in reliability analysis and engineering. The rth conditional moment for a random variable is defined as

$$\mu^r_{r|X>t} = E(X^r | X > t) = \frac{1}{1 - F(t)} \int_t^\infty x^r f(x)\, dx.$$

Now, the rth conditional moment for the $CTPF$ distribution is given as

$$\mu'_{r|X>t} = \frac{1}{1 - \Delta_t(\alpha, \theta, \lambda_1, \lambda_2)} \int_t^\theta x^r \frac{\alpha x^{\alpha-1}}{\theta^\alpha} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \frac{x^{2\alpha}}{\theta^{2\alpha}}\right] dx,$$

where $\Delta_t(\alpha, \theta, \lambda_1, \lambda_2) = (1 + \lambda_1)\frac{t^\alpha}{\theta^\alpha} + (\lambda_2 - \lambda_1)\frac{t^{2\alpha}}{\theta^{2\alpha}} - \lambda_2 \frac{t^{3\alpha}}{\theta^{3\alpha}}$. The conditional moment can be written as

$$\begin{aligned}\mu'_{r|X>t} = \tfrac{1}{1-\Delta_t(\alpha,\theta,\lambda_1,\lambda_2)} \Big[&\int_0^\theta x^r \tfrac{\alpha x^{\alpha-1}}{\theta^\alpha} \Big\{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\tfrac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \tfrac{x^{2\alpha}}{\theta^{2\alpha}}\Big\} dx \\ &- \int_t^t x^r \tfrac{\alpha x^{\alpha-1}}{\theta^\alpha} \Big\{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\tfrac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \tfrac{x^{2\alpha}}{\theta^{2\alpha}}\Big\} dx\Big] \\ = \tfrac{1}{1-\Delta_t(\alpha,\theta,\lambda_1,\lambda_2)} \Big[\mu'_r - &\int_t^t x^r \tfrac{\alpha x^{\alpha-1}}{\theta^\alpha} \Big\{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\tfrac{x^\alpha}{\theta^\alpha} - 3\lambda_2 \tfrac{x^{2\alpha}}{\theta^{2\alpha}}\Big\} dx\Big],\end{aligned}$$

where μ'_r is given in (3.1). Now, solving the integral, we have

$$\mu'_{r|X>t} = \frac{1}{1 - \Delta_t(\alpha, \theta, \lambda_1, \lambda_2)} \left[\mu'_r - \frac{\alpha t^{\alpha+r}}{\theta^{3\alpha}} \left\{\frac{(1+\lambda_1)\theta^{2\alpha}}{(\alpha+r)} + \frac{2(\lambda_2 - \lambda_1)\theta^\alpha t^\alpha}{(2\alpha+r)} - \frac{3\lambda_2 t^{2\alpha}}{(3\alpha+r)}\right\}\right], \tag{3.6}$$

where $\mu_r^{'}$ is given in (3.1). It is to be noted that if $t = 0$ then the conditional moment reduces to the raw moment.

3.5. **Quantile Function.** The quantile function is useful to obtain quantiles of a distribution. This function is also useful to generate a random sample from the distribution. The quantile function is obtained as a solution of $F(x) = p$ for x. Now, for the *CTPF* distribution, the quantile function is obtained by solving

$$(1+\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} + (\lambda_2-\lambda_1)\frac{x^{2\alpha}}{\theta^{2\alpha}} - \lambda_2\frac{x^{3\alpha}}{\theta^{3\alpha}} = p.$$

Writing $(x/\theta)^{\alpha} = w$we have

$$(1+\lambda_1)\,w + (\lambda_2-\lambda_1)\,w^2 - \lambda_2 w^3 = p \textit{ or } \lambda_2 w^3 - (\lambda_2-\lambda_1)\,w^2 - (1+\lambda_1)\,w + p = 0$$

or $c_1 w^3 + c_2 w^2 + c_3 w + p = 0$,
where $c_1 = \lambda_2$; $c_2 = -(\lambda_2 - \lambda_1)$and $c_3 = -(1+\lambda_1)$.
Solving the above cubic equation, the only real root is

$$w = -\frac{c_2}{3c_1} - \frac{2^{1/3}\delta_1}{3c_1\delta_3^{1/3}} + \frac{\delta_3^{1/3}}{3\times 2^{1/3}c_1},$$

where $\delta_1 = -c_2^2 + 3c_1c_3$; $\delta_2 = -2c_2^3 + 9c_1c_2c_3 - 27c_1^2 p$ and $\delta_3 = \delta_2 + \sqrt{4\delta_1^3 + \delta_2^2}$.
The quantile function of *CTPF* distribution is therefore obtained by solving

$$\left(\frac{x}{\theta}\right)^{\alpha} = -\frac{c_2}{3c_1} - \frac{2^{1/3}\delta_1}{3c_1\delta_3^{1/3}} + \frac{\delta_3^{1/3}}{3\times 2^{1/3}c_1},$$

for x and is

$$Q_X(p) = \theta\left(-\frac{c_2}{3c_1} - \frac{2^{1/3}\delta_1}{3c_1\delta_3^{1/3}} + \frac{\delta_3^{1/3}}{3\times 2^{1/3}c_1}\right)^{1/\alpha}. \tag{3.7}$$

The random sample can be generated by replacing p with a uniform random number within [0,1] in the above quantile function. The median can be obtained by using p= 0.5 in (3.7).

3.6. **Shannon Entropy.** The Shannon entropy, [12], is a useful measure to decide about the amount of information in a distribution. The Shannon entropy is defined as

$$I_S = E\left[-\ln\{f(X)\}\right] = \int_0^{\theta} -\ln\{f(x)\}\,f(x)\,dx.$$

Now, for $CTPF$ distribution this expression is written as

$$-\ln f(x) = -\ln\left(\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\right) - \ln\left[1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right],$$

and hence the Shannon entropy for $CTPF$ is

$$\begin{aligned} I_S &= \int_0^{\theta}\left[-\ln\left(\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\right) - \ln\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right] \\ &\quad\times\left[\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]dx \\ &= -\int_0^{\theta}\ln\left(\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\right)\left[\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]dx \\ &\quad-\int_0^{\theta}\ln\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\} \\ &\quad\times\left[\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]dx \end{aligned}$$

or

$$I_S = \ln\left(\frac{\theta}{\alpha}\right) + \frac{(6+3\lambda_1+\lambda_2)}{6\alpha} - I_1, \tag{3.8}$$

where

$$\begin{aligned} I_1 &= \int_0^{\theta}\ln\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\} \\ &\quad\times\left[\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]dx. \end{aligned}$$

The Shannon entropy can be computed for different values of the parameters. An approximation for the Shannon entropy can be obtained by using the expansion of

$$\ln(1+y) = \ln\left\{1+\lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}$$

where

$$y = \lambda_1+2(\lambda_2-\lambda_1)\frac{x^{\alpha}}{\theta^{\alpha}} - 3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}$$

and retaining only the linear term. In this case, the integral becomes

$$\begin{aligned} I_1 &= \int_0^\theta \ln\left\{1+\lambda_1+2\left(\lambda_2-\lambda_1\right)\frac{x^\alpha}{\theta^\alpha}-3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\} \\ &\quad\times\left[\frac{\alpha x^{\alpha-1}}{\theta^\alpha}\left\{1+\lambda_1+2\left(\lambda_2-\lambda_1\right)\frac{x^\alpha}{\theta^\alpha}-3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]dx \\ &= \frac{1}{15}\left(5\lambda_1^2+5\lambda_1\lambda_2+2\lambda_2^2\right) \end{aligned}$$

and an approximate value of Shannon entropy is

$$I_S = \ln\left(\frac{\theta}{\alpha}\right)+\frac{\left(6+3\lambda_1+\lambda_2\right)}{6\alpha}-\frac{1}{15}\left(5\lambda_1^2+5\lambda_1\lambda_2+2\lambda_2^2\right). \tag{3.9}$$

The approximate value of Shannon entropy can be computed for various values of the parameters.

3.7. **Order Statistics.** In this section, a brief description of order statistics for *CTPF* distribution is given. For this, suppose that a random sample of size n is available from *CTPF* distribution and $x_{1:n} \leq x_{2:n} \leq \cdots\cdots \leq x_{n:n}$ be the corresponding order statistics. The distribution of rth order statistics is then given as

$$f_{r:n}\left(x\right) = \frac{1}{B\left(r,n-r+1\right)}f\left(x\right)\left[F\left(x\right)\right]^{r-1}\left[1-F\left(x\right)\right]^{n-r} \quad r=1,2,\ldots,n, \tag{3.10}$$

where $B\left(a,b\right)$ is the complete beta function defined as

$$B\left(a,b\right) = \int_0^1 w^{a-1}\left(1-w\right)^{b-1}dw.$$

More details about the order statistics may be seen in [13] or in [14]. Now, using the density and distribution function of *CTPF* distribution in (3.10), the distribution of rth order statistics is

$$\begin{aligned} f_{r:n}\left(x\right) &= \frac{1}{B\left(r,n-r+1\right)}\frac{\alpha x^{2\alpha-1}}{\theta^{2\alpha}}\left[1+\lambda_1+2\left(\lambda_2-\lambda_1\right)\frac{x^\alpha}{\theta^\alpha}-3\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right] \\ &\left[1+\lambda_1+\left(\lambda_2-\lambda_1\right)\frac{x^\alpha}{\theta^\alpha}-\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right]^{r-1}\times\left[1-\frac{x^\alpha}{\theta^\alpha}\left\{1+\lambda_1+\left(\lambda_2-\lambda_1\right)\frac{x^\alpha}{\theta^\alpha}-\lambda_2\frac{x^{2\alpha}}{\theta^{2\alpha}}\right\}\right]^{n-r} \\ &\qquad x>\theta\ ,\ \alpha>0\ ,\ r=1,2,\ldots\ldots,n. \end{aligned} \tag{3.11}$$

The distribution of minimum can be easily obtained from (3.11) by using $r=1$ and the distribution of maximum can be obtained from (3.11) by using $r=n$.

4. Maximum Likelihood Estimation

In this section, the maximum likelihood estimation for parameters of *CTPF* distribution has been derived. For this, suppose a random sample of size n is available from the *CTPF* distribution. The likelihood function is

$$L(\alpha, \theta, \lambda_1, \lambda_2; x) = \frac{\alpha^n}{\theta^{n\alpha}} \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x_i^{\alpha}}{\theta^{\alpha}} - 3\lambda_2 \frac{x_i^{2\alpha}}{\theta^{2\alpha}}\right].$$

The log of the likelihood function is

$$\ell = \ln L(\alpha, \theta, \lambda_1, \lambda_2; x) = n \ln \alpha - n\alpha \ln \theta + (\alpha - 1)\sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\frac{x_i^{\alpha}}{\theta^{\alpha}} - 3\lambda_2 \frac{x_i^{2\alpha}}{\theta^{2\alpha}}\right]. \tag{4.1}$$

Since the parameter θ appears in the upper domain of the random variable so the maximum likelihood estimator of θ is $\hat{\theta} = \max(x_1, x_2, \ldots, x_n) = x_{n:n}$. In order to obtain the maximum likelihood estimators of other parameters, the partial derivatives of the log-likelihood function is obtained with respect to these parameters. These derivatives are

$$\frac{\partial \ell}{\partial \alpha} = n\left(\frac{1}{\alpha} - \ln \theta\right) + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \frac{2\ln(x_i/\theta)(x_i/\theta)^{\alpha}\{\lambda_1 - \lambda_2 + 3\lambda_3 (x_i/\theta)^{\alpha}\}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)(x_i/\theta)^{\alpha} - 3\lambda_2 (x_i/\theta)^{2\alpha}}, \tag{4.2}$$

$$\frac{\partial \ell}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2(x_i/\theta)^{\alpha}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)(x_i/\theta)^{\alpha} - 3\lambda_2 (x_i/\theta)^{2\alpha}}, \tag{4.3}$$

and

$$\frac{\partial \ell}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2(x_i/\theta)^{\alpha} - 3(x_i/\theta)^{2\alpha}}{1 + \lambda_1 + 2(\lambda_2 - \lambda_1)(x_i/\theta)^{\alpha} - 3\lambda_2 (x_i/\theta)^{2\alpha}} \tag{4.4}$$

The maximum likelihood estimates of α, λ_1 and λ_2 can be obtained by equating (4.2) - (4.4) to zero and numerically solving the resulting equations. Also, as $n \to \infty$

then we know that the asymptotic distribution of maximum likelihood estimates of $\hat{\Theta} = \left(\hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2\right)$ is given as [15],

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix} \sim N_3 \left[\begin{pmatrix} \alpha \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} \right],$$

where $\hat{V}_{ij} = V_{ij}|_{\Theta=\hat{\Theta}}$. Also, the entries V_{ij} can be obtained by inverting the Hessian matrix; see Appendix.

5. Numerical Study

In this section, a numerical study for the proposed *CTPF* distribution has been conducted. The study is carried out in two different ways; namely simulation and real data application. These numerical studies are given in the following subsections.

5.1. **Simulation.** In this section, a simulation for *CTPF* distribution is performed by generating random samples from the proposed *CTPF* distribution. The simulation algorithm is given below:

(1) Generate random samples of sizes 50, 100, 200, 500 and 1000 from *CTPF* distribution by using different values of the parameters.
(2) Obtain maximum likelihood estimates of the parameters α, λ_1 and λ_2. Also set $\hat{\theta} = x_{n:n}$ for each sample.
(3) Repeat the process for, say, 20000 times.
(4) Obtain average estimate as

$$\hat{\hat{\alpha}} = \frac{1}{20000}\sum_{j=1}^{20000}\hat{\alpha}_j \; ; \; \hat{\hat{\theta}} = \frac{1}{20000}\sum_{j=1}^{20000}\hat{\theta}_j \; ; \; \hat{\hat{\lambda}}_1 = \frac{1}{20000}\sum_{j=1}^{20000}\hat{\lambda}_{1j} \; and \; \hat{\hat{\lambda}}_2 = \frac{1}{20000}\sum_{j=1}^{20000}\hat{\lambda}_{2j}.$$

(5) Obtain mean square error of the estimate as

$$MSE\left(\hat{\alpha}\right) = \tfrac{1}{20000}\sum_{j=1}^{20000}\left(\hat{\alpha}_j - \hat{\hat{\alpha}}\right)^2 \; ; \; MSE\left(\hat{\theta}\right) = \tfrac{1}{20000}\sum_{j=1}^{20000}\left(\hat{\theta}_j - \hat{\hat{\theta}}\right)^2$$
$$MSE\left(\hat{\lambda}_1\right) = \tfrac{1}{20000}\sum_{j=1}^{20000}\left(\hat{\lambda}_{1j} - \hat{\hat{\lambda}}_1\right)^2 \; ; \; MSE\left(\hat{\lambda}_2\right) = \tfrac{1}{20000}\sum_{j=1}^{20000}\left(\hat{\lambda}_{2j} - \hat{\hat{\lambda}}_2\right)^2$$

The estimated parameters and the mean square errors are given in Table 3, below. From the table, it can be seen that the estimated values of the parameters are close to actual values and hence the estimation process is consistent. Also the mean square

TABLE 3. **Simulation Results**

$\alpha = 1.5\,, \theta = 2.0\,, \lambda_1 = 0.5\,, \lambda_2 = -0.25$								
Sample Size	**Estimates**				**Mean Square Errors**			
	α	θ	λ_1	λ_2	α	θ	λ_1	λ_2
50	1.489	2.000	0.504	-0.244	0.097	0.095	0.045	0.043
100	1.519	1.988	0.503	-0.246	0.089	0.089	0.044	0.041
200	1.506	1.985	0.480	-0.264	0.088	0.086	0.040	0.042
500	1.517	1.991	0.503	-0.239	0.073	0.079	0.024	0.039
1000	1.512	1.995	0.516	-0.266	0.042	0.076	0.023	0.038
$\alpha = 2.5\,, \theta = 3.0\,, \lambda_1 = -0.5\,, \lambda_2 = 0.25$								
Sample Size	**Estimates**				**Mean Square Errors**			
	α	θ	λ_1	λ_2	α	θ	λ_1	λ_2
50	2.506	2.987	-0.511	0.266	0.096	0.092	0.048	0.056
100	2.494	2.995	-0.500	0.258	0.094	0.087	0.040	0.048
200	2.504	2.987	-0.513	0.246	0.092	0.087	0.031	0.034
500	2.490	3.005	-0.520	0.251	0.087	0.083	0.028	0.030
1000	2.512	2.988	-0.515	0.231	0.085	0.080	0.027	0.029

TABLE 4. Summary Measures for Two Data Sets

Data	n	**Min**	**Mean**	Q_1	**Median**	Q_3	**Skew**	**Max**
Mammal Brain	84	0.45	106.9261	9.8250	50.5000	183.500	1.1019	442.00
Bladder Cancer	128	0.08	9.3656	3.3475	6.3950	11.8375	3.2481	79.05

error of the estimates reduces with an increase in the sample size and hence the estimates become more efficient with an increase in the sample size.

5.2. **Real Data Applications.** A real data application of the proposed *CTPF* distribution is conducted in this section which has been done by using mammal brain data; obtained from [16]; containing mammal brain weight in grams and the bladder cancer data; obtained from [17]; containing remission time from bladder cancer. The summary measures of these two data sets are given below. Different competitive distributions along with the proposed *CTPF* distribution are fitted on these data sets, namely, the cubic transmuted power function distribution, by [11], the transmuted power function distribution, by [10] and the power function distribution for the purpose of comparison. The distributions have been fitted by computing the maximum likelihood estimates of the parameter, obtained by using the *maxLik* function of *R*, [18].

TABLE 5. **MLE's and Goodness of Fit Measures for the Mammal Brain Data**

Parameters	CTPF (New)	CTPF (Ansari)	TPF (Haq)	Power Function
$\hat{\theta}$	442 -	442 -	442 -	442 -
$\hat{\alpha}$	0.5272 (0.0733)	0.5191 (0.0764)	0.4456 (0.0459)	0.3823 (0.0417)
$\hat{\lambda}_1$	0.9997 (0.5036)	0.9995 (0.5071)	1.0000 (0.3103)	
$\hat{\lambda}_2$	0.6838 (0.6238)			
Log-likelihood	(0.6238)	(0.5071)	(0.3103)	
AIC	874.7988	914.0048	886.4986	915.4038
BIC	882.0913	918.8664	891.3602	917.8346
KS (pvalue)	0.9781	0.8392	0.0111	0.5732

The maximum likelihood estimates for various distributions and their standard error for the mammal brain data are given in Table 5. This table also contains the values of log-likelihood function, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) for various distributions for the mammal brain data sets.

From the above table, it is observed that the proposed $CTPF$ distribution is the best fit for the mammal brain data as this distribution has the largest value of the log-likelihood function. Also, the proposed $CTPF$ distribution has the smallest values of AIC and BIC. This distribution also has largest p-value for the Kolmogorov-Smirnov test and hence the proposed CTPF distribution is the best fit to the data.

The results for various distributions for the bladder cancer data are given in Table 6 below. This table also shows that the proposed $CTPF$ distribution is the best fit for the bladder cancer data as this distribution has the largest value of the log-likelihood function and the smallest values of AIC and BIC alongside the highest p-value for the Kolmogorov-Smirnov test.

Table 6. **MLEs and Goodness of Fit Measures for the Bladder Cancer Data**

Parameters	CTPF (New)	CTPF (Ansari)	TPF (Haq)	Power Function
$\hat{\theta}$	79.05 -	79.05 -	79.05 -	79.05 -
$\hat{\alpha}$	0.4016 (0.0856)	0.2664 (0.1846)	0.4589 (0.0374)	0.3822 (0.0388)
$\hat{\lambda}_1$	0.0428 (0.0512)	1.0000 (0.3486)	1.0000 (0.2307)	
$\hat{\lambda}_2$	0.9995 (0.2248)			
Log-likelihood	450.5112	459.7019	453.2533	475.5632
AIC	907.0224	923.4038	910.5066	953.1264
BIC	915.5785	929.1079	916.2107	955.9784
KS (pvalue)	0.4528	0.3748	0.0248	0.2587

The plots of actual data and fitted distributions for two data sets are shown in figure 2 below.

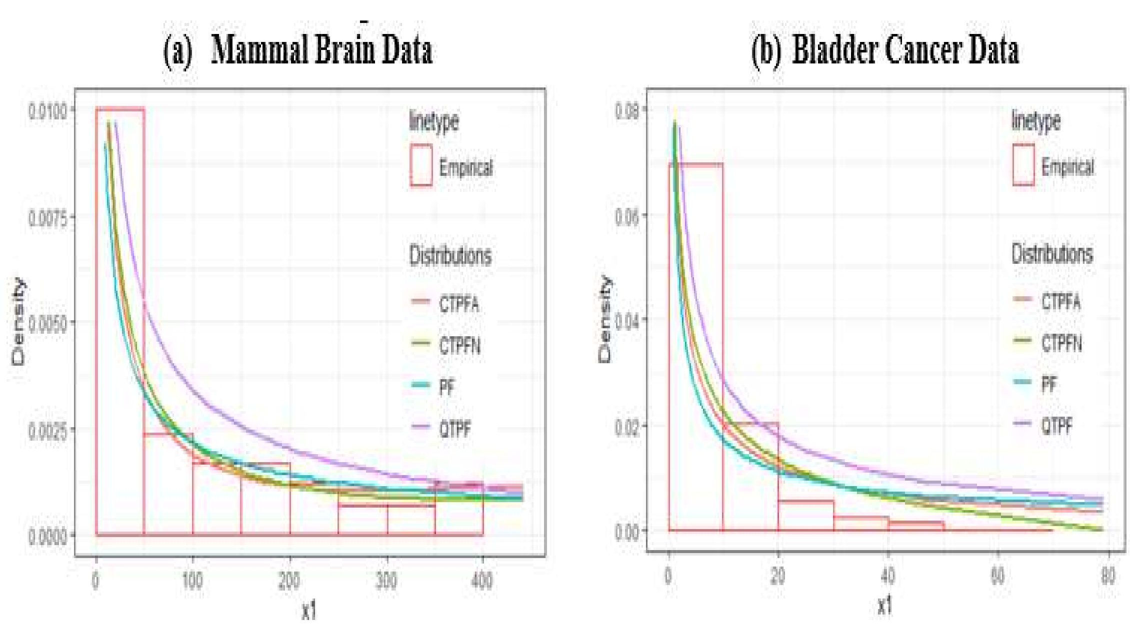

Figure 2. **Actual Data and Fitted Distributions**

6. Conclusions

In this paper, a new cubic transmuted power function ($CTPF$) distribution is proposed by using the cubic transmuted family of distributions suggested by Rahman et al. (2018). The proposed distribution provides the cubic transmuted power function distribution of [11] and the transmuted power function distribution of [10] as a special case. Some useful properties of the proposed $CTPF$ distribution have also been studied. These properties include moments, generating functions, quantile function and random number generation. The geometric mean, harmonic mean and conditional moments of the distribution are also obtained. The maximum likelihood estimation of the parameters of the proposed $CTPF$ distribution has also been done. The simulation study to see the consistency of the estimation is also conducted. The proposed $CTPF$ distribution provides better fit on two real data sets. The proposed $CTPF$ distribution can be used to model the data sets for reliability analysis.

Appendix

The Hessian matrix for the proposed $CTPF$ distribution is

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}.$$

The variance - covariance matrix of the estimated parameters is given as

$$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}^{-1}.$$

The entries of $\mathbf{H}$ are given below.

We have used $\Delta\left(\lambda_1, \lambda_2\right) = 1 + \lambda_1 + 2\left(\lambda_2 - \lambda_1\right)\left(\frac{x}{\theta}\right)^{\alpha} - 3\lambda_3\left(\frac{x}{\theta}\right)^{2\alpha}$.

$$\begin{aligned} H_{11} &= \frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum\nolimits_{i=1}^{n}\left[\frac{1}{\Delta^2\left(\lambda_1, \lambda_2\right)}\left(2\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)\right.\right. \\ &\left.\left\{\lambda_1 - \lambda_2 + 3\lambda_2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\right)^2 - \frac{2}{\Delta\left(\lambda_1, \lambda_2\right)} \\ &\left.\left(\frac{x_i}{\theta}\right)^{\alpha}\ln^2\left(\frac{x_i}{\theta}\right)\left\{\lambda_1 - \lambda_2 + 6\lambda_2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\right] \end{aligned}$$

$$H_{12} = \frac{\partial^2 \ell}{\partial\alpha\partial\lambda_1} = \sum\nolimits_{i=1}^{n} \left[\frac{1}{\Delta^2(\lambda_1,\lambda_2)}\left(2\left(\frac{x_i}{\theta}\right)^{\alpha}\left\{1-2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\right.\right.$$
$$\left.\left.\times \ln\left(\frac{x_i}{\theta}\right)\left\{\lambda_1-\lambda_2+3\lambda_2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\right)^2 - \frac{2}{\Delta(\lambda_1,\lambda_2)}\ln\left(\frac{x_i}{\theta}\right)\right]$$

$$H_{13} = \frac{\partial^2 \ell}{\partial\alpha\partial\lambda_2} = \sum\nolimits_{i=1}^{n} \left[\frac{1}{\Delta^2(\lambda_1,\lambda_2)}\left(2\left(\frac{x_i}{\theta}\right)^{\alpha}\left\{2\left(\frac{x_i}{\theta}\right)^{\alpha}-3\left(\frac{x_i}{\theta}\right)^{2\alpha}\right\}\right.\right.$$
$$\left.\times \ln\left(\frac{x_i}{\theta}\right)\left\{\lambda_1-\lambda_2+3\lambda_2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\right)^2 + \frac{2}{\Delta(\lambda_1,\lambda_2)}$$
$$\left.\times\left(\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)\left\{1-3\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\right)\right]$$

$$H_{22} = \frac{\partial^2 \ell}{\partial\lambda_1^2} = -\sum\nolimits_{i=1}^{n}\left[\frac{1}{\Delta^2(\lambda_1,\lambda_2)}\left\{1-2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}^2\right]$$

$$H_{23} = \frac{\partial^2 \ell}{\partial\alpha\partial\lambda_2} = -\sum\nolimits_{i=1}^{n}\left[\frac{1}{\Delta^2(\lambda_1,\lambda_2)}\left\{1-2\left(\frac{x_i}{\theta}\right)^{\alpha}\right\}\left\{2\left(\frac{x_i}{\theta}\right)^{\alpha}-3\left(\frac{x_i}{\theta}\right)^{2\alpha}\right\}\right]$$

$$H_{33} = \frac{\partial^2 \ell}{\partial\lambda_2^2} = -\sum\nolimits_{i=1}^{n}\left[\frac{1}{\Delta^2(\lambda_1,\lambda_2)}\left\{2\left(\frac{x_i}{\theta}\right)^{\alpha}-3\left(\frac{x_i}{\theta}\right)^{2\alpha}\right\}^2\right]$$

It is to be noted that $H_{21} = H_{12}$, $H_{31} = H_{13}$ and $H_{32} = H_{23}$.

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